Given a graph $G=(V,E)$,

let Ω = all matchings of G.

Goal: sample from uniform(Ω).

MC: From $X_t \in \Omega$,

1. Choose $e=(y,z)\in E$ uar.

(add) 2. If y & z are unmatched in X_t, then let $X' = X_t \cup e$.

(remove) 3. If $e \in X_t$, then let $X' = X_t \setminus e$.

(slide) 4. If y is unmatched in X_t & $(z,w) \in X_t$, then let $X' = X_t \cup (y,z) \setminus (z,w)$

5. With prob. $\frac{1}{2}$ let $X_{t+1} = X'$ (if defined) & otherwise let $X_{t+1} = X_t$.

This is ergodic MC. Symmetric & thus π = uniform(Ω).

Use canonical paths to bound mixing time.

For all $I, F \in \Omega$, define path γ_{IF} in (Ω, P).

Consider a pair I, F.

For edges in $I \cap F$: nothing to do.

For edges in $I \setminus F$ or $F \setminus I$

need to drop need to add.

Let $I \oplus F = (I \setminus F) \cup (F \setminus I)$.

The symmetric difference $I \oplus F$ of 2 matchings consists of components which are either:

- alternating paths
- augmenting paths
- alternating cycles.

Fix arbitrary order $V = \{v_1, \ldots, v_n\}$.

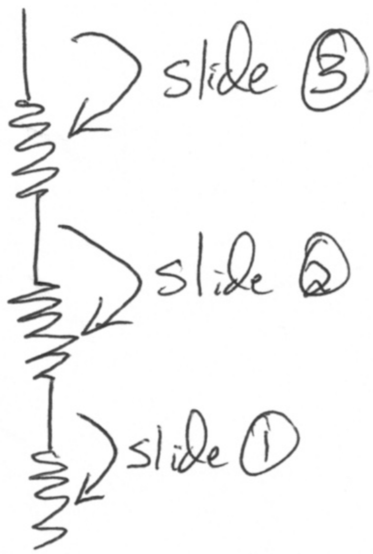
Order components in $I \oplus F$ by min vertex # in each component.

To go $I \rightsquigarrow F$:

For components in $I \oplus F$ in order:

- "unwind" from I to F

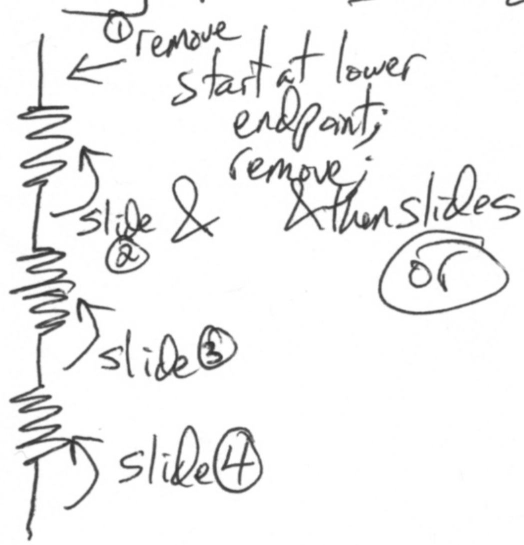
For alternating path: I & F



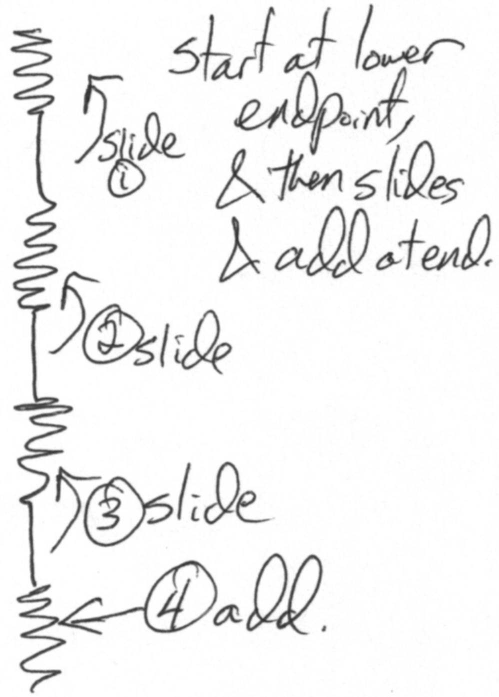
Series of slides starting at end with edge in F.

For augmenting path: I & F

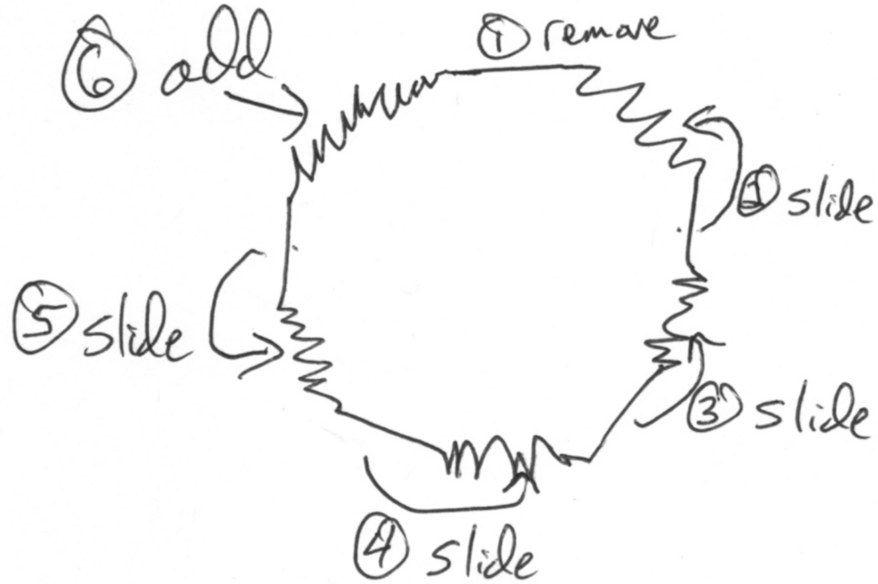
either:



or



For alternating cycle: $\underline{I}$ & F



start at lowest vtx.
remove edge in I.
then slide &
add at end.

That defines γ_{IF}.

Given a transition $M \to M'$,
need to bound congestion.

Let $P_{M,M'} = \{(I,F) \in \Omega^2 : \gamma_{IF} \ni M \to M'\}$

then congestion $\rho(\overrightarrow{MM'}) = \dfrac{|P_{M,M'}|}{\frac{1}{2m}|\Omega|} = \dfrac{2|P_{M,M'}|m}{|\Omega|}$

where $m = |E|$

Since $P(M,M') = \dfrac{1}{2m}$ [choose random e, then move to X' w. prob. $1/2$)

Encoding technique:

We'll define $\eta_{M,M'} : P_{M,M'} \to \Omega$ which is injective & thus $|P_{M,M'}| \le |\Omega|$,

so $\rho(\overrightarrow{MM'}) \le 2m.$ ← this is $m = |E|$ (not M)

Say $M' = M \cup e' \setminus e$ (so "sliding")

Let $\eta(I,F) = (I \cap F) \cup (I \oplus F \setminus (M \cup e'))$

common edges

the opposite edges in $I \oplus F$ & drop e' to maintain a matching.

Let $E = \eta(I,F)$.

Given E & $M \to M'$ Then

$M \cap E = I \cap F$ (same common edges)

$M \oplus E = I \oplus F$ (same symmetric difference)

but need to determine which edges of $M \oplus E$ belong to I & which to F.

Since $M \oplus E = I \oplus F$, they have the same components & same ordering on components.

Thus, given $M \to M'$ we know which component currently working on, say i^{th} component.

For components $< i$, we know:

M matches F (finished already)

& thus E matches I

For components $> i$ we know:

M matches I (haven't started yet)

& E matches F.

For i^{th} component:

sliding edge $e \to e'$

So earlier portion of this component:

$M = F$ & ~~E~~ $E = I$

& on later portion:

$M = I$ & $E = F$

Thus, from E & $M \to M'$, can uniquely decode (I, F).

Since η is injective

$$\Rightarrow |\mathcal{P}_{M,M'}| \leq |\Omega|$$

So $\rho(M,M') \leq 2m$

& $T_{mix} = O(\rho^2 \log(|\Omega|)) = O(m^2 n \log n)$

since $|\Omega| \leq n!$

(can improve to $O(mn^2 \log n)$

Using $T_{mix} = O(\rho \, l_{max} \log(|\Omega|))$

length of longest γ_{IF} $\leq n$.

More general, parameter $\lambda > 0$,

Matching $M \in \Omega$, has weight $w(M) = \lambda^{|M|}$

<u>Goal:</u> Sample from $\pi(M) = \dfrac{w(M)}{Z}$

where $Z = \sum_{M' \in \Omega} w(M')$.

Use same MC as before but change step 5:

5. With prob. $\frac{1}{2}\min\left\{1, \frac{w(X')}{w(X_t)}\right\}$ let $X_{t+1} = X'$

else let $X_{t+1} = X_t$.

Metropolis filter.

Note, $\pi(M)P(M,M') = \pi(M')P(M',M)$

So π is the stationary dist.

Same canonical paths as before.

for $M \to M'$, congestion defined as:

$$\rho(M,M') = \frac{1}{\pi(M)P(M,M')} \sum_{(I,F)\in \rho_{M,M'}} \pi(I)\pi(F)$$

Since can consider $M \to M'$ or $M' \to M$,

assume $\pi(M') \geq \pi(M)$ & hence

$$P(M,M') = \frac{1}{2m}$$

Then $\rho(M,M') = \frac{2m\lambda^{|M|}}{|\Omega|} \sum_{(I,F)\in \rho_{M,M'}} \pi(I)\pi(F) = \frac{2m}{|\Omega|}\sum \lambda^{|I|+|F|-|M|}$

So we have: $\rho(M,M') = \frac{2m}{|\Omega|} \sum_{(I,F)\in P_{M,M'}} \lambda^{|I|+|F|-|M|}$

In E & $M \to M'$, we have:

$$|E| + |M| \geq |I| + |F| - 2$$

lose 1^{st} edge in current component & extra edge for current slide.

Hence $\rho(M,M') \leq 2m\hat{\lambda}^2$

where $\hat{\lambda} = \max\{1, \lambda\}$

Thus, $T_{mix} = O(\hat{\lambda}^2 m^2 n \log n)$.

$\mathcal{P}$ = perfect matchings

& $\mathcal{N}$ = near-perfect matchings = exactly 2 unmatched vertices

Define same chain as before but restrict to $\mathcal{P} \cup \mathcal{N}$, so $\Omega = \mathcal{P} \cup \mathcal{N}$.

Claim 1: For a graph $G=(V,E)$, on n vertices if the min degree is $> \frac{n}{2}$, then

$$\frac{|\mathcal{P}|}{|\Omega|} \geq \frac{1}{n^2}.$$

Thus, for $\lambda = 1$, $\pi(\mathcal{P}) \geq \frac{1}{n^2}$

Claim: if $\frac{|\mathcal{P}|}{|\Omega|} \geq \frac{1}{n^2}$ then the MC with $\lambda=1$,

$$T_{mix} = \text{Poly}(n).$$