The basis-exchange walk

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1 December 2021

Matroids

Definition

$\mathcal{M} = (E, \mathcal{I})$, where $E = \{1, \ldots, n\}$, and $\mathcal{I} \subseteq 2^E$ (independent sets) such that:

- $\emptyset \in \mathcal{I}$;
- if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$;
- if $I, J \in \mathcal{I}$ and $|I| < |J|$, then $\exists j \in J \setminus I$ such that $I \cup \{j\} \in \mathcal{I}$.

The Hasse diagram of $(\mathcal{I}, \subseteq)$ might look like this:

$\mathcal{B}$ = maximal independent sets (bases).

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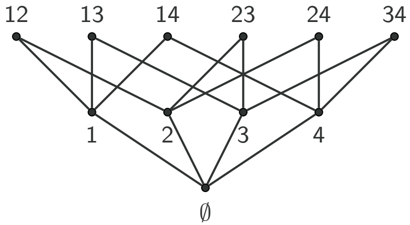

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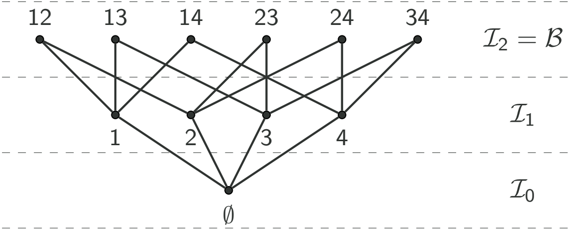

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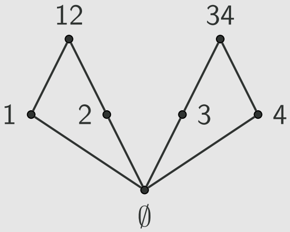
Examples

A *non*-example $\mathcal{B} = \{12, 34\}$



An example $\mathcal{B} = \{12, 13\}$

What to notice:

- The third axiom implies that the induced subgraph of two consecutive levels, $\mathcal{I}_{k-1}$ and $\mathcal{I}_k$, is connected.
- We can first drop and then add an element to move through independent sets of the same level.

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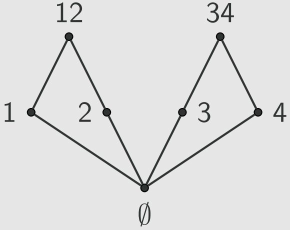

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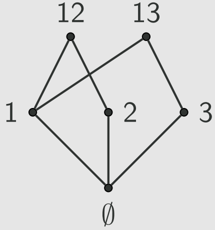

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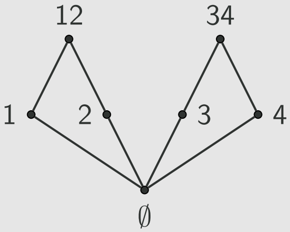

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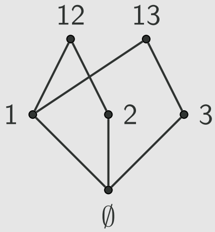

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Matroids

Classes & operations

Some types of matroids:

- Linear/representable ($\mathcal{I} = \{\text{lin. ind. vectors/columns of a matrix } A\}$)
- Graphic ($\mathcal{I} = \{\text{forests of a graph } G\}$, $\mathcal{B} = \{\text{spanning trees of } G\}$)
- Non-representable (almost all matroids [Nelson, 2016])

Matroids are closed under

- Deletion
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The basis-exchange walk

Suppose the walk is at a basis $B \in \mathcal{B}$.

Step 1. Remove an element $i \in B$ u.a.r.
Step 2. Add an element $j \in E$ u.a.r. such that $B \backslash \{i\} \cup \{j\} \in \mathcal{B}$.

This basis-exchange walk over $\mathcal{B}$ is aperiodic and reversible with respect to the uniform distribution, and so it converges to the uniform distribution.

Is it fast mixing?

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Basis-exchange graph

Conjecture [Mihail and Vazirani, 1989]

The basis-exchange graph has (cutset) expansion at least 1, i.e.

$$\forall S, \quad |E(S, S^c)| \geq min(|S|, |S^c|).$$

Theorem [Feder and Mihail, 1992]

True for *balanced* matroids, for which all minors satisfy

$$\forall i \neq j, \quad P(i \in B \mid j \in B) \leq P(i \in B).$$

The last condition is called negative correlation and there exist matroids that do not satisfy it.

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Mixing time

Exchange walks

Theorem [Anari et al., 2019]

Mihail and Vazirani conjecture is true for all matroids, and

$$t_{mix}(P_r^\vee, \epsilon) \le r\left(\log\frac{1}{\pi_{r,min}} + \log\frac{1}{\epsilon}\right).$$

Achieved by lower bounding the Poincaré constant (spectral gap),

$$1 - \lambda_2(P) = \lambda(P) := \inf\left\{\frac{\mathcal{E}_P(f,f)}{Var_\pi(f)} \mid f : \Omega \to \mathbb{R},\ Var_\pi(f) \neq 0\right\},$$

where

$$\mathcal{E}_P(f,f) = \frac{1}{2}\sum_{x,y\in\Omega}\pi(x)P(x,y)(f(x)-f(y))^2,$$

$$Var_\pi(f) = \frac{1}{2}\sum_{x,y\in\Omega}\pi(x)\pi(y)(f(x)-f(y))^2.$$

Simplicial complexes
Definition

An *abstract simplicial complex* $\mathcal{C} = (E, \mathcal{S})$ consists of a ground set of elements E, and a nonempty downwards closed collection of sets $\mathcal{S}$ (faces):

- $\emptyset \in \mathcal{S}$;
- if $S \in \mathcal{S}$, $T \subseteq S$, then $T \in \mathcal{S}$.

Simplicial Complexes = Matroids - augmentation axiom.
Matroids = Simplicial Complexes for which the greedy algorithm works.

We can encode a variety of combinatorial structures and distributions within the maximal faces of a simplicial complex.
Examples: bases of a matroid, independent sets of a graph,
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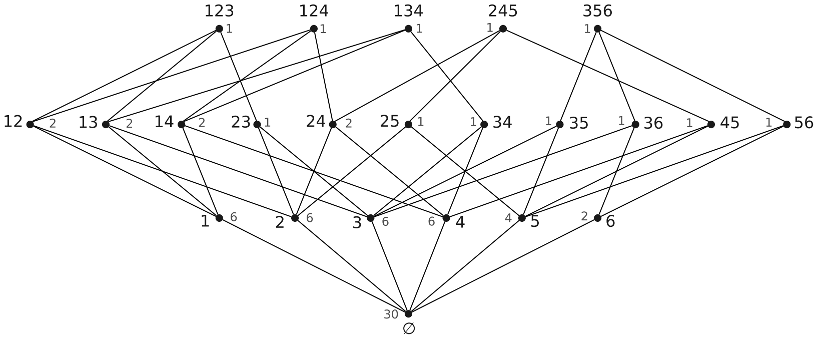
Simplicial complexes

Example

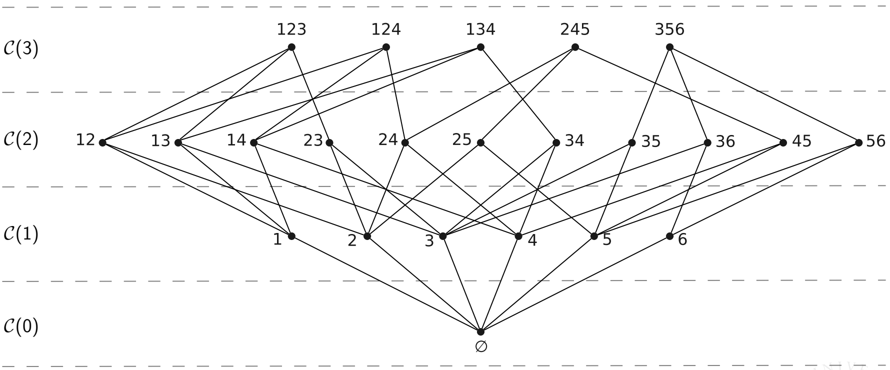
A visualization of a weighted simplicial complex $\mathcal{C}$.



Simplicial complexes

Example

The Hasse diagram $(\mathcal{S}, \subseteq)$.



Simplicial complexes

Example

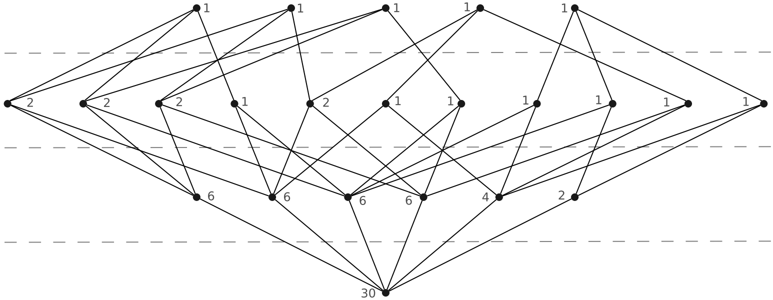
The distributions π_k, $0 \leq k \leq d$.

$\pi_3 \propto$

$\pi_2 \propto$

$\pi_1 \propto$

$\pi_0 \propto$



Simpicial complexes

Exchange walks

Two operators:

- "Going-up", $P_k^{\uparrow}$; starting from a set $S \in \mathcal{C}(k)$, we add an element $i \in E \setminus S$ with probability $\propto \pi_{k+1}(S \cup i)$.
- "Going-down", $P_k^{\downarrow}$; starting from a set $S \in \mathcal{C}(k)$, we remove an element $i \in S$ uniformly at random.

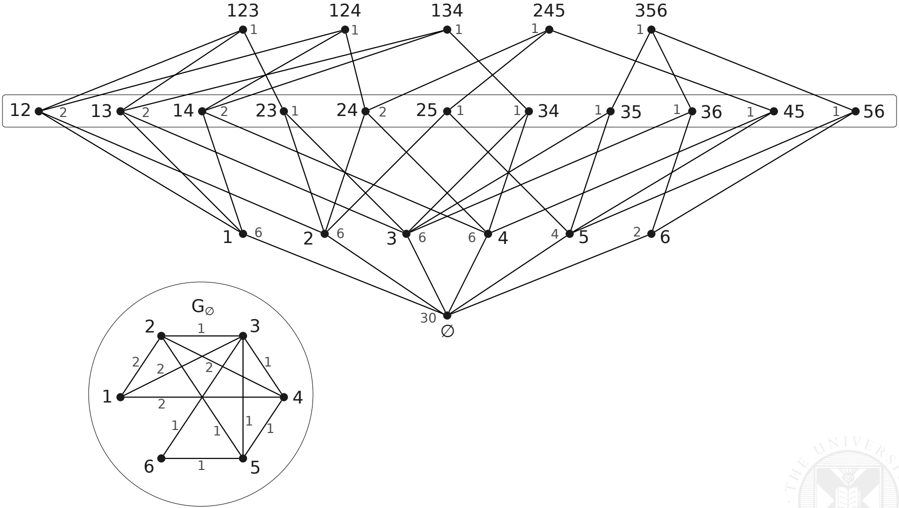
We can now define the exchange walks over $\mathcal{C}(k)$ as

$$P_k^{\wedge} = P_k^{\uparrow} P_{k+1}^{\downarrow},$$
$$P_k^{\vee} = P_k^{\downarrow} P_{k-1}^{\uparrow}.$$

Our main goal is to study properties of the global walk at $\mathcal{C}(d)$, $P_d^{\vee}$. This can be done by looking at properties of some "local" walks G_S.

Simplicial complexes

Local walks G_S - one for every face S, $|S| \leq d-2$.



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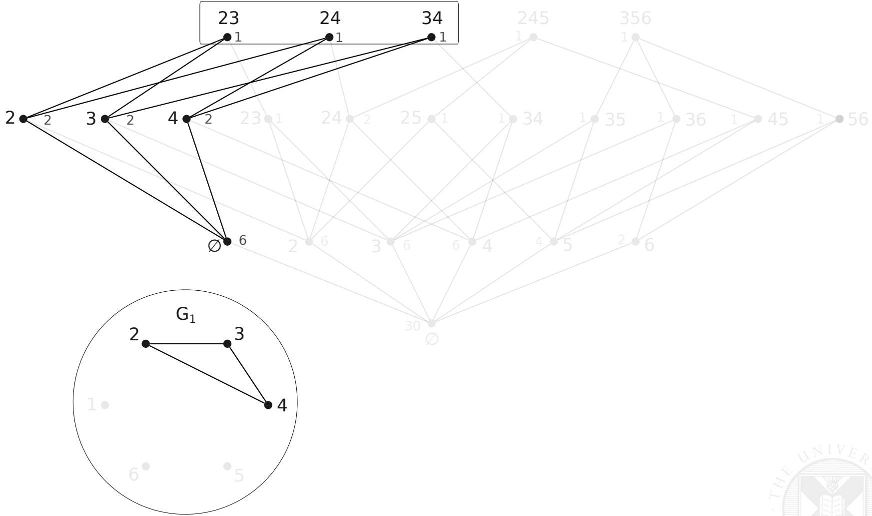

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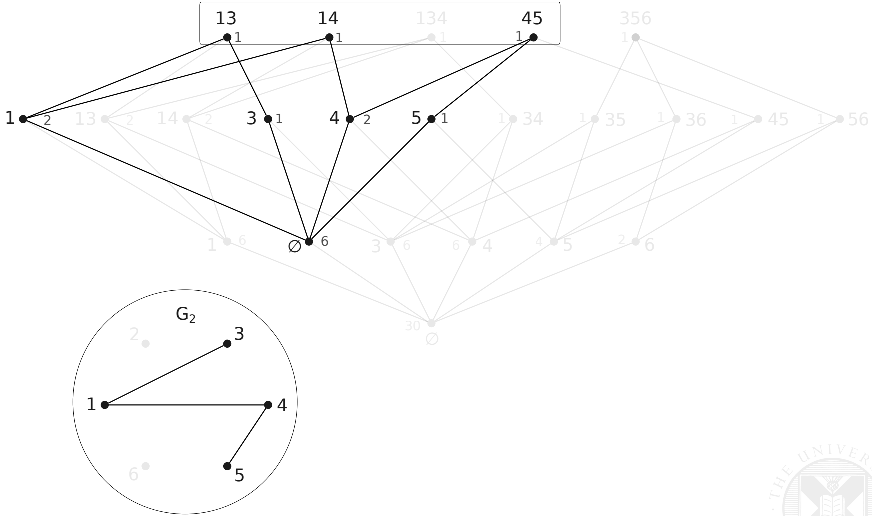

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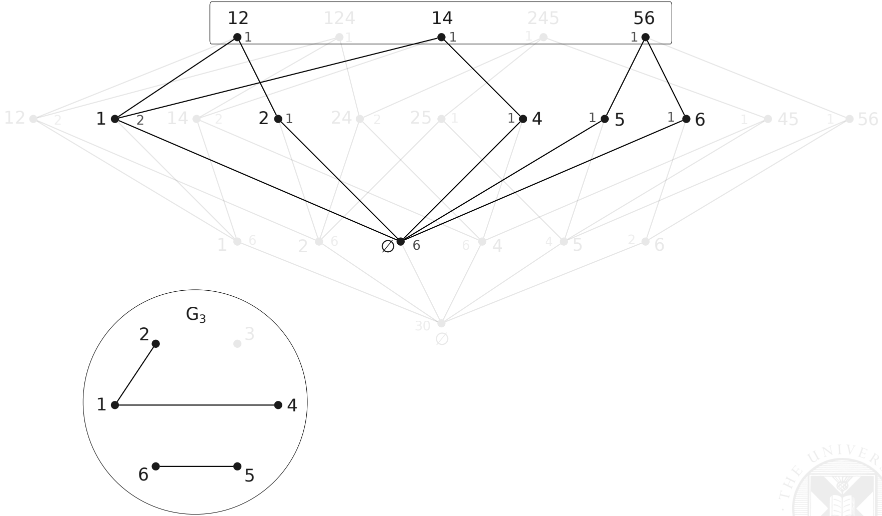

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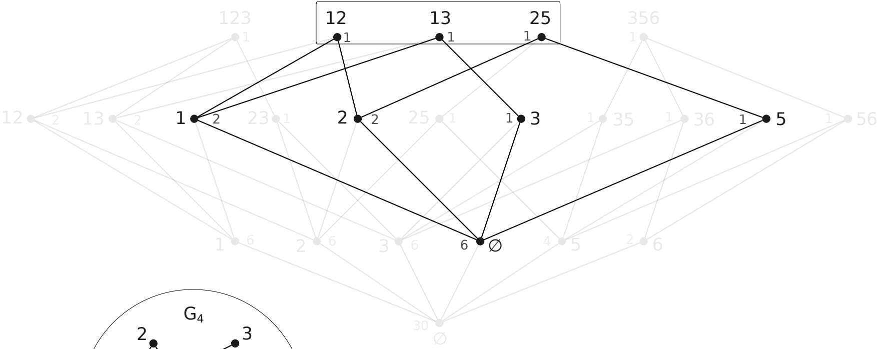

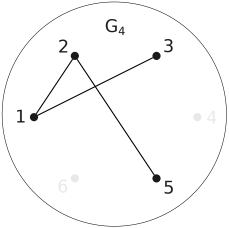

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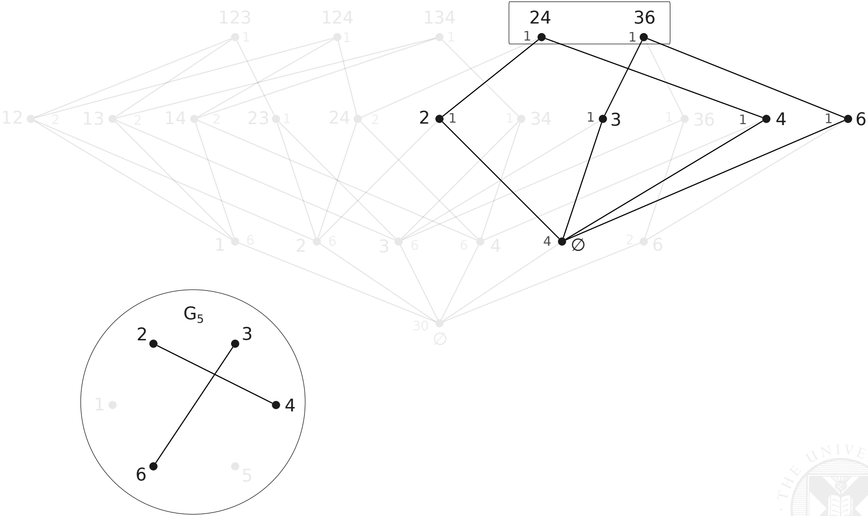

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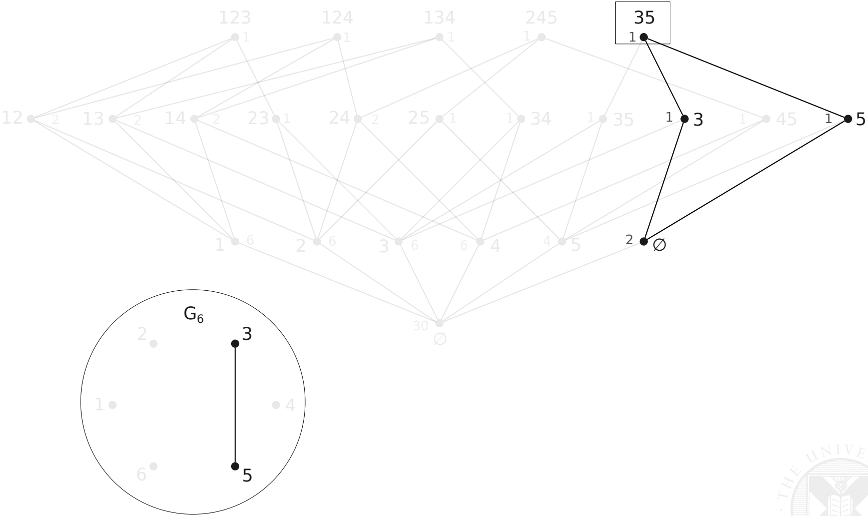

Trickling down theorem

Theorem [Oppenheim, 2018]

Let $\mathcal{C}$ be a simplicial complex and suppose that for all $v \in \mathcal{C}(1)$ we have that $\lambda_2(G_v) \leq \gamma$. Then, if $G_\emptyset$ is connected,

$$\lambda_2(G_\emptyset) \leq \frac{\gamma}{1-\gamma}.$$

Proof.

$$\begin{aligned}
\mathcal{E}_{G_\emptyset}(f,f) &= \sum_{v \in \mathcal{C}(1)} \pi_1(v) \mathcal{E}_{G_v}(f_v, f_v) \\
&\geq (1-\gamma) \sum_{v \in \mathcal{C}(1)} \pi_1(v) \mathrm{Var}_{\pi_{v,1}}(f_v) \qquad \text{(because } \lambda_2(G_v) \leq \gamma\text{)} \\
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Now choose $f = v_2$, where $G_\emptyset v_2 = \lambda_2 v_2$. Then,

$$\mathcal{E}_{G_\emptyset}(v_2, v_2) \geq (1-\gamma)\left[\mathrm{Var}_{\pi_1}(v_2) - \mathrm{Var}_{\pi_1}(\lambda_2 v_2)\right],$$

which simplifies into

$$(1-\lambda_2)\mathrm{Var}_{\pi_1}(v_2) \geq (1-\gamma)(1-\lambda_2^2)\mathrm{Var}_{\pi_1}(v_2).$$

Thus, $(1-\lambda_2) \geq (1-\gamma)(1-\lambda_2^2)$. In particular, if $\lambda_2(G_\emptyset) < 1$,

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Local-to-global theorem

Theorem [Alev and Lau, 2020]

Let $\mathcal{C}$ be a simplicial complex that is a $(\alpha_0, ..., \alpha_{d-2})$-local-spectral expander. Then, for any $2 \leq k \leq d$,

$$\lambda(P_k^\vee) = \lambda(P_{k-1}^\wedge) \geq \frac{1}{k} \prod_{i=0}^{k-2} (1 - \alpha_i) .$$

Proof (by induction).

Base case ($k = 2$): From the local-spectral assumption, and because $P_1^\wedge = \frac{I+G_\emptyset}{2}$,

$$\lambda(P_2^\vee) = \lambda(P_1^\wedge) = \frac{1}{2}\lambda(G_\emptyset) \geq \frac{1}{2}(1 - \alpha_0).$$

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Local-to-global theorem

Proof

For the inductive step we will need this inequality:

$$\begin{aligned}
\mathcal{E}_{P_k^\wedge}(f,f) &= \frac{k}{k+1}\mathcal{E}_{\widetilde{P}_k^\wedge}(f,f) \\
&= \frac{k}{k+1}\sum_{S\in\mathcal{C}(k-1)}\pi_{k-1}(S)\mathcal{E}_{G_S}(f_S,f_S) \\
&\geq \frac{k}{k+1}(1-\alpha_{k-1})\sum_{S\in\mathcal{C}(k-1)}\pi_{k-1}(S)\mathrm{Var}_{\pi_{S,1}}(f_S) \\
&\qquad\qquad (\lambda_2(G_S)\leq\alpha_{k-1}) \\
&= \frac{k}{k+1}(1-\alpha_{k-1})\mathcal{E}_{P_k^\vee}(f,f)\,.
\end{aligned}$$

Local-to-global theorem

Proof

Inductive step. Suppose the theorem holds for level $k-1$. Then, starting by the previous inequality,

$$\begin{aligned}
\mathcal{E}_{P^{\wedge}_{k-1}}(f,f) &\geq \frac{k-1}{k}(1-\alpha_{k-2})\mathcal{E}_{P^{\vee}_{k-1}}(f,f) \\
&\geq \frac{k-1}{k}(1-\alpha_{k-2})\frac{1}{k-1}\prod_{i=0}^{k-3}(1-\alpha_i)\operatorname{Var}_{\pi_{k-1}}(f) \\
&= \frac{1}{k}\prod_{i=0}^{k-2}(1-\alpha_i)\operatorname{Var}_{\pi_{k-1}}(f),
\end{aligned}$$

which implies that

$$\lambda(P^{\vee}_k) = \lambda(P^{\wedge}_{k-1}) \geq \frac{1}{k}\prod_{i=0}^{k-2}(1-\alpha_i). \quad \square$$

An application to matroids

Spectral gap of the basis-exchange walk

For a matroid, every walk G_S is connected (augmentation property).

If $S \in C(r-2)$, then G_S is the transition matrix of a complete k-partite graph (matroid partition property).
For the uniform distribution over the bases, $\lambda_2(G_S) \leq 0$.

Applying the trickling down theorem, the matroid complex is a $(0, \dots, 0)$-local-spectral expander.

Finally, by the local-to-global theorem,

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Applying the trickling down theorem, the matroid complex is a $(0, \ldots, 0)$-local-spectral expander.

Finally, by the local-to-global theorem,

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An application to matroids

Spectral gap of the basis-exchange walk

For a matroid, every walk G_S is connected (augmentation property).

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Mixing time of the basis-exchange walk

The spectral gap gives the following bound for the mixing time (for P PSD),

$$t_{mix}(P,\epsilon) \leq \frac{1}{\lambda(P)}\left(\frac{1}{2}\log\frac{1}{\pi_{min}} + \log\frac{1}{2\epsilon}\right).$$

Applying this to the basis exchange walk for the uniform distribution, where $\lambda(P_r^\vee) \geq \frac{1}{r}$ and $\frac{1}{\pi_{r,min}} \leq \binom{n}{r} \leq n^r$, we get the mixing time bound of [Anari et al., 2019]:

$$t_{mix}(P_r^\vee) := t_{mix}(P_r^\vee, 1/4) = O\left(r^2 \log n\right).$$

In followup work [Cryan et al., 2019, Anari et al., 2021], by using the modified log-Sobolev constant, this bound was improved to

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Conclusion

The basis-exchange walk is fast mixing!

- we can produce approximately random samples of bases;
- we can approximately count the number of bases;
- we have concentration of measure results over the basis-exchange graph.

Similar techniques (with simplicial complexes) have recently produced more great results:

- Very efficient approximate sampling of random spanning trees;
- Optimal mixing of exchange walks (Glauber Dynamics) for a variety of models.