Motivation

Mixing time analysis: Analyze mixing behavior of common MCMC algorithms for sampling from Ising and Potts models with coupling heuristic.

Methods

Background

- Ising model: A probability distribution on \((-1, +1)^V\), where \(V\) is the set of vertices of a graph \(G = (V, E)\). A configuration of the Ising model assigns one of two spins, either -1 or 1, to every vertex.
- Potts model: The generalization of the Ising model, allowing vertices to take on one of \(q\) colors instead of two spins.
- \(\beta\) (beta): Parameter dictating strength of bonds between sites. In ferromagnetic case \(\beta > 0\), configurations with more monochromatic edges, or edges with matching endpoint colors, are more likely to occur.
- Markov Chain Monte Carlo (MCMC): A class of algorithms which use Markov chains to randomly sample from high-dimensional distributions.

Results

1. We conducted experimental trials of the heat bath dynamics and the Swendsen-Wang dynamics for both the two-dimensional lattice and the mean field to observe mixing behavior.
2. Forward coupling of \(q + 1\) chains for heat bath: \(q\) chains with solid-color starting states (one for each color), last chain starts with random colors.
3. Coupling from the past for heat bath to exactly sample from Ising: two chains, one with all -1’s, and one with all +1’s.
4. Forward coupling of two chains for Swendsen-Wang: one chain of a solid color, and another with a checkerboard pattern.

Evaluation

- Mixing behavior of Glauber and Swendsen-Wang as \(\beta\) approaches critical \(\beta_c\) supports conjectured slowdown of the Potts model.
- Growth in mixing time on the complete graph nicely fits proven bounds for mixing time of Glauber dynamics.
- Growth in mixing time on the lattice consistent with proven upper bounds on mixing time; suggests possibility of separate, tighter bounds for Glauber and Swendsen-Wang dynamics.

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References


Figure 1. A simple Markov chain with states \{A, B, C, D\}. Figure 2. Complete graph on 5 vertices with 2 monochromatic edges.

Figure 3. Forward coupling vs. CFTP of Glauber for Ising model.

Figure 4. Glauber mixing on mean field at uniqueness with \(q = 2, 3, 4\).
- Glauber dynamics on lattice at critical point \(\beta_c\) mixes with time at most...
- Growth curves for colors 2-5.

Figure 5. Glauber mixing on lattice at criticality with \(q = 2, 3, 4, 5\).
- Glauber dynamics on lattice at critical point \(\beta_c\) mixes with time at most...
- Growth curves for colors 2-5.

Figure 6. Glauber mixing on mean field in regions around critical points with \(q = 3\).
- Glauber dynamics on mean field has mixing time...
- Growth curves for colors 2-5.

Figure 7. Swendsen-Wang mixing on lattice at criticality with \(q = 2, 3, 4, 5\).
- Swendsen-Wang mixes on lattice at critical point \(\beta_c\) with time at most...
- Growth curves for 2-5, \(q = 2\) appears to grow especially slowly.

Figure 8. Swendsen-Wang slowdown at critical point for Ising and \(q = 3\) Potts.
- Mixing time of Swendsen-Wang should be fast for \(\beta < \beta_c\) and \(\beta > \beta_c\), but slows down at \(\beta_c\).
- We notice slowdown near theoretical critical points \(\beta = \beta_0\) and \(\beta = \beta_1\) for Ising and \(q = 3\) Potts, respectively.
- \(q = 3\) asymptotes early, possibly due to limited graph size.