Last class: MC for generating a random matching for any graph $G=(V,E)$.

Today: Random perfect matching of any bipartite graph $G=(L\cup R, E)$
↑ all edges b/w L & R.

— Estimating # of perfect matchings
Corresponds to 0-1 permanent.
(extends to non-negative permanents)

For a graph $G=(V,E)$,
let $\mathcal{M}$ = all matchings of G.

To generate a random matching, defined a Markov chain on $\mathcal{M}$.

To generate a random perfect matching, we'll define a MC on $\mathcal{P} \cup \mathcal{N}$ where
$\mathcal{P}$ = all perfect matchings
& $\mathcal{N}$ = all near-perfect matchings
= matchings with exactly 2 unmatched vertices

MC on $\Omega = P \cup N$:

From $X_t \in \Omega$,

1. Choose $e = (v,w) \in E$ u.a.r.
2. If $X_t \in N(v,w)$ then set $X' = X_t \cup e$.
3. If $e \in X_t$ then set $X' = X_t \setminus e$.
4. If $X_t \in N(v,z)$ for $z \neq w$, then set $X' = X_t \cup e \setminus (w,y)$ for $(w,y) \in X_t$.
5. If X' is defined & $X' \in \Omega$ then with prob. $\frac{1}{2}$ set $X_{t+1} = X'$ else $X_{t+1} = X_t$.

We argued (roughly) that if $\pi(P) \geq \frac{1}{n^2}$ then $T_{mix} = poly(n)$ & we can generate a random perfect matching in poly(n) time.

Moreover, if degree$(v) > \frac{n}{2}$ for all $v \in V$, then $\pi(P) \geq \frac{1}{n^2}$.

Introduce weights on matchings which depends on the hole pattern.
Say every $M \in \Omega$, has a weight $w(M) > 0$.
How to sample from $\pi(M) = \frac{w(M)}{Z}$?

Same chain as before change step 5 to:

5. Set $X_{t+1} = X'$ w.p. $\min\left\{1, \frac{w(X')}{w(X_t)}\right\}$
& o/w $X_{t+1} = X_t$.

Metropolis filter

What weights?
Bipartite graph with $V = L \cup R$.

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For $v \in L, w \in R$, let $N(v,w) =$ near-perfect matchings with unmatched vertices v & w.

thus, $N = \bigcup_{v \in L, w \in R} N(v,w)$.

& $\Omega = P \cup_{v,w} N(v,w)$

For $M \in \mathcal{P}$, let $\omega(M) = 1$

For $M \in N(y,z)$, let $\omega(M) = \omega(y,z) = \dfrac{|\mathcal{P}|}{|N(y,z)|}$

So n^2+1 different weights for graph with $n+n$ vertices.

Note, $\omega(N(y,z)) = \sum_{M \in N(y,z)} \omega(M) = |N(y,z)| \times \dfrac{|\mathcal{P}|}{|N(y,z)|} = |\mathcal{P}|$.

& $\omega(\mathcal{P}) = |\mathcal{P}|$.

Thus, $\pi(N(y,z)) = \pi(\mathcal{P}) = \dfrac{1}{n^2+1}$.

Moreover with these weights,
$T_{mix} = O(n^5 \log n)$, for any bipartite graph.

Thus, in Poly(n) time can find a random perfect matching.

Why is it rapid mixing?

Congestion for $M \to M'$:

$$\rho(M,M') = \frac{1}{\pi(M)P(M,M')} \sum_{(I,F)\in \mathcal{P}_{M,M'}} \pi(I)\pi(F)$$

Note, $\pi(M)P(M,M') = \pi(M')P(M',M)$

assume $\pi(M) \le \pi(M')$

then $P(M,M') = \frac{1}{2m}$ ← $m = |E|$.

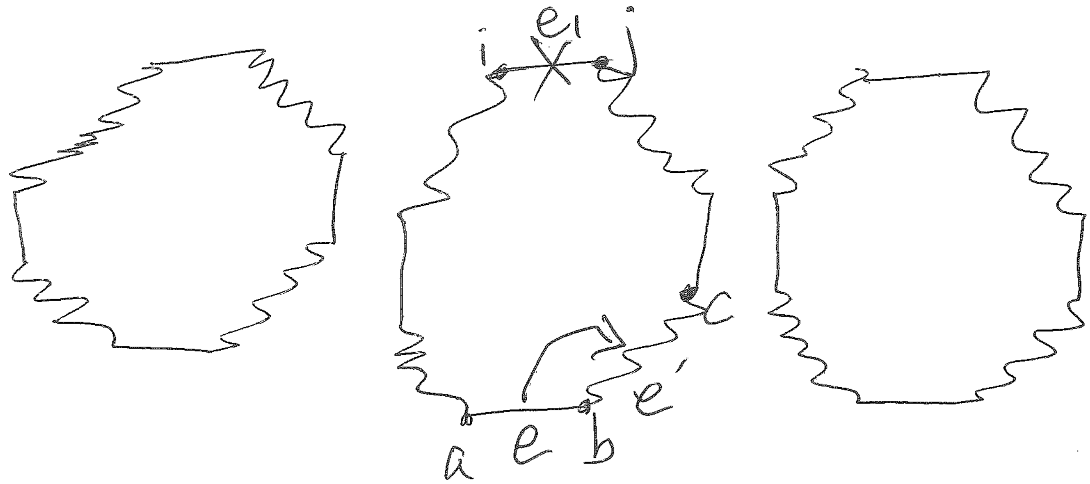
So, $$\rho(M,M') = \frac{1}{2\pi(M)\, m} \sum_{(I,F)\in \mathcal{P}_{M,M'}} \pi(I)\pi(F)$$

define $\cancel{C}\ \eta : \mathcal{P}_{M,M'} \to \Omega$ where

encoding C where $\pi(I)\pi(F) \le \pi(M)\pi(C)$,

then $$\rho(M,M') \le 2m \sum_{C\in\Omega} \pi(C) \le 2m.$$

Consider $\underline{I}, \underset{\sim\sim}{F} \in \mathcal{P}$ so $I \oplus F$:



Note $M \oplus C$ has same edges as $I \oplus F$

except 1^{st} edge e_1 of current cycle

& one edge e' in current slide.

Thus, $\pi(M)\,\pi(C) \geq \lambda^2 \pi(I)\,\pi(F)$

& hence $\rho \leq O(m\lambda^2)$.

too much! Since $\lambda \approx \frac{1}{n!}$ at end.

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Suppose $\lambda=0$, so $\omega(y,z)=\dfrac{|P|}{|N(y,z)|}$

Consider $I, F \in P$ so $\omega(I)\omega(F)=1$.

For $M \to M'$, we have $M \in N(i,c)$

$\& \; C \in N(j,b)$

Note, $(i,j) \in E \;\&\; (b,c) \in E$

& it also holds that:

$$\underbrace{|N(i,c)| \times |N(j,b)|}_{\text{when } (i,j),(b,c)\in E} \leq |P|^2$$

Thus, $\omega(i,c)\,\omega(j,b) \geq 1$

$\& \; \omega(M)\,\omega(C) \geq \omega(I)\,\omega(F)$.

$\blacksquare$

In general, for $\lambda > 0$,

$$\lambda(N(i,c))\,\lambda(N(j,b))\,\lambda(i,j)\,\lambda(b,c) \leq \lambda(P)^2,$$

Thus, $\omega(I)\omega(F) = 1 \leq \dfrac{\omega(i,c)\,\omega(j,b)}{\lambda(i,j)\,\lambda(b,c)}$.

How to find these weights?

Consider $K_{n,n}$ (complete bipartite graph)

where edges of G have activity 1

& non-edges of G have activity λ

for some $0 \leq \lambda \leq 1$.

In other words, for $v \in L, z \in R$,

$$\lambda(v,z) = \begin{cases} 1 & \text{if } (v,z) \in E \\ \lambda & \text{if } (v,z) \notin E \end{cases}$$

then for matching $M \in \Omega$,

$$\text{let } \lambda(M) = \prod_{e \in M} \lambda(e).$$

If $\lambda = 1$, then all $n!$ perfect matchings of $K_{n,n}$ have activity 1.

If $\lambda = 0$, then every $M \in \mathcal{P}$ has activity 1
& $M \notin \mathcal{P}$ has activity 0

So $\lambda(\mathcal{P}) = |\mathcal{P}|$ when $\lambda = 1$.

Generalize weights to:

$$\omega(y,z) = \frac{\lambda(\mathcal{P})}{\lambda(N(y,z))}$$

where $\lambda(\mathcal{P}) = \sum_{M \in \mathcal{P}} \lambda(M) = \sum_{M \in \mathcal{P}} \prod_{e \in M} \lambda(e)$

$$\& \ \lambda(N(y,z)) = \sum_{M \in N(y,z)} \lambda(M).$$

thus if $\lambda = 0$ then $\lambda(\mathcal{P}) = |\mathcal{P}|$

$$\& \ \lambda(N(y,z)) = |N(y,z)|$$

$$\text{so } \omega(y,z) = \frac{|\mathcal{P}|}{|N(y,z)|} \quad \text{if } \lambda = 0$$

& if $\lambda = 1$ then $\omega(y,z) = \frac{n!}{(n-1)!} = n.$

Hence, weights are easy to compute when $\lambda = 1$

& when $\lambda = 0$ then they give

a random perfect matching.

(Note $\lambda = \frac{1}{n!}$ is roughly equivalent to $\lambda = 0$.)

Suppose we had wrong weights $\hat{\omega}(y,z)$

$$\left(\text{So } \hat{\omega}(y,z) \neq \frac{\lambda(\rho)}{\lambda(N(y,z))}\right)$$

What is the stationary distribution $\hat{\pi}$ for MC run with these $\hat{\omega}$?

Then $\hat{\pi}(\rho) = \lambda(\rho) = \sum_{m\in\rho} \lambda(m)$

$$\hat{\pi}(N(y,z)) = \sum_{m\in N(y,z)} \lambda(m)\hat{\omega}(y,z)$$

$$= \hat{\omega}(y,z)\,\lambda(N(y,z))$$

Run the MC many times & look at fraction of samples that are in ρ to those in $N(y,z)$,

then $$\frac{\hat{\pi}(\rho)}{\hat{\pi}(N(y,z))} = \frac{\lambda(\rho)}{\hat{\omega}(y,z)\lambda(N(y,z))} = \frac{\omega(y,z)}{\hat{\omega}(y,z)}$$

So this ratio tells us how far $\hat{\omega}$ is off from ω.

Again, $$w(y,z) = \hat{w}(y,z)\frac{\hat{\pi}(P)}{\hat{\pi}(N(y,z))}.$$

Suppose, $$\frac{w(y,z)}{2} \le \hat{w}(y,z) \le 2w(y,z) \quad (*)$$

then $$\frac{\hat{\pi}(P)}{4} \le \hat{\pi}(N(y,z)) \le 4\hat{\pi}(P)$$

So with $O(n^2 \log n)$ samples we can estimate $w(y,z)$ using

Start with $\lambda = 1$. Compute $w(y,z)$ for all $y \in L, z \in R$.

- Set $\hat{w}(y,z) = w(y,z)$ for all $y \in L, z \in R$.
- Set $\lambda = \lambda\left(1 - \frac{1}{2n}\right)$
- Run MC with λ & $\hat{w}$ & then generate $O(n^2 \log n)$ samples from $\hat{\pi}$ & estimate $w(y,z)$.

Repeat, until $\lambda \le \frac{1}{n!}$.

Overall running time: $O^*()$ ignores log factors.

- $O^*(n^4)$ time per sample
- $O^*(n^2)$ samples per round
- $O^*(n^2)$ rounds

$\Rightarrow O^*(n^8)$ time.

(Can improve this to $O^*(n)$ rounds.)

$O^*(n^7)$ time is still the best known.