The basis-exchange walk

Giorgos Mousa

School of Informatics
University of Edinburgh

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Matroids

Definition

\[ \mathcal{M} = (E, \mathcal{I}) \], where \( E = \{1, \ldots, n\} \), and \( \mathcal{I} \subseteq 2^E \) (independent sets) such that:

- \( \emptyset \in \mathcal{I} \);
- if \( I \in \mathcal{I} \) and \( J \subseteq I \), then \( J \in \mathcal{I} \);
- if \( I, J \in \mathcal{I} \) and \( |I| < |J| \), then \( \exists j \in J \setminus I \) such that \( I \cup \{j\} \in \mathcal{I} \).

The Hasse diagram of \((\mathcal{I}, \subseteq)\) might look like this:

\[ \mathcal{B} = \text{maximal independent sets (bases)}. \]
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The Hasse diagram of \( (\mathcal{I}, \subseteq) \) might look like this:

\[ \begin{align*}
12 & \quad 13 & \quad 14 & \quad 23 & \quad 24 & \quad 34 \\
1 & \quad 2 & \quad 3 & \quad 4 & \quad \emptyset
\end{align*} \]

\( \mathcal{B} \) = maximal independent sets (bases).
Matroids

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Examples

A non-example \( B = \{12, 34\} \)

An example \( B = \{12, 13\} \)

What to notice:

- The third axiom implies that the induced subgraph of two consecutive levels, \( \mathcal{I}_{k-1} \) and \( \mathcal{I}_k \), is connected.
- We can first drop and then add an element to move through independent sets of the same level.
Matroids

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34 & & & \\
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Matroids

Classes & operations

Some types of matroids:
- Linear/representable ($\mathcal{I} = \{\text{lin. ind. vectors/columns of a matrix } A\}$)
- Graphic ($\mathcal{I} = \{\text{forests of a graph } G\}$, $\mathcal{B} = \{\text{spanning trees of } G\}$)
- Non-representable (almost all matroids [Nelson, 2016])

Matroids are closed under
- Deletion
- Contraction
- Truncation
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The basis-exchange walk

Suppose the walk is at a basis $B \in \mathcal{B}$.

Step 1. Remove an element $i \in B$ u.a.r.
Step 2. Add an element $j \in E$ u.a.r. such that $B \setminus \{i\} \cup \{j\} \in \mathcal{B}$.

This basis-exchange walk over $\mathcal{B}$ is aperiodic and reversible with respect to the uniform distribution, and so it converges to the uniform distribution.

Is it fast mixing?
Matroids
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Is it fast mixing?
Conjecture [Mihail and Vazirani, 1989]
The basis-exchange graph has (cutset) expansion at least 1, i.e.

$$\forall S, \ |E(S, S^c)| \geq \min(|S|, |S^c|).$$

Theorem [Feder and Mihail, 1992]
True for balanced matroids, for which all minors satisfy

$$\forall i \neq j, \ P(i \in B \mid j \in B) \leq P(i \in B).$$

The last condition is called negative correlation and there exist matroids that do not satisfy it.
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Mixing time

Exchange walks

**Theorem [Anari et al., 2019]**

Mihail and Vazirani conjecture is true for all matroids, and

\[ t_{\text{mix}}(P_r^\vee, \epsilon) \leq r \left( \log \frac{1}{\pi_{r,\text{min}}} + \log \frac{1}{\epsilon} \right). \]

Achieved by lower bounding the Poincaré constant (spectral gap),

\[ 1 - \lambda_2(P) = \lambda(P) := \inf \left\{ \frac{\mathcal{E}_P(f, f)}{\text{Var}_\pi(f)} \mid f : \Omega \to \mathbb{R}, \text{Var}_\pi(f) \neq 0 \right\}, \]

where

\[ \mathcal{E}_P(f, f) = \frac{1}{2} \sum_{x, y \in \Omega} \pi(x)P(x, y)(f(x) - f(y))^2, \]

\[ \text{Var}_\pi(f) = \frac{1}{2} \sum_{x, y \in \Omega} \pi(x)\pi(y)(f(x) - f(y))^2. \]
Simplicial complexes

Definition

An abstract simplicial complex \( C = (E, S) \) consists of a ground set of elements \( E \), and a nonempty downwards closed collection of sets \( S \) (faces):

- \( \emptyset \in S \);
- if \( S \in S \), \( T \subseteq S \), then \( T \in S \).

Simplicial Complexes = Matroids - augmentation axiom.
Matroids = Simplicial Complexes for which the greedy algorithm works.

We can encode a variety of combinatorial structures and distributions within the maximal faces of a simplicial complex.
Examples: bases of a matroid, independent sets of a graph, configurations of a multi-spin system, etc.
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Simplicial complexes

Example

A visualization of a weighted simplicial complex $\mathcal{C}$.
Simplicial complexes

Example

The Hasse diagram \((S, \subseteq)\).
The distributions $\pi_k$, $0 \leq k \leq d$. 

\begin{itemize}
  \item $\pi_3 \propto$
  \item $\pi_2 \propto$
  \item $\pi_1 \propto$
  \item $\pi_0 \propto$
\end{itemize}
Simplicial complexes

Exchange walks

Two operators:

- “Going-up”, $P^\uparrow_k$; starting from a set $S \in C(k)$, we add an element $i \in E \setminus S$ with probability $\propto \pi_{k+1}(S \cup i)$.

- “Going-down”, $P^\downarrow_k$; starting from a set $S \in C(k)$, we remove an element $i \in S$ uniformly at random.

We can now define the exchange walks over $C(k)$ as

\[
P^\wedge_k = P^\uparrow_k P^\downarrow_k,
\]

\[
P^\vee_k = P^\downarrow_k P^\uparrow_{k-1}.
\]

Our main goal is to study properties of the global walk at $C(d)$, $P^\vee_d$. This can be done by looking at properties of some “local” walks $G_S$. 
Simplicial complexes

Local walks $G_S$ - one for every face $S$, $|S| \leq d - 2$. 
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Trickling down theorem

Theorem [Oppenheim, 2018]

Let $C$ be a simplicial complex and suppose that for all $v \in C(1)$ we have that $\lambda_2(G_v) \leq \gamma$. Then, if $G_\emptyset$ is connected,

$$\lambda_2(G_\emptyset) \leq \frac{\gamma}{1 - \gamma}.$$

Proof.

$$\mathcal{E}_{G_\emptyset} (f, f) = \sum_{v \in C(1)} \pi_1(v) \mathcal{E}_{G_v} (f_v, f_v)$$

$$\geq (1 - \gamma) \sum_{v \in C(1)} \pi_1(v) \text{Var}_{\pi_1, v} (f_v)$$

(because $\lambda_2(G_v) \leq \gamma$)

$$= (1 - \gamma) [\text{Var}_{\pi_1} (f) - \text{Var}_{\pi_1} (G_\emptyset f)].$$
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$$E_{G_\emptyset}(f, f) = \sum_{v \in C(1)} \pi_1(v)E_{G_v}(f_v, f_v) \geq (1 - \gamma) \sum_{v \in C(1)} \pi_1(v) \text{Var}_{\pi_1(v)}(f_v) \quad \text{(because } \lambda_2(G_v) \leq \gamma)$$

$$= (1 - \gamma) [\text{Var}_{\pi_1}(f) - \text{Var}_{\pi_1}(G_\emptyset f)].$$
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Proof

We have that

\[ \mathcal{E}_{G_\emptyset} (f, f) \geq (1 - \gamma) \left[ \operatorname{Var}_{\pi_1} (f) - \operatorname{Var}_{\pi_1} (G_\emptyset f) \right]. \]

Now choose \( f = v_2 \), where \( G_\emptyset v_2 = \lambda_2 v_2 \). Then,

\[ \mathcal{E}_{G_\emptyset} (v_2, v_2) \geq (1 - \gamma) \left[ \operatorname{Var}_{\pi_1} (v_2) - \operatorname{Var}_{\pi_1} (\lambda_2 v_2) \right], \]

which simplifies into

\[ (1 - \lambda_2) \operatorname{Var}_{\pi_1} (v_2) \geq (1 - \gamma)(1 - \lambda_2^2) \operatorname{Var}_{\pi_1} (v_2). \]

Thus, \( (1 - \lambda_2) \geq (1 - \gamma)(1 - \lambda_2^2) \). In particular, if \( \lambda_2 (G_\emptyset) < 1 \),

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Theorem [Alev and Lau, 2020]

Let $C$ be a simplicial complex that is a $(\alpha_0, ..., \alpha_{d-2})$-local-spectral expander. Then, for any $2 \leq k \leq d$,

$$\lambda(P_k^\vee) = \lambda(P_{k-1}^\wedge) \geq \frac{1}{k} \prod_{i=0}^{k-2} (1 - \alpha_i).$$

Proof (by induction).

Base case ($k = 2$): From the local-spectral assumption, and because $P_1^\wedge = \frac{I + G_\emptyset}{2}$,

$$\lambda(P_2^\vee) = \lambda(P_1^\wedge) = \frac{1}{2} \lambda(G_\emptyset) \geq \frac{1}{2} (1 - \alpha_0).$$
Local-to-global theorem

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Local-to-global theorem

Proof

For the inductive step we will need this inequality:

\[ \mathcal{E}_{P_k^\wedge} (f, f) = \frac{k}{k+1} \mathcal{E}_{\tilde{P}_k^\wedge} (f, f) \]

\[ = \frac{k}{k+1} \sum_{S \in C(k-1)} \pi_{k-1}(S) \mathcal{E}_{G_S} (f_S, f_s) \]

\[ \geq \frac{k}{k+1} (1 - \alpha_{k-1}) \sum_{S \in C(k-1)} \pi_{k-1}(S) \Var_{\pi_S, 1} (f_S) \]

\[ (\lambda_2(G_S) \leq \alpha_{k-1}) \]

\[ = \frac{k}{k+1} (1 - \alpha_{k-1}) \mathcal{E}_{P_k^\vee} (f, f). \]
Theorem 1.2: Local-to-global theorem

Proof

Inductive step. Suppose the theorem holds for level \( k - 1 \). Then, starting by the previous inequality,

\[
\mathcal{E}_{P_{k-1}^\uparrow} (f, f) \geq \frac{k-1}{k} (1 - \alpha_{k-2}) \mathcal{E}_{P_{k-1}^\downarrow} (f, f)
\]

\[
\geq \frac{k-1}{k} (1 - \alpha_{k-2}) \frac{1}{k-1} \prod_{i=0}^{k-3} (1 - \alpha_i) \text{Var}_{\pi_{k-1}} (f)
\]

\[
= \frac{1}{k} \prod_{i=0}^{k-2} (1 - \alpha_i) \text{Var}_{\pi_{k-1}} (f),
\]

which implies that

\[
\lambda (P_{k}^\downarrow) = \lambda (P_{k-1}^\uparrow) \geq \frac{1}{k} \prod_{i=0}^{k-2} (1 - \alpha_i).
\]
An application to matroids
Spectral gap of the basis-exchange walk

For a matroid, every walk $G_S$ is connected (augmentation property).

If $S \in C(r - 2)$, then $G_S$ is the transition matrix of a complete $k$-partite graph (matroid partition property).

For the uniform distribution over the bases, $\lambda_2(G_S) \leq 0$.

Applying the trickling down theorem, the matroid complex is a $(0, ..., 0)$-local-spectral expander.

Finally, by the local-to-global theorem,

$$\lambda(P_r^\vee) \geq \frac{1}{r} \prod_{i=0}^{r-2} (1 - 0) = \frac{1}{r}.$$
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An application to matroids

Mixing time of the basis-exchange walk

The spectral gap gives the following bound for the mixing time (for $P$ PSD),

$$
t_{mix}(P, \epsilon) \leq \frac{1}{\lambda(P)} \left( \frac{1}{2} \log \frac{1}{\pi_{min}} + \log \frac{1}{2\epsilon} \right).
$$

Applying this to the basis exchange walk for the uniform distribution, where $\lambda(P_r^\vee) \geq \frac{1}{r}$ and $\frac{1}{\pi_{r,\min}} \leq \binom{n}{r} \leq n^r$, we get the mixing time bound of [Anari et al., 2019]:

$$
t_{mix}(P_r^\vee) := t_{mix}(P_r^\vee, 1/4) = O \left( r^2 \log n \right).
$$

In followup work [Cryan et al., 2019, Anari et al., 2021], by using the modified log-Sobolev constant, this bound was improved to

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Applying this to the basis exchange walk for the uniform distribution, where $\lambda(P^\lor_r) \geq \frac{1}{r}$ and $\frac{1}{\pi_{r, min}} \leq \binom{n}{r} \leq n^r$, we get the mixing time bound of [Anari et al., 2019]:

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$$t_{mix}(P^\lor_r) = O \left( r \log r \right).$$
Conclusion

The basis-exchange walk is fast mixing!
▶ we can produce approximately random samples of bases;
▶ we can approximately count the number of bases;
▶ we have concentration of measure results over the basis-exchange graph.

Similar techniques (with simplicial complexes) have recently produced more great results:
▶ Very efficient approximate sampling of random spanning trees;
▶ Optimal mixing of exchange walks (Glauber Dynamics) for a variety of models.