A Survey of MCMC for Ferromagnetic Spin Systems

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Motivation

Mixing time analysis: Analyze mixing behavior of common MCMC algorithms for sampling from Ising and Potts models with coupling heuristic.

Background

- **Ising model**: A probability distribution on $\{-1,+1\}^V$, where V is the set of vertices of a graph $G=(V,E)$. A configuration of the Ising model assigns one of two spins, either -1 or 1, to every vertex.
- **Potts model**: The generalization of the Ising model, allows vertices to take on one of q colors instead of two spins.
- **β (beta)**: Parameter dictating strength of bonds between sites. In **ferromagnetic** case $\beta > 0$, configurations with more monochromatic edges, or edges with matching endpoint colors, are more likely to occur.
- **Markov Chain Monte Carlo (MCMC)**: A class of algorithms which use Markov chains to randomly sample from high-dimensional distributions.
- **Heat bath Glauber dynamics**: A simple Markov chain that makes local moves, changing the state of one vertex at a time.
- **Swendsen-Wang dynamics**: A more sophisticated Markov chain that makes global moves by changing groups of vertices in each step.
- **Mixing time**: Number of steps needed for a Markov chain to converge to stationary distribution, expressed in terms of number of vertices $n=|V|$.
- **Coupling**: Heuristic method for bounding mixing time that runs multiple correlated chains.
- **Mean field model**: Potts model on the complete graph, serves as a good approximation of mixing behavior on high-dimensional lattices.

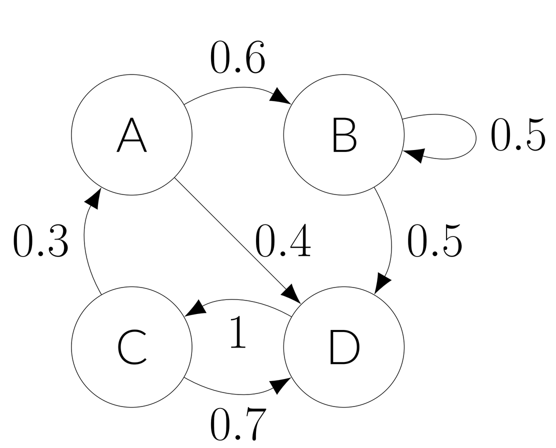

Figure 1. A simple Markov chain with states $\{A,B,C,D\}$.

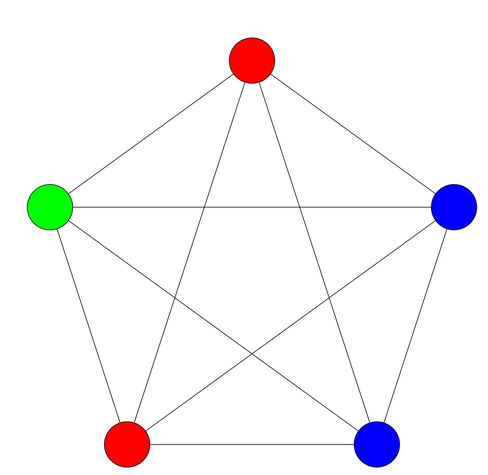

Figure 2. Complete graph on 5 vertices with 2 monochromatic edges.

Methodology

1. We conducted experimental trials of the **heat bath dynamics** and the **Swendsen-Wang dynamics** for both the two-dimensional lattice and the mean field to observe mixing behavior.
2. **Forward coupling** of $q+1$ chains for **heat bath**: q chains with solid-color starting states (one for each color), last chain starts with random colors.
3. **Coupling from the past** for **heat bath** to exactly sample from Ising: two chains, one with all -1's, and one with all +1's.
4. **Forward coupling** of two chains for **Swendsen-Wang**: one chain of a solid color, and another with a checkerboard pattern.

Results

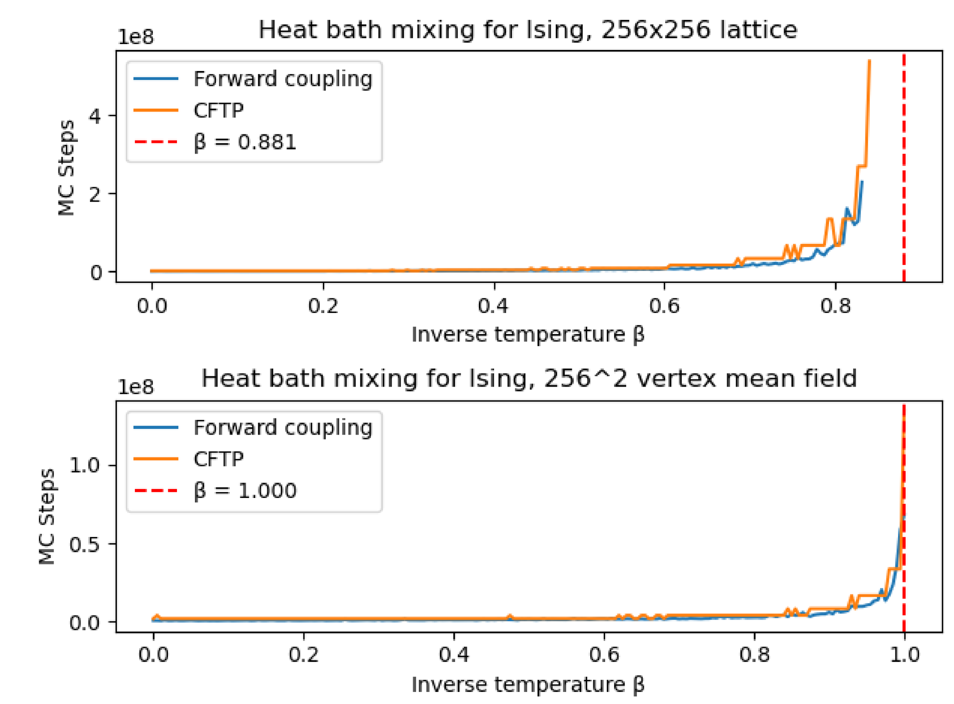

Figure 3. Forward coupling vs. CFTP of Glauber for Ising model

- Exists a critical point β_c where mixing is fast for $\beta \le \beta_c$, slow for $\beta > \beta_c$.
- Slowdown near theoretical $\beta_c \approx .88$ and $\beta_c = 1$ for lattice and mean field, respectively.
- Run forward coupling and CFTP to check consistency of mixing heuristics.

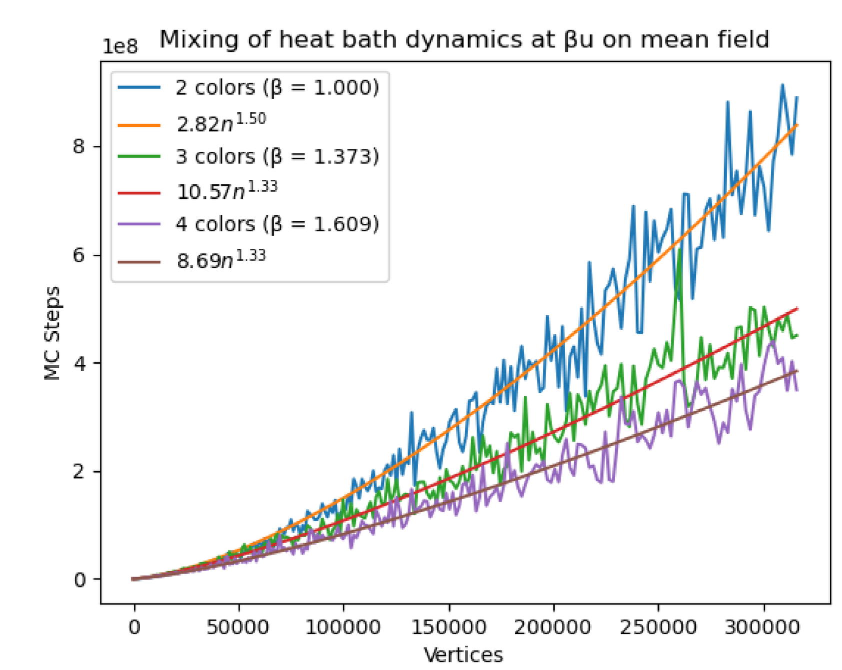

Figure 4. Glauber mixing on mean field at uniqueness with $q=2,3,4$

- Glauber on mean field mixes fast at uniqueness threshold β_u.

$$\begin{aligned} &\Theta(n^{3/2}) && \text{for } q=2 \\ &\Theta(n^{4/3}) && \text{for } q\ge 3 \end{aligned}$$

- We fit curves $cn^{3/2}$ and $cn^{4/3}$ to show growth in mixing time.

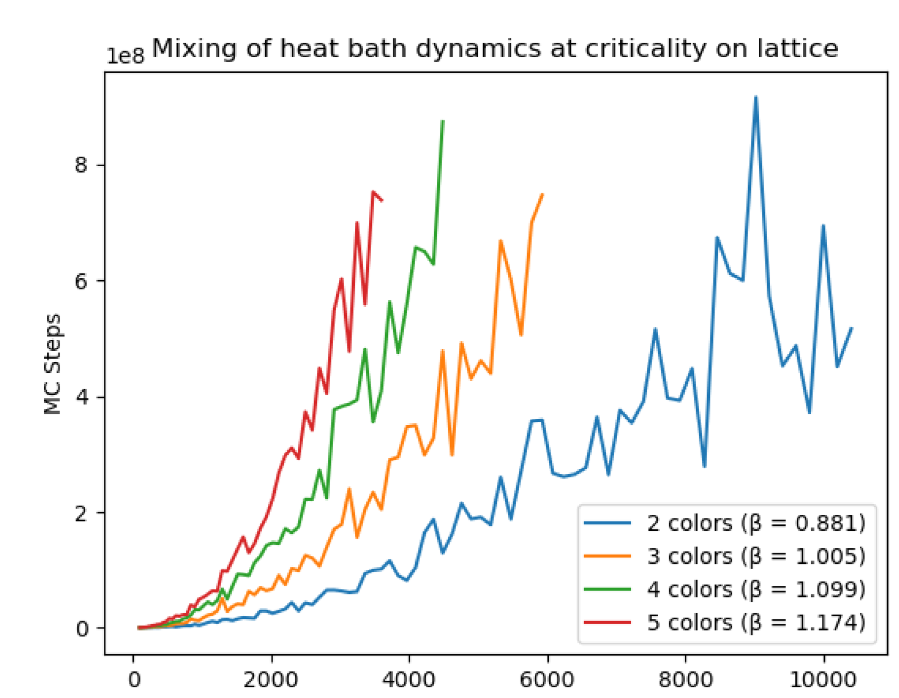

Figure 5. Glauber mixing on lattice at criticality with $q=2,3,4,5$

- Glauber dynamics on lattice at critical point β_c mixes with time at most...

$$\begin{aligned} &\mathrm{poly}(n) && \text{for } q<4 \\ &n^{O(\log n)} && \text{for } q=4 \\ &\exp(n) && \text{for } q\ge 5 \end{aligned}$$

- Growth curves for colors 2-5.

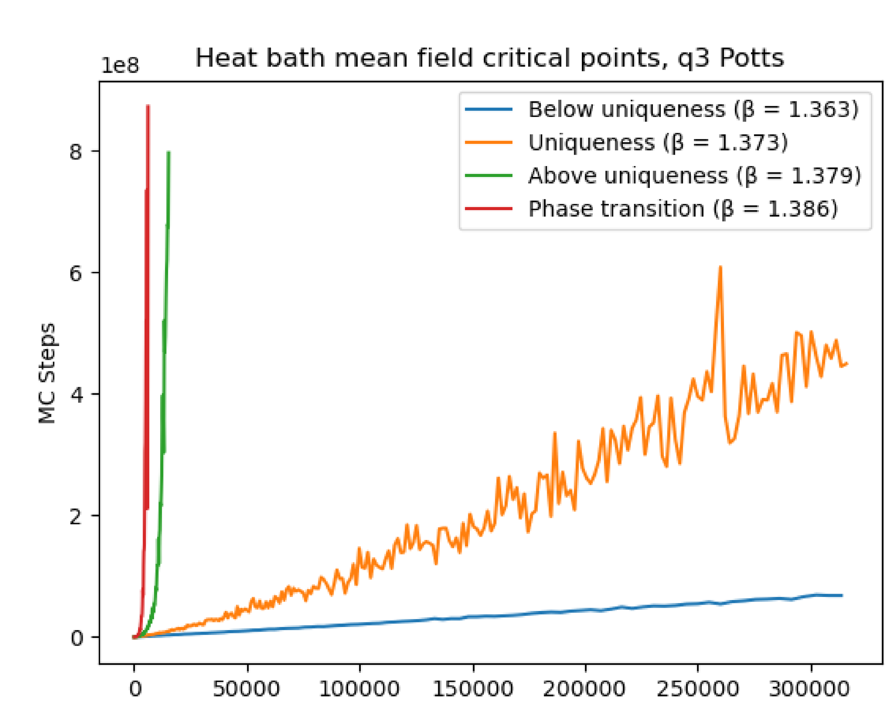

Figure 6. Glauber mixing on mean field in regions around critical points with $q=3$

- Glauber dynamics on mean field has mixing time...

$$\begin{aligned} &\Theta(n\log n) && \text{for } \beta<\beta_u \\ &\mathrm{poly}(n) && \text{at } \beta=\beta_u \\ &\exp(n) && \text{for } \beta>\beta_u \end{aligned}$$

- Run couplings around critical points: one below β_u, one at β_u, one above β_u, and one at the phase transition point β_c.

Results cont.

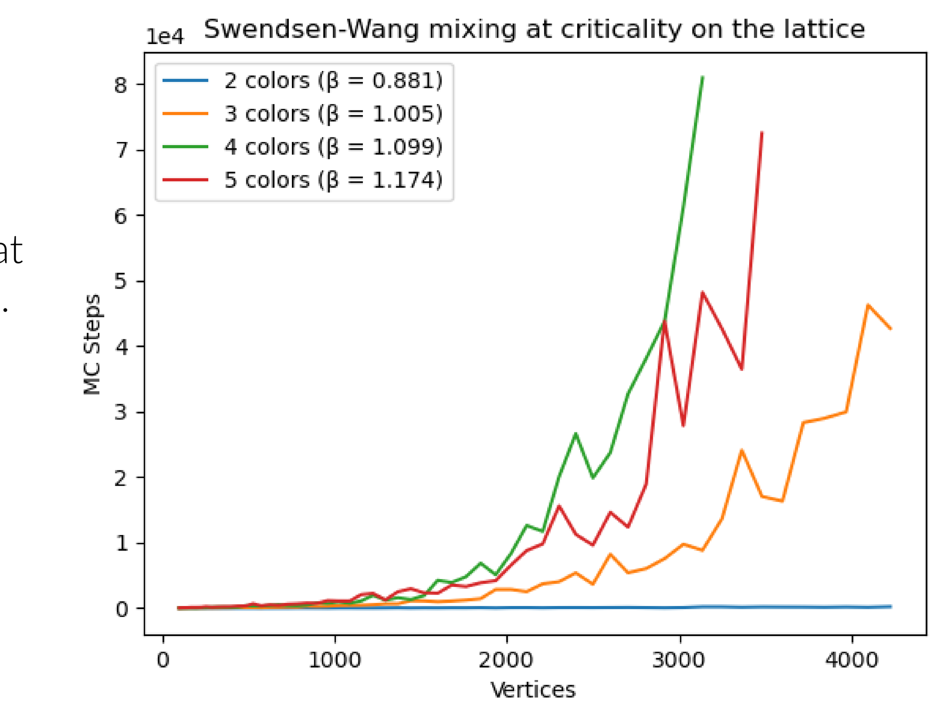

Figure 7. Swendsen-Wang mixing on lattice at criticality with $q=2,3,4,5$

- Swendsen-Wang mixes on lattice at critical point β_c with time at most...

$$\begin{aligned} &\mathrm{poly}(n) && \text{for } q=2,3 \\ &n^{O(\log n)} && \text{for } q=4 \\ &\exp(n) && \text{for } q=5 \end{aligned}$$

- Growth curves for 2-5, $q=2$ appears to grow especially slowly.

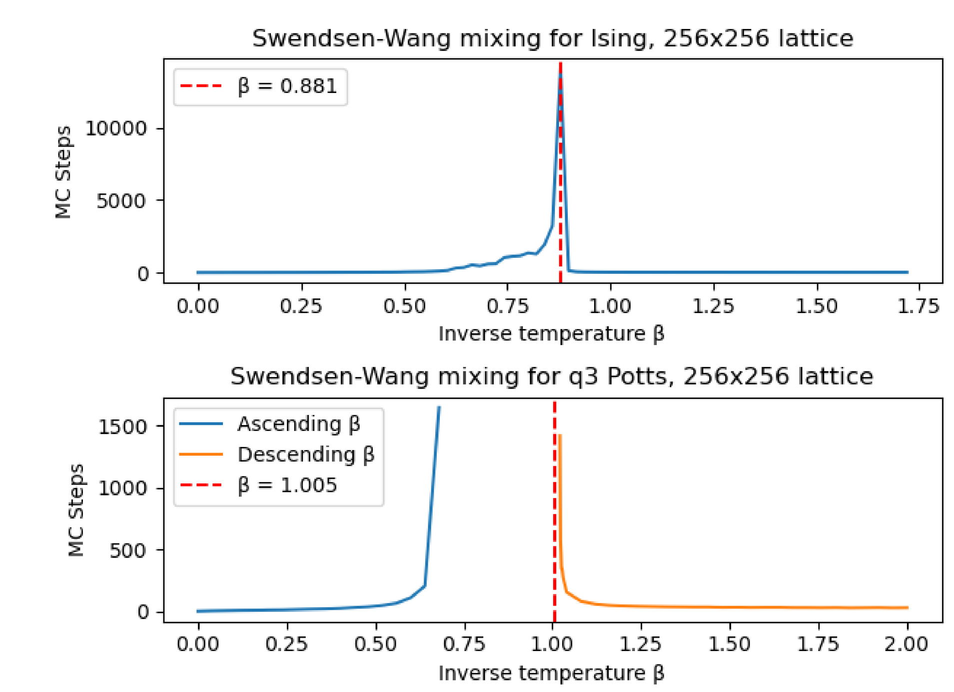

Figure 8. Swendsen-Wang slowdown at critical point for Ising and $q=3$ Potts

- Mixing time of Swendsen-Wang should be fast for $\beta<\beta_c$ and $\beta>\beta_c$, but slows down at b_c.
- We notice slowdown near theoretical critical points $\beta_c = .88$ and $\beta_c = 1.005$ for Ising and $q=3$ Potts, respectively.
- $q=3$ asymptotes early, possibly due to limited graph size.

Evaluation

- Mixing behavior of Glauber and Swendsen-Wang as β approaches critical β_c supports conjectured slowdown of the Potts model.
- Growth in mixing time on the complete graph nicely fits proven bounds for mixing time of Glauber dynamics.
- Growth in mixing time on the lattice consistent with proven upper bounds on mixing time; suggests possibility of separate, tighter bounds for Glauber and Swendsen-Wang dynamics.

References

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Acknowledgements

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Github: https://github.com/ucsb/kevinl-f22-dimap

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