## Colorings:

Undirected $G=(V,E)$

For vertex $w \in V$, $\deg(w) = \#\text{of neighbors} = |\{(w,z) \in E\}|$

Maximum degree $\Delta = \max_{w} \deg(w)$

For integer $k \geq 2$,

Proper vertex $k$-coloring is $\sigma: V \to \{1,2,\dots,k\}$

where $\forall (w,z) \in E \;\; \sigma(w) \neq \sigma(z)$

(no monochromatic edges)

When $k > \Delta$, always exists a $k$-coloring

$k \leq \Delta$, there are graphs with no $k$-coloring.

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Given $G$ & $k$, let $\Omega$ = all proper $k$-colorings of $G$.

<u>Goal</u>: compute $|\Omega|$ via FPRAS

or sample uniformly at random from $\Omega$.

**Glauber Dynamics:** (or Gibbs sampler)

From $X_t \in \Omega$,

1. Choose $v \in V$ u.a.r. & $c \in \{1, \dots, k\}$ u.a.r.
2. For all $w \neq v$, set $X_{t+1}(w) = X_t(w)$.
3. Set $X_{t+1}(v) = \begin{cases} c & \text{if } c \notin X_t(N(v)) \\ X_t(v) & \text{o/w} \end{cases}$

Ergodic:

- aperiodic since $P(\sigma, \sigma) \geq \frac{1}{k}$ (choose current color)
- irreducible when $k \geq \Delta + 2$:

Let $V = \{v_1, \dots, v_n\}$.

Fix $\sigma$ & $\tau$. To go $\sigma \rightsquigarrow \tau$,

go through $i = 1 \rightarrow n$:

for $v_i$, if some neighbor $w$ has $\tau(v_i)$ then recolor $w$ to any free color

& change $v_i$ to $\tau(v_i)$.

Symmetric so $\pi = \text{uniform}(\Omega)$ is unique stationary distribution.

Mixing time?

Let's prove $T_{mix} = O(n \log n)$ when $k > 3\Delta$.

<u>Coupling</u>: For $X_t, Y_t$,

- choose same $v$ & $c$ to (try for) update

(identity coupling)

Let $A_t = \{v : X_t(v) = Y_t(v)\}$ = "agree"

$D_t = \{v : X_t(v) \neq Y_t(v)\}$ = "disagree"

As before, since if $X_t \neq Y_t$ then ~~$E[|D_t|] \geq 1$~~ $|D_t| \geq 1$

So: $\Pr(X_t \neq Y_t) \leq E[|D_t|]$

Let's look at $E[|D_t|]$

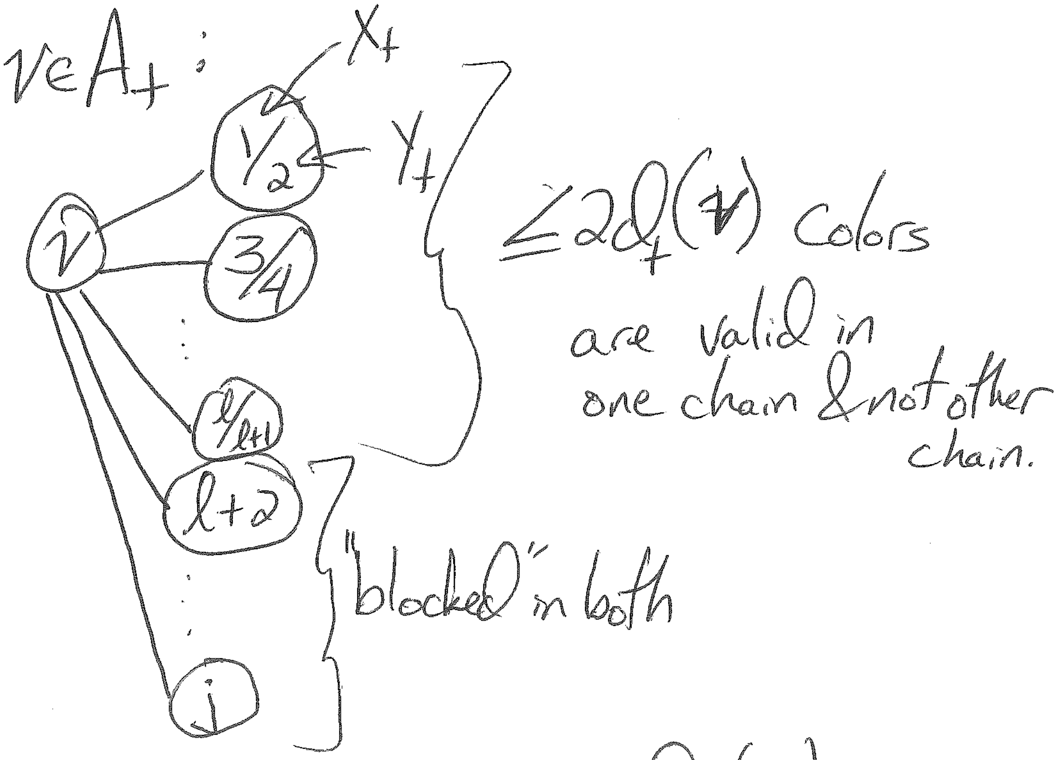
& compare $E[|D_{t+1}|]$ vs. $E[|D_t|]$.

Let $a_t(v) = A_t \cap N(v)$ = agree neighbors of $v$

$d_t(v) = D_t \cap N(v)$ = disagree neighbors of $v$.

Note, $$\sum_{v \in A_t} d_t(v) = \sum_{v \in D_t} a_t(v) = \text{\# of agree-disagree edges.}$$

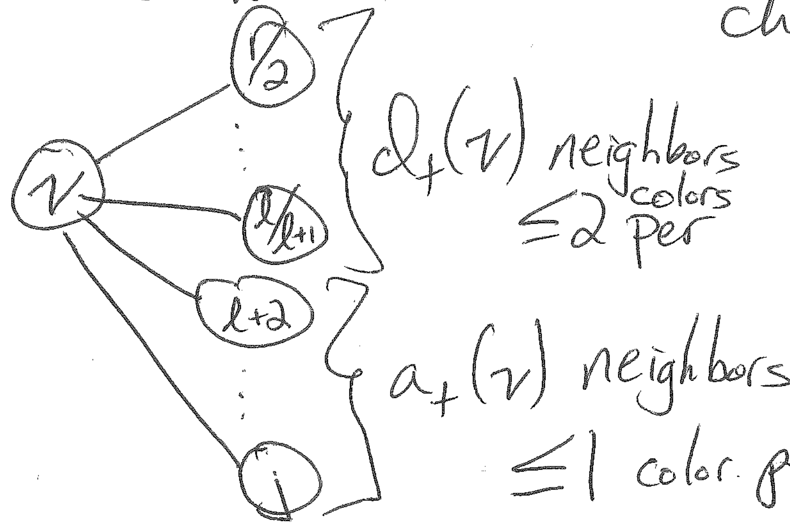
Suppose $v \in A_t$:




$$\Pr(v \in D_{t+1} \mid v \in A_t) \leq \frac{2d_t(v)}{nk}$$

Suppose $v \in D_t$:

how many colors valid for $v$ in both chains?




$\leq 2\Delta - a_t(v)$ colors in $N(v)$

$$\left|X_t(N(v)) \cup Y_t(N(v))\right| \leq 2\Delta - a_t(v)$$

Colors available for $v$ in $X_t$ & $Y_t$ $= \left|[k] \setminus \left(X_t(N(v)) \cup Y_t(N(v))\right)\right|$

$$\geq k - (2\Delta - a_t(v)) = k - 2\Delta + a_t(v).$$

$$\Pr(v \in A_{t+1} \mid v \in D_t) \geq \frac{k - 2\Delta + a_t(v)}{nk}.$$

$$E\left[|D_{t+1}|\right] \le |D_t| + \sum_{v\in A_t} \frac{2d_t(v)}{nk} - \sum_{v \in D_t} \frac{(k-2\Delta+a_t(v))}{nk}$$

$$\le |D_t| + \sum_{v\in A_t} \cancel{\frac{2d_t(v)}{nk}} + \sum_{v\in D_t} \frac{-k+3\Delta \cancel{+2a_t(v)}}{nk}$$

$$= |D_t|\left(1 - \frac{(k-3\Delta)}{nk}\right)$$

$$\le |D_t|\left(1-\frac{1}{nk}\right) \quad \text{for } k>3\Delta.$$

$$\Pr(X_t \ne Y_t) \le E\left[|D_t|\right]$$

$$\le |D_0|\left(1-\frac{1}{nk}\right)^t$$

$$\le ne^{-t/nk}$$

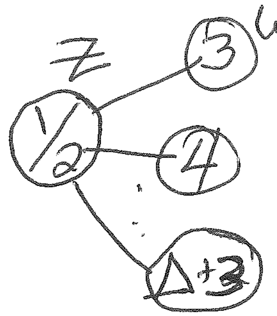
$$\le \frac{1}{4} \text{ for } t \ge nk\log(4n)$$

$$\Rightarrow T_{mix} = O(n\log n) \text{ for } k>3\Delta.$$

Note, worst-case pair has $|D_t| = 1$.

<u>Path coupling:</u> Consider $X_t, Y_t$ where

$$D_t = X_t \oplus Y_t = \{z\}.$$




$\leq \Delta$ colors in $N(z)$
(since all agree)

If update $v = z$ then $\geq k - \Delta$ color choices for $z$.

$$\Pr\left(X_{t+1}(z) = Y_{t+1}(z)\right) \geq \frac{k-\Delta}{nk}$$

If update $w \in N(z)$ then $\leq 2$ colors are "bad" for $w$.

$$\Pr\left(X_{t+1}(w) \neq Y_{t+1}(w)\right) \leq \frac{2}{nk}$$

$$\Rightarrow E\left[\,|D_{t+1}| \;\middle|\; |D_t| = 1\right] \leq 1 + \frac{2\Delta}{nk} - \frac{(k-\Delta)}{nk}$$

$$= 1 + \frac{-k+3\Delta}{nk} \leq 1 - \frac{1}{nk} \text{ for } k \geq 3\Delta + 1.$$

Better coupling:

if update $v = w \in N(z)$,

then pair color $c = 1 = X_t(z)$

with color $c' = 2 = Y_t(z)$

& $c'$ with $c$.

then only $\frac{c'}{c}$ for $w$ is "bad"

$$\Rightarrow E\left[|D_{t+1}| \,\middle|\, |D_t| = 1\right]$$

$$\leq 1 + \frac{\Delta}{nk} - \frac{(k-\Delta)}{nk} \leq 1 - \frac{1}{nk} \text{ for } k \geq 2\Delta + 1.$$

What about pairs $X_t, Y_t$ where $|D_t| > 1$?

Path coupling [Bubley-Dyer '97].

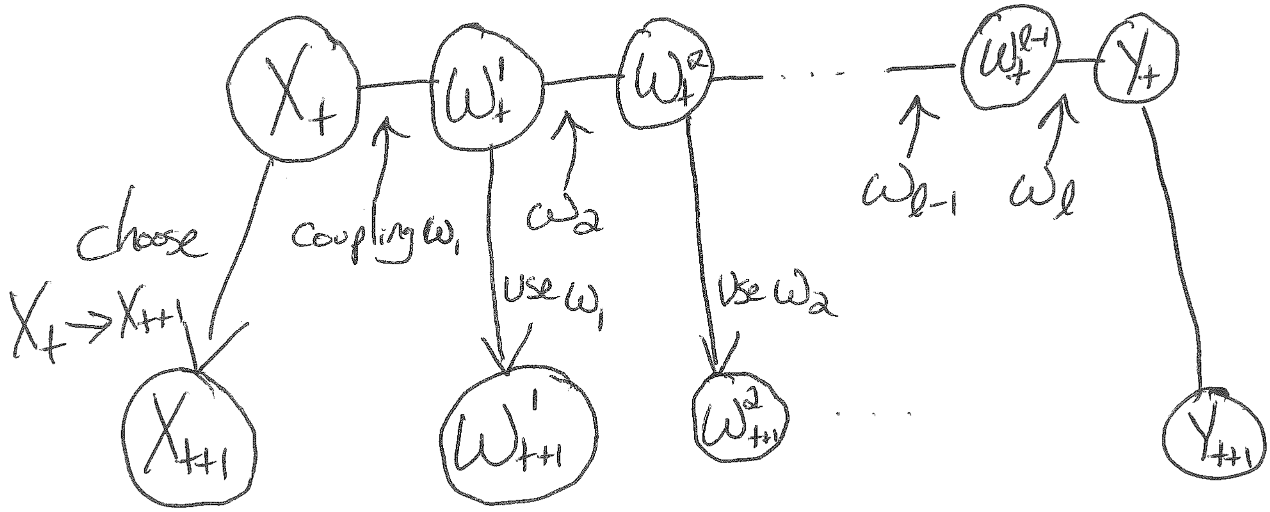
Consider $X_t, Y_t$ where $D_t = \ell > 1$:

Define sequence: $\omega_t^0, \omega_t^1, \ldots, \omega_t^\ell$ where:

$\omega_t^0 = X_t,\ \omega_t^\ell = Y_t$

& $|\omega_t^i \oplus \omega_t^{i+1}| = 1$

So it's a path on colorings of length $\ell$.




Coupling for $X_t, Y_t$ is $\omega_1 \circ \omega_2 \circ \cdots \circ \omega_\ell$

(couplings compose)

choose $X_t \to X_{t+1}$, apply $\omega_1$ to get $\omega_t^1 \to \omega_{t+1}^1$

Use $\omega_t^1 \to \omega_{t+1}^1$ & $\omega_2$ to get $\omega_t^2 \to \omega_{t+1}^2$

$\vdots$

Use $\omega_t^{\ell-1} \to \omega_{t+1}^{\ell-1}$ & $\omega_\ell$ to get $Y_t \to Y_{t+1}$

i.e., $\omega = \omega_1 \circ \omega_2 \circ \cdots \circ \omega_\ell$

Let $H(X_t, Y_t) = |D_t| =$ Hamming distance.

$$E[|D_{t+1}|] = E[H(X_{t+1}, Y_{t+1})]$$
$$\leq E\left[\sum_{i=0}^{l-1} H(\omega_{t+1}^{i}, \omega_{t+1}^{i+1})\right]$$
$$= \sum_{i=0}^{l-1} E[H(\omega_{t+1}^{i}, \omega_{t+1}^{i+1})]$$
$$\leq l \times \left(1 - \frac{1}{nk}\right) \quad \text{for } k \geq 2\Delta + 1$$
$$= |D_t| \left(1 - \frac{1}{nk}\right)$$

Thus,

$$\Pr(X_t \neq Y_t) \leq E[|D_t|] \leq n\left(1 - \frac{1}{nk}\right)^t \leq ne^{-t/nk}$$
$$\leq 1/4$$
$$\text{for } t \geq nk\ln(4n)$$

So, $T_{mix} = O(nk\log n)$. ∎

## Path coupling [Bubley-Dyer '97]:

For ergodic MC on $\Omega$ with stationary dist. $\pi$.

Let $S \subset \Omega \times \Omega$ s.t. $(\Omega, S)$ is connected.

For $(X, Y) \in \Omega \times \Omega$, let

$\mathrm{dist}(X,Y)$ = length of shortest path b/w $X$ & $Y$ in $(\Omega, S)$

If there exists $\beta < 1$ s.t. for all $(X_t, Y_t) \in S$, there is a coupling $(X_t, Y_t) \to (X_{t+1}, Y_{t+1})$

where $E[\mathrm{dist}(X_{t+1}, Y_{t+1})] \leq \beta\, \mathrm{dist}(X_t, Y_t)$

then
$$T_{mix} = O\left(\frac{\log D_{max}}{1-\beta}\right)$$

where $D_{max} = \max_{(X,Y) \in \Omega^2} \mathrm{dist}(X, Y)$