Fundamental Theorem of Markov Chains:

For a finite, ergodic MC defined by P on Ω, there is a unique stationary distribution π. Moreover, for all $i,j \in \Omega$, $\lim_{t \to \infty} P^t(i,j) = \pi(j)$.

Ergodic: $\exists\, t^*$ s.t. $\forall i,j \in \Omega$, $P^{t^*}(i,j) > 0$

Irreducible + aperiodic $\Rightarrow$ ergodic

Irreducible: $\forall i,j\ \exists\, t = t(i,j)$, $P^t(i,j) > 0$

aperiodic: $\forall i$, $\gcd(T_i) = 0$ where $T_i = \{t : P^t(i,i) > 0\}$

stationary distribution π means $\pi P = \pi$.

1. Existence of a stationary distribution:

Fix a state $k \in \Omega$.

Let $X_0 = k$.

Let T_k be the 1st return time to state k.

(Note, T_k is a random variable.)

For $i \neq k$, let N_i = # of visits to state i before T_k (or at T_k).

& let $\rho_i(k) = E[N_i]$

and set $\rho_k(k) = 1$.

We'll show that: $\rho(k) P = \rho(k)$

where $\rho(k) = (\rho_1(k), \rho_2(k), \ldots, \rho_N(k))$

where $N = |\Omega|$.

then by setting $\mu(i) = \dfrac{\rho_i(k)}{\sum_j \rho_j(k)}$

we have that $\mu P = \mu$ so μ is a stationary distribution.

Fix a state $i \neq k$.

Then
$$\rho_i(k) = \sum_{t=1}^{\infty} \Pr(X_t = i, T_k \geq t+1 \mid X_0 = k)$$

$$= \sum_{t=1}^{\infty} \Pr(X_t = i, T_k \geq t \mid X_0 = k) \quad (\text{since } i \neq k)$$

$$= P(k,i) + \sum_{t=2}^{\infty} \sum_{j \neq k} \Pr(X_t = i, X_{t-1} = j, T_k \geq t \mid X_0 = k) \cancel{P(j}$$

$$= P(k,i) + \sum_{t=2}^{\infty} \sum_{j \neq k} \Pr(X_{t-1} = j, T_k \geq t \mid X_0 = k) P(j,i)$$

$$= P(k,i) + \sum_{j \neq k} \sum_{t=2}^{\infty} \Pr(X_{t-1} = j, T_k \geq t \mid X_0 = k) P(j,i)$$

$$= P(k,i) + \sum_{j \neq k} \sum_{t=1}^{\infty} \Pr(X_t = j, T_k \geq t+1 \mid X_0 = k) P(j,i)$$

$$= \cancel{P(} \rho_k(k) P(k,i) + \sum_{j \neq k} \rho_j(k) P(j,i)$$

$$= \sum_{j} \rho_j(k) P(j,i)$$

Thus, $\rho_i(k) = \sum_j \rho_j(k) P(j,i)$ so $\rho(k) = \rho(k) P$

We know the existence of a stationary distribution, how to prove uniqueness?

& convergence (i.e., that it's a limiting distribution)?

Consider 2 copies (X_t) & (Y_t) with arbitrary initial states $X_0, Y_0 \in \Omega$.

We'll define a coupling so that $\lim_{t \to \infty} \Pr(X_t \neq Y_t) = 0$

By ergodicity we know there exists t^* such that:

$$P^{t^*}(x,y) > 0 \quad \text{for all } x,y \in \Omega.$$

Thus, there exists $\epsilon > 0$ s.t. for all $x,y \in \Omega$,

$$P^{t^*}(x,y) \geq \epsilon.$$

We'll define a t^*-step coupling for X_t, Y_t.

Given X_{kt^*} & Y_{kt^*} for integer $k \geq 0$,

1. Run X_{kt^*} for t^* steps.
2. If $Y_{kt^*} = X_{kt^*}$ then use same $X_t = Y_t$ for all $kt^* \leq t \leq (k+1)t^*$
3. Else, choose independent t^* step evolution $Y_{kt^*} \to Y_{(k+1)t^*}$ for Y_t

$$\Pr\left(X_{(k+1)t^*} \neq Y_{(k+1)t^*} \mid X_{kt^*} \neq Y_{kt^*}\right)$$
$$\leq 1-\epsilon$$

thus, $\Pr\left(X_{kt^*} \neq Y_{kt^*} \mid X_0, Y_0\right) \leq (1-\epsilon)^k$

$$\leq e^{-\epsilon k}$$
$$\to 0 \text{ as } k \to \infty$$

Setting $Y_0 \sim \mu$ (the stationary distribution we proved exist)

then $Y_t \sim \mu$ for all $t \geq 0$,

& $\lim_{t\to\infty} X_t = \cancel{\mu}$ so $X_t \to \mu$ for any $X_0 \in \Omega$,

& for all $X_0 \in \Omega$, all $j \in \Omega$,

$$\lim_{t\to\infty} \Pr\left(X_t = j \mid X_0 = i\right) = \mu(j)$$

i.e., $\lim_{t\to\infty} P^t(i,j) = \mu(j)$

Note, if we know $T_{mix}(\frac{1}{4})$ then let $t^* = T_{mix}(1/4)$.

For any $X_0 \in \Omega$, for $Y_0 \sim \mu$ then

$$d_{TV}(X_{t^*}, Y_{t^*}) \leq 1/4$$

Moreover for any $X_0 \in \Omega$, $Z_0 \in \Omega$,

$$d_{TV}(X_{t^*}, Z_{t^*}) \leq 1/2$$

& $\exists$ a t^*-step coupling of $(X_0, Z_0) \to (X_{t^*}, Z_{t^*})$

So that $Pr(X_{t^*} \neq Z_{t^*}) \leq 1/2$.

Hence, for any integer $k \geq 0$

$$Pr(X_{(k+1)t^*} \neq Z_{(k+1)t^*} \mid X_{kt^*} \neq Z_{kt^*}) \leq \frac{1}{2}$$

& $Pr(X_{kt^*} \neq Z_{kt^*} \mid X_0, Z_0) \leq \frac{1}{2^k}$

Thus, $T_{mix}(\epsilon) \leq \underbrace{T_{mix}(\frac{1}{4})}_{= t^*} \log_2(1/\epsilon)$.

or $T_{mix}(\epsilon) \leq T_{mix}(\frac{1}{2e}) \ln(1/\epsilon)$.

Also, let $\tau_j = E[T_j]$ = mean return time for state j

then $\mu(j) = \frac{1}{\tau_j}$

(Can verify that this is a stationary distribution.)