For undirected $G=(V,E)$,

let $\Omega(G)$ = all spanning trees of G

$= \{S \subset E : S \text{ is acyclic \& connected}\}$

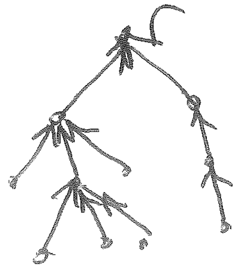
thus, $|S| = |V|-1$.

Goal: sample uniformly from Ω.

For undirected G, define $\overleftrightarrow{G}$ to be directed $\overleftrightarrow{G}=(V,\overleftrightarrow{E})$
where: for $(v,w) \in E$, add $\overrightarrow{vw}$ & $\overrightarrow{wv}$ to $\overleftrightarrow{E}$

For directed $\overrightarrow{G}$, for vertex $r \in V$,

arborescence = in-trees rooted at r:



$S \subset E$ s.t. $|S| = |V|-1$
& every $v \neq r$ has exactly 1 edge
directed away from v
(i.e., it's a directed tree rooted at r
pointing towards r

Let $A_r(\overleftrightarrow{G})$ = set of arborescences rooted at r

in-trees

& $A(\overleftrightarrow{G}) = \bigcup_{r \in V} A_r(\overleftrightarrow{G})$ = all arborescences.

<u>Observation:</u> $|\Omega(G)| = |A_r(\overleftrightarrow{G})|$ for any $r \in V$.

# of spanning trees of G = # arborescences of $\overleftrightarrow{G}$ rooted at $r \in V$.

<u>Proof:</u>

$\Rightarrow$ for spanning tree S, point edges toward root r.

$\Leftarrow$ for $S' \in A_r(\overleftrightarrow{G})$, drop orientations.

$\blacksquare$

MC1 on $A(\overleftrightarrow{G})$: MC on all in-trees (any root)

From $X_t \in A(\overleftrightarrow{G})$,

1. Let r be the current root.
2. Choose a random out-neighbor w of r.
   ($\overrightarrow{rw} \in \overleftrightarrow{E}$)
3. Let $\overrightarrow{wz}$ be the unique edge away from w in X_t.
   (this points towards r)
4. Let $X' = X_t \cup \overrightarrow{rw} \setminus \overrightarrow{wz}$.
   (note $X' \in A_w(\overleftrightarrow{G})$)
5. Let $X_{t+1} = \begin{cases} X' & \text{with Prob. } 1/2 \\ X_t & \text{o/w} \end{cases}$

If G is d-regular then MC1 is symmetric
& thus π is uniform over $A(\overleftrightarrow{G})$.

In general, for $S \in A(\overleftrightarrow{G})$,
$\pi(S)$ depends on its root r (depends on its degree(r))
So all $S \in A_r(\overleftrightarrow{G})$ have same π.

New MC, called MC_r on $A_r(\overleftrightarrow{G})$ for a fixed $r \in V$.

From $Y_t \in A_r(\overleftrightarrow{G})$,

1. Run MC1 with $X_0 = Y_t$ & stop at 1st time t' where $X_{t'} \in A_r(\overleftrightarrow{G})$ (i.e., 1st time when root returns to r).

2. Set $Y_{t+1} = X_{t'}$

<u>Note</u>: MC_r is symmetric & thus $\pi = \text{uniform}(A_r(\overleftrightarrow{G}))$.

What's the mixing time?

Consider lazy random walk on G.

From $v \in V$, with prob. $\frac{1}{2}$ stay at v

& with prob $\frac{1}{2}$ move to a random neighbor.

Cover time = # of steps until random walk visits every vertex at least once from worst initial state

denote as T_{cover}

for MC2

<u>Lemma</u>: $T_{mix} \leq 4\, E[T_{cover}]$

& for all G,

$$n \log n \leq E[T_{cover}] = O(nm)$$

We'll use coupling.

For $Y_t, Y_t' \in Ar(\overleftrightarrow{G})$, define coupling:

$$(Y_t, Y_t') \to (Y_{t+1}, Y_{t+1}')$$

where for all Y_0, Y_0',

$$\Pr(Y_T \neq Y_T') \leq 1/4$$

for $T = 4E[T_{cover}]$

For Y_t, Y_t', let $H(Y_t, Y_t') = |Y_t \setminus Y_t'|$

$=$ # edges in Y_t & not in Y_t'

Thus if $H(Y_t, Y_t') = 0$ then $Y_t = Y_t'$

Note, $Y_t \to Y_{t+1}$ uses excursion:

$$X_0 \to X_1 \to X_2 \to \cdots \to X_{t'}$$

root: $r \to v_1 \to v_2 \to \cdots \to r$

$$\underbrace{\overrightarrow{rv_1},\ \overrightarrow{v_1v_2},\ \ldots,\ \overrightarrow{v_{t'-1}r}}$$

for $Y'_t \to Y'_{t+1}$ use same sequence of edges

then share these edges, so

for all v on this excursion,

let i be last time saw v

then $\overrightarrow{v_i v_{i+1}} \in Y_{t+1} \cap Y'_{t+1}$

Once every vertex is visited at least once then

$Y_t = Y'_t$,

but these excursions are just a lazy random walk on G.

Let $T = 4E[T_{cover}]$.

$$\Pr(Y_T \neq Y'_T) \leq \Pr(T_{cover} > T)$$
$$= \Pr(T_{cover} > 4E[T_{cover}])$$
$$\leq 1/4 \text{ by Markov's ineq.} \quad \blacksquare$$

For the upper bound on the cover time of $O(n^3)$,

see:

Aleliunas, Karp, Lipton, Lovasz, Rackoff,
Random walks, universal traversal sequences,
and the complexity of maze problems
in FOCS '79.

Kirchoff's matrix-tree theorem:

For undirected G,

let A be the 0-1 adjacency matrix

& $K = D - A$ = Laplacian matrix of G

so $$K(i,j) = \begin{cases} \deg(i) & \text{if } i=j \\ -1 & \text{if } (i,j) \in E \\ 0 & \text{o/w} \end{cases}$$

K is known Kirchoff in-degree matrix.

for $r \in V$, let $K_{r,r}$ = K with row r & column r deleted

then $|\mathcal{T}| = |Ar(\overleftrightarrow{G})| = \det(K_{r,r})$.