Swendsen–Wang algorithm:

For $G = (V, E)$,
let $\mathcal{J} = \{+1, -1\}$

Configuration $\sigma \in \mathcal{J}$ is an assignment $\sigma : V \rightarrow \{+1, -1\}$.

Gibbs Distribution: $\Pi_{\beta}(\sigma) = \frac{e^{\beta \sum_{(v,w) \in E} \delta_{\sigma(v)\sigma(w)}}}{Z_\beta}$

where partition function $Z_\beta = \sum_{\sigma \in \mathcal{J}} e^{\beta \sum_{(v,w) \in E} \delta_{\sigma(v)\sigma(w)}}$

Note: $\sum_{(v,w) \in E} \delta_{\sigma(v)\sigma(w)} = |M(\sigma)| - |D(\sigma)|$

- $M(\sigma)$: # of monochromatic edges
- $D(\sigma)$: # of disagree edges

Ferromagnetic: $\beta > 0$ so most likely configurations are all $+$ & all $-$(called ground state)

but only 2 such whereas \( \binom{N}{N/2} \) balanced configurations (half $+$, half $-$)

Tradeoff betw energy vs. entropy
- all $+$ all $-$ \( \binom{N}{N/2} \) balanced

(called attractive in ML)
\( \beta = 0 \) (corresponding to infinite temperature as \( \beta = \frac{1}{\text{temp}} \))

then balanced dominates as all configurations have the same measure.

\( \beta = \infty \) (zero temperature)

then all +, all - dominate (everything else has zero measure)

Look at magnetization \( n(\sigma) = \sum_n \sigma(n) \)

Small \( \beta \)

large \( \beta \)

For \( \sqrt{n} \times \sqrt{n} \) grid, recall \( \mathbb{E} \beta_c \) where:

\( \forall \beta < \beta_c \): Glauber dynamics has \( O(\log n) \) mixing time

\( \forall \beta > \beta_c \): Glauber has mixing time \( \exp(\Omega(\sqrt{n})) \)

i.e., \( \geq e^{cn} \) for constant \( c > 0 \).
Note, $|M(\sigma)| + |D(\sigma)| = 1|E|$.

Thus, $2|M(\sigma)| - 1|E| = 1|M(\sigma)| - |D(\sigma)|$

Hence, $\Pi_1(\sigma) = \frac{e^{\beta(M(\sigma) - D(\sigma))}}{Z_{\text{I}}} = \frac{e^{2\beta M(\sigma)}}{e^{\beta|E|} Z_{\text{I}}} = \frac{e^{2\beta M(\sigma)}}{Z'}$

Generalization from $q=2$ spins to integer $q \geq 2$ spins.

Potts model: spins $\{1, 2, \ldots, q\}$

State space $\mathcal{Z}_p = \{1, 2, \ldots, q\}^V$

Configurations $\sigma \in \mathcal{Z}_p$ is assignment $\sigma : V \rightarrow \{1, 2, \ldots, q\}$

Gibbs distribution: $\Pi_p(\sigma) = \frac{e^{\beta M(\sigma)}}{Z_p}$

$M(\sigma) = \sum_{(v,w) \in E} \delta(\sigma(v) = \sigma(w))$

$q=2$ is the Ising model with $\beta_1 \rightarrow \beta_2$.

$\beta_2(8) = \frac{\beta_1}{2} \ln (1 + \sqrt{g})$ [Dumain-Capin-Betara]

$q=2$ was from [Onsager '44]
Random cluster model:

For $G=(V,E)$, $RC$ has 2 parameters:

$0 < p < 1$ & $q > 0$

Configurations are subsets of edges $SCE$

Gibbs distribution for $RC$:

$$\mu_{RC}(S) = \frac{P^{|S|}(1-p)^{|E\backslash S|} q^{c(S)}}{Z}$$

where $c(S) =$ # of components in graph $(V,S)$

(isolated vertices count as a component)

When $q=1$: Percolation—keep edge $e$ with prob $p$

delete $e$ w.r.p. $1-p$

For integer $q \geq 2$, set $p = 1-e^{-\beta}$ & use same $q$

then $Z_{RC} = Z_{Potts}$

Note, $\mu(S) = \frac{P^{|S|}(1-p)^{|E\backslash S|} q^{c(S)}}{Z} = \frac{(1-p)^{|E|} \left( \frac{p}{1-p} \right)^{|S|} c(S)}{Z}$

$(1-p = e^{-\beta})$

$$= \frac{p^{|S|}(1-p)^{-|E|} q^{c(S)}}{e^{\beta |E|} Z} = \frac{1}{Z_{RC}} \left( \frac{p}{1-p} \right)^{|S|} c(S).$$
**Swendsen-Wang algorithm:** For integer \( q \geq 2 \)

MC on \( \mathcal{G} = \{1, \ldots, q\}^V \) = Potts configurations.

**Idea:** From \( X_t \in \mathcal{G} \)

1. Transform to \( A_t \subset E \) which is RC configuration.

2. Perform step \( A_t \rightarrow A_{t+1} \) in RC model.

3. Map \( A_{t+1} \) back to \( X_{t+1} \in \mathcal{G} \) which is a Potts configuration.

Formally, from \( X_t \in \mathcal{G} = \{1, \ldots, q\}^V \)

1. Let \( M_t = \{(v,w) \in E : X_t(v) = X_t(w)\} \)
   = set of monochromatic edges.

2. For each \( (v,w) \in M_t \), independently
   add \( (v,w) \) to \( A_{t+1} \) w.p. \( p = 1 - e^{-\beta} \)
   & leave out \( (v,w) \) from \( A_{t+1} \) w.p. \( 1 - p = e^{-\beta} \).

   Let \( A_{t+1} \) be the remaining edges
   ignore the colors in \( X_t \) so \( A_{t+1} \subset E \).

3. For each connected component \( C \in (V, A_{t+1}) \),
   independently choose a color \( c \in \{1, \ldots, q\} \) var
   & assign every \( v \in C \) with color \( c \).

4. Let \( X_{t+1} \in \{1, \ldots, q\}^V \) be the resulting coloring (ignore edges)
Recall, $T_p$ = Gibbs distribution for Potts model
& $M_{RC}$ = Gibbs dist. for RC model

Claim 1: If $X_t \sim T_p$ then after steps 1 & 2,
$A_{t+1} \sim M_{RC}$

Claim 2: If $A_{t+1} \sim M_{RC}$ then after steps 3 & 4,
$X_{t+1} \sim T_p$

Therefore, if $X_t \sim T_p$ then $X_{t+1} \sim T_p$

So $T_p$ is a stationary distribution
for the SW dynamics,
and since SW is ergodic it's the unique stationary dist.

[Guo-Jerrum '16] For all $G$, all $\beta$, $T_{mix} = O(n^{10})$
for the SW dynamics,
big polynomial but its $\text{Poly}(n)$ for all $G$, all $\beta$ for Ising
Proof of Claim 1: \( X_t \sim \Pi \Rightarrow A_{t+1} \sim \mathcal{U} \)

For \( S \subset E \),

\[
Pr(A_{t+1} = S \mid X_t \sim \Pi) = \sum_{\xi \in \mathcal{B}_t} \frac{e^{\beta_1 M(\sigma)}}{Z_p} \prod_{s \in S} (1 - p)^{1 - M(\sigma)}
\]

\[
= \sum_{\xi : M(\sigma) = 2A(\sigma) = S} \frac{1}{Z_p} \prod_{s \in S} (1 - p)^{1 - M(\sigma)}
\]

\[
(1 - p = e^{-\beta})
\]

\[
= \frac{1}{Z_p} \sum_{\xi} \left( \frac{p}{1 - p} \right)^{|S|} \prod_{s \in S} (1 - p)^{1 - M(\sigma)}
\]

\[
= \frac{1}{Z_p} \left( \frac{p}{1 - p} \right)^{|S|} \sum_{\xi : M(\sigma) = 2S} \prod_{s \in S} (1 - p)
\]

\[
= \frac{1}{Z_p} \left( \frac{p}{1 - p} \right)^{|S|} c(s)
\]
Proof of claim 2: \( A_{t+1} \sim U_{RC} \Rightarrow X_{t+1} \sim \Pi_P \)

For \( \sigma \in \{1, \ldots, R \} \),

\[
\Pr(X_{t+1} = \sigma | A_{t+1} \sim U_{RC})
= \sum_{S : S \in \text{M}(\sigma)} \mu(S) \frac{1}{q^c(S)}
= \sum_{S : S \in \text{M}(\sigma)} \frac{1}{Z_{RC}} \left( \frac{P}{1-P} \right)^{|S|} \frac{q^c(S)}{q(S)}
= \frac{1}{Z_{RC}} \sum_{k=1}^{\infty} \sum_{S : S \in \text{M}(\sigma), |S| = k} \left( \frac{P}{1-P} \right)^k
= \frac{1}{Z_{RC}} \sum_{k} \left( IM(\sigma) \right)^{k} \left( \frac{P}{1-P} \right)^k
= \frac{1}{Z_{RC}} \left( \frac{P}{1-P} + 1 \right)^{IM(\sigma)}
= \frac{1}{Z_{RC}} \left( \frac{1}{1-P} \right)^{IM(\sigma)}
\]

\( 1-P = e^{-\lambda} \)

\[
e \frac{e^{\lambda IM(\sigma)}}{Z_{RC}} = \Pi_P(\sigma),
\]

\( \Box \)
SW is poly(n) mixing for all G, all $\beta$

when $q = 2$

What if $q \geq 3$?

Examples where exponentially slow.

For $\sqrt{n} \times \sqrt{n}$ grid, $\beta = \beta_c$,

$q = 2, 3: \quad T_{mix} = O(\text{poly}(n))$

$q = 4: \quad T_{mix} = o(\log n) \quad$ (conjecture should be poly(n))

$q \geq 5: \quad T_{mix} = \exp\left(J2(\sqrt{n})\right)$

Key: $q < 4$: magnetization = # of spins of dominant color is continuous but derivative is discontinuous

$\beta_c\beta$ $\rightarrow$ $q \geq 5$: magnetization is discontinuous

$q = 2$: [Lubetzky-Sly '10]

$q \geq 3$: [Chassay, Lubetzky '16]