# Functional analysis:

Consider finite, ergodic MC defined by $P$ on $\Omega$ with stationary distribution $\pi$.

For a function $f: \Omega \to \mathbb{R}$,

let its expectation $E_\pi(f) = \sum_{x \in \Omega} \pi(x) f(x)$

& variance $Var_\pi(f) = \sum_{x \in \Omega} \pi(x)\left(f(x) - E_\pi f\right)^2$ (V1)

Note, $Var_\pi f = \sum_x \pi(x)\left(f(x)^2 - 2f(x)E_\pi f + (E_\pi f)^2\right)$

$$= \sum_x \pi(x) f(x)^2 - 2\underbrace{\left(\sum_x \pi(x) f(x)\right)}_{E_\pi f} E_\pi f + (E_\pi f)^2 \underbrace{\sum_x \pi(x)}_{1}$$

$$= \sum_x \left(\pi(x) f(x)^2\right) - (E_\pi f)^2$$ (V2)

Moreover, $Var_\pi f = \sum_x \pi(x) f(x)^2 - (E_\pi f)^2$

$$= \sum_x \pi(x) f(x)^2 \sum_y \pi(y) - \sum_x \pi(x) f(x) \sum_y \pi(y) f(y)$$

$$= \sum_{x,y \in \Omega} \left(\pi(x)\pi(y) f(x)^2 - \pi(x)\pi(y) f(x) f(y)\right)$$

$$= \frac{1}{2} \sum_{x,y \in \Omega} \pi(x)\pi(y)\left(f(x) - f(y)\right)^2$$ (V3)

$\mathrm{Var}_\pi f$ measures global variation.

Local variation is measured by the Dirichlet form:

$$\mathcal{E}_P(f,f) = \frac{1}{2} \sum_{x,y \in \Omega} \pi(x) P(x,y) \left(f(x) - f(y)\right)^2$$

So $\mathrm{Var}_\pi f$ measures how $f$ changes over all pairs $x,y$

& $\mathcal{E}_P(f,f)$ measures over transitions.

Recall congestion: $\rho = \max_{\vec{ij} \in P} \frac{1}{\pi(i)P(i,j)} \sum_{(I,F) \in \mathcal{P}_{ij}} \pi(I)\pi(F)|\gamma_{IF}|$

$\mathcal{P}_{ij} = \{(I,F) : \gamma_{IF} \ni \vec{ij}\}$

<u>Lemma</u>: for any $f$, $\frac{\mathcal{E}_P(f,f)}{\mathrm{Var}_\pi f} \geq \frac{1}{\rho}$

<u>Proof</u>:

V3

$$2\mathrm{Var}_\pi f = \sum_{x,y \in \Omega} \pi(x)\pi(y)\left(f(x) - f(y)\right)^2 = \sum_{x,y} \pi(x)\pi(y) \left( \sum_{\vec{ij} \in \gamma_{xy}} 1 \cdot \left(f(i) - f(j)\right)\right)^2$$

$$\leq \sum_{x,y} \pi(x)\pi(y)|\gamma_{xy}| \sum_{\vec{ij} \in \gamma_{xy}} \left(f(i) - f(j)\right)^2$$

$$= \sum_{i,j \in \Omega} \left(f(i) - f(j)\right)^2 \sum_{x,y : \vec{ij} \in \gamma_{xy}} \pi(x)\pi(y)|\gamma_{xy}| \leq \sum_{i,j} \pi(i)P(i,j)\left(f(i) - f(j)\right)^2 \rho$$

$$= 2\rho\, \mathcal{E}_P(f,f) \quad \blacksquare$$

Cauchy-Schwarz: $|\langle u,v \rangle|^2 \leq \langle u,u \rangle \cdot \langle v,v \rangle$

[illegible]

Assume $P$ is lazy so that $P = \frac{1}{2}(I + P')$, i.e., there is a self-loop of $\geq \frac{1}{2}$ for every state.

Theorem: $\mathrm{Var}_\pi(Pf) \leq \mathrm{Var}_\pi f - \frac{1}{2}\mathcal{E}_P(f,f)$

where
$$(Pf)(x) = \sum_{y \in \Omega} P(x,y) f(y)$$

So $Pf$ is averaging of $f$ after one step of $P$.

For ergodic $P$, $P^t \to \pi$ so $P^t f \to E_\pi f$ which doesn't change with $t$.

So, can measure mixing time by looking at how fast
$$\mathrm{Var}_\pi(P^t f) \to 0 \text{ as } t \to \infty,$$
and this theorem says the variance decays by the Dirichlet form at each step.

Note, the theorem implies:
$$\mathrm{Var}_\pi(P^t f) \leq \left(1 - \frac{\alpha}{2}\right)^t \mathrm{Var}_\pi f$$
where $\alpha = \inf_f \dfrac{\mathcal{E}_P(f,f)}{\mathrm{Var}_\pi f}$.

Let $$\alpha = \inf_{\substack{f:\Omega\to\mathbb{R}, \\ (f \text{ not constant}, \\ \text{so } Var_\pi f \neq 0)}} \frac{\mathcal{E}_P(f,f)}{Var_\pi f}$$

$\alpha$ is the Poincare constant.

<u>Theorem:</u> $$T_{mix}^{x_0}(\epsilon) \leq \frac{2}{\alpha}\left(2\ln(1/\epsilon) + \cancel{\tfrac{1}{2}}\ln\left(\cancel{4}\pi(x_0)\right)^{-1}\right)$$

thus, $$T_{mix} \leq O\left(\frac{1}{\alpha}\log\left(\frac{1}{\pi_{min}}\right)\right)$$

<u>Proof:</u> For any $A \subset \Omega$, let $f(x) = 1(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$

Note, $Var_\pi f \leq 1$ so:

$$Var_\pi(P^t f) \leq \left(1-\frac{\alpha}{2}\right)^t \leq e^{-t\alpha/2}$$

let $$t = \frac{2}{\alpha}\left(2\ln\left(\frac{1}{\epsilon}\right) + \ln\frac{1}{\pi(x_0)}\right) = \frac{2}{\alpha}\ln\left(\frac{1}{\epsilon^2\pi(x_0)}\right)$$

then $$Var_\pi(P^t f) \leq \epsilon^2\pi(x_0)$$

Note, $$Var(P^t f) \geq \pi(x_0)\left((P^t f)(x_0) - E_\pi P^t f\right)^2$$
$$= \pi(x_0)\left((P^t f)(x_0) - E_\pi f\right)^2$$

So $\epsilon \geq |(P^t f)(x_0) - E_\pi f| = |P^t(x_0, A) - \pi(A)|$ for all $A \subset \Omega$.

## Connection to eigenvalues:

$P$ has eigenvalues $1 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N > -1$.

corresponding to $\pi$

for lazy $P$, then eigenvalues are $\geq 0$.

$$\text{inverse spectral gap} = (1-\lambda_2)^{-1}$$

$$1-\lambda_2 = \inf_f \frac{\mathcal{E}_P(f,f)}{\mathrm{Var}_\pi f}$$

Thus,

$$\Omega\left(\frac{1}{1-\lambda_2}\right) \leq T_{mix} \leq O\left(\frac{1}{1-\lambda_2} \log\left(\frac{1}{\pi_{min}}\right)\right)$$

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Recall, conductance, for $S \subset \Omega$,

$$\Phi(S) = \frac{\sum_{x \in S, y \in \bar{S}} \pi(x) P(x,y)}{\pi(S)\pi(\bar{S})}$$

Note, for $f = 1_S = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{o/w} \end{cases}$

then $\frac{\mathcal{E}_P(f,f)}{\mathrm{Var}_\pi f} = \Phi(S)$

Recall, theorem says: $\mathrm{Var}_\pi(Pf) \le \mathrm{Var}_\pi f - \frac{1}{2}\mathcal{E}_P(f,f)$

Proof: Since $P$ is lazy, can write as $P = \frac{1}{2}(I + \hat{P})$.

First, note

$$(Pf)(x) = \sum_y P(x,y) f(y) = \frac{1}{2} f(x) + \sum_y \hat{P}(x,y) f(y) = \frac{1}{2}\sum_y \hat{P}(x,y)\big(f(x) + f(y)\big)$$

Assume $E_\pi f = 0$ (o/w can shift $f$ & $\mathrm{Var}_\pi f$ stays same)

Then, $\mathrm{Var}_\pi(Pf) = \sum_x \pi(x)\big((Pf)(x)\big)^2 = \frac{1}{4}\sum_x \pi(x)\Big(\sum_y \hat{P}(x,y)\big(f(x)+f(y)\big)\Big)^2$

$$\le \frac{1}{4}\sum_{x,y\in\Omega} \pi(x)\hat{P}(x,y)\big(f(x)+f(y)\big)^2$$

And, $\mathrm{Var}_\pi f = \sum_{x\in\Omega} \pi(x) f(x)^2$

$$= \frac{1}{2}\sum_x \pi(x) f(x)^2 + \frac{1}{2}\sum_y \pi(y) f(y)^2$$

$$= \frac{1}{2}\sum_{x,y} \pi(x)\hat{P}(x,y) f(x)^2 + \frac{1}{2}\sum_{x,y} \pi(x)\hat{P}(x,y) f(y)^2$$

b/c $\sum_x \pi(x) P(x,y) = \pi(y)$

$$= \frac{1}{2}\sum_{x,y} \pi(x)\hat{P}(x,y)\big(f(x)^2 + f(y)^2\big)$$

Thus, $\mathrm{Var}_\pi f - \mathrm{Var}_\pi(Pf) \ge \frac{1}{4}\sum_{x,y} \pi(x)\hat{P}(x,y)\big(f(x) - f(y)\big)^2$

$$= \frac{1}{2}\mathcal{E}_{\hat{P}}(f,f).$$

Alternative functional quantity, modified log-Sobolev constant

$$\rho = \inf_{\substack{f \geq 0, \\ Ent_\pi f \neq 0}} \frac{\mathcal{E}_P(f, \log f)}{Ent_\pi f}$$

where $\mathcal{E}_P(f, \log f) = \sum_{x,y} \pi(x) P(x,y) \left(f(x) - \log f(x)\right)^2$

& $Ent_\pi f = \sum_x \pi(x) f(x) \log\left(\frac{f(x)}{E_\pi f}\right)$

<u>Theorem</u>: $T_{mix} = O\left(\frac{1}{\rho} \log\log\left(1/\pi_{min}\right)\right)$.

## Comparison inequality:

Consider 2 MC's with same state space $\Omega$

& stationary $\pi$

transition matrices $P$ & $\hat{P}$.

Suppose we know $\alpha$ for $P$, and want to know $\hat{\alpha}$ for $\hat{P}$.

for each $\vec{ij} \in P$, define a path $\gamma_{ij}$ along $\hat{P}$,

then let $$\hat{\rho} = \max_{\vec{k\ell} \in \hat{P}} \frac{1}{\pi(k)\hat{P}(k,\ell)} \sum_{(i,j) \in \mathcal{P}_{k\ell}} |\gamma_{ij}| \, \pi(i) P(i,j)$$

& then $$\hat{\mathcal{E}}_{\hat{P}}(f,f) \geq \frac{1}{\hat{\rho}} \mathcal{E}_P(f,f)$$

& $$\hat{T}_{mix} \leq \cancel{O(\tfrac{1}{\rho}} \; O\left(\rho \, T_{mix} \log\left(\frac{1}{\pi_{min}}\right)\right)$$

# Matroids:

Matroid $M=(E,I)$ consists of:

- ground set $E$
- set of subsets $I \subseteq 2^E$ called <u>independent sets</u>

where:

P1. Downward closure: (every subset of an independent set is also an independent set)

$$\text{if } S \in I, T \subseteq S \text{ then } T \in I$$

P2. Augmentation property: (Indpt. set exchange property)

$$\text{if } S, T \in I \ \& \ |S| > |T|$$
$$\text{then } \exists e \in S \setminus T \text{ s.t. } T \cup e \in I.$$

Note, $\emptyset$ is always an independent set.

A set $R \subset E$ which is not an independent set is called <u>dependent</u>.

A basis is a maximal independent set, ie, an independent set that becomes dependent if we add any $e \in E$.

All bases have the same size, this size is the rank of $M$.

Example:

1. Graphic matroid: Let $G=(V,E)$ be a connected graph.
then $M=(E,I)$ where:
- $E$ = edges of $G$
- $I$ = forests of $G$ (i.e., acyclic subgraphs)

& the bases are maximal forests which are spanning trees of $G$.

2. Transversal matroids: Let $G=(L,R,E)$ be a bipartite graph.
then $M=(E,I)$ where:
- $E=L$
- $I$ is set of subsets of $L$ that are endpoints of some matching of $G$.

bases are subsets of $L$ covered by maximum matchings (note, an independent set can correspond to multiple matchings

Goal: generate a random bases of a given matroid
(or approx. count the # of bases)

More generally, approx. count the # of independent sets of a given size.

the matroid is given by a membership oracle:
given $S \subset E$, the oracle says whether or not $S$ is an independent set.

Basis exchange process:

Let $\Omega = \{S : S \text{ is a basis for } M\}$

From $B \in \Omega$,

1. Pick $e \in B$ u.a.r.
2. Let $F = \{f \in E : B \setminus e \cup f \in \mathcal{I}\}$ denote the set of elements we can add to $B \setminus e$ while remaining independent.
3. Pick $f \in F$ u.a.r. & move to $B \setminus e \cup f$.

Irreducible: Consider $B, B' \in \Omega$, so $|B| = |B'| = r$.

Let $B \oplus B' = \{e_1, \ldots, e_k\} \cup \{e_1', \ldots, e_k'\}$

where: $\uparrow \subseteq B$ $\quad \uparrow \subseteq B'$

For $i = 1 \to k$,

Remove $e_i$ from $B$. Then by P2, $\exists e_i' \in B'$

so that $B \setminus e_i \cup e_i'$ is independent set.

The basis exchange walk is symmetric

so $\pi = \text{uniform}(\Omega)$.

[ALOV18] showed $T_{mix} = O(r^2 \log n) = O(n^2 \log n)$

using spectral gap.

[CGM19] showed $T_{mix} = O(n \log n)$

using modified log-Sobolev.