Recap of last class (in a simplified setting):

Consider ergodic MC on Ω with transition matrix P, and stationary π.

Let's assume $\pi = \text{uniform}(\Omega)$

& entries of P are p or 0,

so if $P(u,v) > 0$ then $P(u,v) = p$.

Then we defined conductance as:

for $S \subset \Omega$,

$$\Phi(S) = \Pr(X_{t+1} \notin S \mid X_t \in S, X_t \sim \pi)$$

$$= \frac{\sum_{y \in S, z \notin S} \pi(y) P(y,z)}{\pi(S)}$$

So ~~defn~~ edge-expansion

$$= \frac{|E(S,\bar{S})| \times p}{|S|} \quad \text{(in our simplified setting)}$$

let $\Phi_* = \min_{S \subset \Omega : |S| \le \frac{|\Omega|}{2}} \Phi(S)$

Theorem:

$$\Omega\left(\frac{1}{\Phi_*}\right) = T_{mix} = O\left(\frac{1}{\Phi_*^2} \log(|\Omega|)\right)$$

Again in our simplified setting:

— Canonical paths:

For all $x, y \in \Omega$, define a path γ_{xy}

where ~~&~~ $\gamma_{xy} = (z_0, z_1, \ldots, z_\ell)$,

$z_0 = x$, $z_\ell = y$, & $P(z_i, z_{i+1}) > 0 \; \forall i$.

(So a path in (Ω, P))

For transition $\overrightarrow{uv}$ (i.e., $P(u,v) > 0$)

then: Congestion is defined as:

$$\rho(\overrightarrow{uv}) = \frac{|\mathcal{P}_{uv}|}{p|\Omega|}$$

where $\mathcal{P}_{uv} = \{(x,y) \in \Omega^2 : \gamma_{xy} \ni \overrightarrow{uv}\}$

= set of paths γ_{xy} that go through $\overrightarrow{uv}$

Let $\rho = \max_{\overrightarrow{uv}} \rho(\overrightarrow{uv})$

then: $T_{mix} = O(\rho^2 \log(|\Omega|))$

because $\cancel{\Phi(S)}$

if $\rho \leq \beta$ then $|\mathcal{P}_{uv}| \leq \beta p |\Omega|$ & $\cancel{\Phi(S)}$ $|E(S,\bar{S})| \geq \frac{|S||\bar{S}|}{\beta p |\Omega|} \geq \frac{|S|}{2\beta p}$ so $\Phi(S) \geq \frac{1}{2\beta}$.

What's an example of canonical paths?

Consider RW on n-dimensional hypercube $\{0,1\}^n$.

For $I, F \in \Omega$, we'll define $\gamma_{IF} = (z_0, z_1, \dots, z_\ell)$
where $\ell \leq n$.

Let $z_0 = I$

For $i = 1 \to n$:

for all $j \neq i$, $z_i(j) = z_{i-1}(j)$

& $z_i(i) = F(i)$

(so set i^{th} bit to F & rest same)

Note, $z_n = F$, & z_i, z_{i-1} differ in ≤ 1 bit

so $P(z_{i-1}, z_i) = \frac{1}{2n}$

(& skip if $z_{i-1} = z_i$)

How to bound $\rho(\overrightarrow{uv})$?

— Suppose $\overrightarrow{uv}$ flips bit i.

We'll define a map $\eta_{\overrightarrow{uv}} : \mathcal{P}_{uv} \rightarrow \Omega$

which will be injective $\left(\text{for } \tau \in \Omega, \ |\eta^{-1}(\tau)| \leq 1 \right)$

& hence $|\mathcal{P}_{uv}| \leq |\Omega|$

& thus $\rho(\overrightarrow{uv}) = O(n)$

think of η as "encoding": from η & $\overrightarrow{uv}$

we can uniquely determine $(I,F) \in \mathcal{P}_{uv}$.

Let $E = \eta_{\overrightarrow{uv}}(I,F)$

j^{th} bit of $\eta_{uv}(I,F)$

& $E(j) = \begin{cases} F(j) & \text{for } j > i \\ I(j) & \text{for } j \leq i \end{cases}$ $\left(\begin{matrix} \text{bit } i \\ \text{is flipped} \\ \text{by } \overrightarrow{uv} \end{matrix} \right)$

Given $E = \eta_{uv}(I, F)$ & $\overrightarrow{uv}$

then we can determine I, F:

I agrees with $\begin{cases} E & \text{on bits} \le i \\ U & \text{on bits} > i \end{cases}$

F agrees with $\begin{cases} v & \text{on bits} \le i \\ E & \text{on bits} > i \end{cases}$

Therefore, $|\eta^{-1}(E)| \le 1$ so η is injective

& hence $\rho = O(n)$. ~~where n = |E|~~

Since $\pi_{min} = 2^{-n}$ so $\log\left(\frac{1}{\pi_{min}}\right) = O(n)$

then we get $T_{mix} = O(n^3)$.

(Note, we already know $T_{mix} = O(n \log n)$ from a coupling argument.)

Next class: Use this approach for a random matching (of any G).

Random matchings:

Given $G=(V,E)$,

let Ω = all matchings of G (any size)

Goal: Sample from uniform(Ω).

From $X_t \in \Omega$,

1. Choose $e \in E$ u.a.r.
2. Let $X' = X_t \oplus e = \begin{cases} X_t \setminus e & \text{if } e \in X_t \\ X_t \cup e & \text{if } e \notin X_t \end{cases}$
3. If $X' \in \Omega$, set $X_{t+1} = X'$ with prob. $1/2$ & otherwise set $X_{t+1} = X_t$.

Ergodic & symmetric $\Rightarrow \pi = $ uniform(Ω).

Let's analyze a slightly different chain: (not essential, but simplifies a bit)

From $X_t \in \Omega$,

1. Choose $e=(v,w) \in E$ u.a.r.

2a. If $e \in X_t$, $X' = X_t \setminus e$.

2b. If w & v are unmatched in X_t, then $X' = X_t \cup e$

2c. If w is unmatched in X_t & $(v,z) \in X_t$ then $X' = X_t \cup (v,w) \setminus (v,z)$

3. Set $X_{t+1} = X'$ with prob. $1/2$ & else $X_{t+1} = X_t$.

Canonical paths for matchings chain (with add/delete/slide)

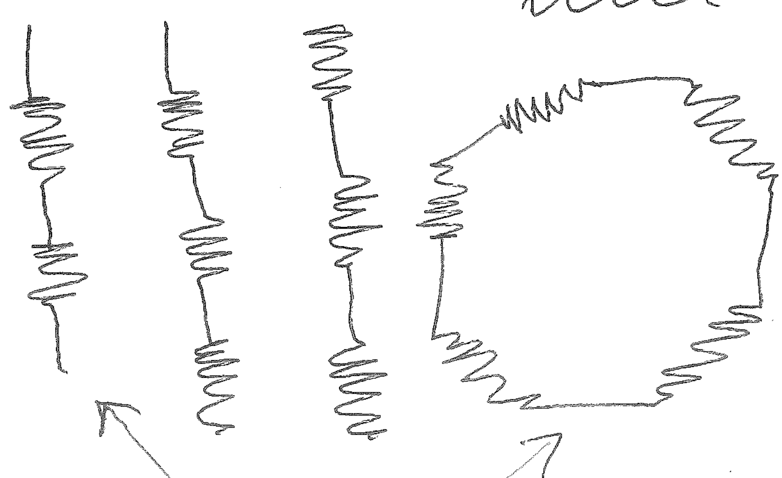
For $I, F \in \Omega$, define path γ_{IF} in (Ω, P).

Fix arbitrary order $V = \{v_1, v_2, \ldots, v_n\}$.

Consider $I \oplus F = (I \setminus F) \cup (F \setminus I)$.

Consists of augmenting paths,
deaugmenting paths,
alternating paths,
& alternating cycles.

For example, $\underline{I}$ & F (wavy)



Order components by min vtx. in each component.
Then "unwind" components in order.
For deaug. path, remove smaller end & slide.
For alt. cycle, remove edge at smallest vtx, slide & then add.

Consider transition $t = M \to M'$.

Suppose t is a slide move,

So $M' = M \cup (v,w) \setminus (v,z)$

Want to bound $|\mathcal{P}_{M,M'}|$

Define $\eta_t : \mathcal{P}_{M,M'} \to \Omega$

Let $E = \eta_t(I,F)$

we'll define as:

$$E = (I \cup F) \setminus (M \cup (v,w) \cup (v,z)) \cup (I \cap F)$$

$$= (I \cap F) \cup (I \oplus F \setminus t)$$

Look at $M \oplus E$ where $t = M \to M'$.

Note, $M \oplus E = (I \oplus F) \setminus (v,w)$

just need to figure out which "parity" is I & which is F for each component.

Can order components in $M \oplus E$ as in $I \oplus F$

& in earlier components, M matches F
E matches I

in later components, E matches F
M matches I

in current component, earlier part of M matches F (rest is I)
& later part of E matches F

One catch: if current component in γ_{IF} is an alternating cycle C, then E has 2 edges incident 1^{st} vertex in C.

So $E \notin \Omega$.

Need to drop an extra edge.

Need to add an extra bit of "memory" to record if current component is a path or cycle in $I \oplus F$.

Thus, $|P_{M,M'}| \leq 2|\Omega|$

& $\rho(t) \leq \dfrac{|P_{M,M'}|}{p|\Omega|} \leq \dfrac{2|\Omega|}{\frac{1}{2m}|\Omega|} = 4m$

$\Rightarrow T_{mix} = O(m^2 n \log n)$

since $|\Omega| \leq n!$ so $\log(|\Omega|) = O(n \log n)$.

Can improve bound from $T_{mix} = O(\rho^2 \log|\Omega|)$

to $T_{mix} = O(\rho\, l_{max} \log|\Omega|)$

length of longest γ_{IF}

this gives $T_{mix} = O(nm \log n)$