Given a graph $G = (V, E)$,

let $\mathcal{Z}$ = all matchings of $G$.

Goal: sample from uniform($\mathcal{Z}$).

MC: From $X_t \in \mathcal{Z}$,

1. Choose $e = (y, z) \in E$.

(add) 2. If $y$ & $z$ are unmatched in $X_t$,
then let $X' = X_t \cup e$.

(remove) 3. If $e \in X_t$,
then let $X' = X_t \backslash e$.

(slide) 4. If $y$ is unmatched in $X_t$ & $(z, w) \in X_t$,
then let $X' = X_t \cup (y, z) \backslash (z, w)$

5. With prob. $\frac{1}{2}$ let $X_{t+1} = X'$ (if defined)
else otherwise let $X_{t+1} = X_t$.

This is ergodic MC. Symmetric & thus $\Pi = \text{uniform}(\mathcal{Z})$. 
Use canonical paths to bound mixing time.
For all $I, F \in \mathcal{S}$, define path $\gamma_{IF}$ in $(\mathcal{S}, \mathcal{F})$.

Consider a pair $I, F$.
For edges in $I \cap F$: nothing to do.
For edges in $I \setminus F$ or $F \setminus I$
\[ \text{need to drop} \quad \text{need to add.} \]

Let $I \oplus F = (I \setminus F) \cup (F \setminus I)$.
The symmetric difference $I \oplus F$ of 2 matchings consists of components which are either:
- alternating paths
- augmenting paths
- alternating cycles.

Fix arbitrary order $V = \{v_1, \ldots, v_n\}$.
Order components in $I \oplus F$ by min vertex # in each component.
To go \( I \Rightarrow F \):
For components in \( I \& F \) in order:
- "unwind" from \( I \) to \( F \)

For alternating path: \( I \& F \)

1. slide 3
2. slide 2
3. slide 0

Series of slides starting at end with edge in \( F \).

For augmenting path: \( I \& F \)

either:

1. Remove start at lower endpoint, remove slides & then slides or
2. slide &
3. slide 0
4. slide 4

Start at lower endpoint, & then slides & add at end.

5. slide
6. slide
7. slide
8. add.
For alternating cycle: $I \& F$

1. remove
2. add
3. slide
4. slide
5. slide

Start at lowest vtx. remove edge in $I$, then slide & add at end.

That defines $X_{IF}$.

Given a transition $M \rightarrow M'$, need to bound congestion.

Let $P_{m,m'} = \sum (I,F) e \in \Sigma^2 : X_{IF} \in M \rightarrow M'$

Then congestion $\rho(M,M') = \frac{1}{2m} \frac{|P_{m,m'}|}{1521} = \frac{2|P_{m,m'}|}{1521}$

where $m = |E|$

Since $P(M,M') = \frac{1}{2m}$ [choose random $e$, then move to $X$ w/ prob. $\frac{1}{2}$]
Encoding technique:

We'll define \( \gamma : P_n \times P_n \to P_2 \)
which is injective & thus \( |P_n \times P_n| \leq |P_2| \)

\[ \rho(\gamma) \leq 2m \quad \text{(this is)} \]

Say \( M = M U e e \) (so "sliding")

let \( \gamma(x, y) = (x, y) U (x \neq y \setminus (M U e e)) \)

\[ \uparrow \text{Common edges} \]
\[ \uparrow \text{the opposite edges in I\&F} \& \text{drop } e e \text{ to maintain a matching.} \]

Let \( E = \gamma(I, I) \).

Given \( E \& M \to M' \) then

\[ M \cap E = I \cap F \quad \text{(same common edges)} \]
\[ M \oplus E = I \oplus F \quad \text{(same symmetric difference)} \]

but need to determine which edges of \( M \oplus E \) belong to \( I \) & which to \( F \).
Since $M \oplus E = I \oplus F$, they have the same components & same ordering on components.

Thus, given $M \rightarrow M'$ we know which component currently working on

Say $i$th component.

For components $< i$, we know:

$M$ matches $F$ (finished already)

& thus $E$ matches $I$

For components $> i$, we know:

$M$ matches $I$ (haven't started yet)

& $E$ matches $F$.

For $i$th component:

Sliding edge $e \rightarrow e'$

So earlier portion of this component:

$M = F$ & $E = I$

& on later portion:

$M = I$ & $E = F$

Thus, from $E \& M \rightarrow M'$, can uniquely decode $E$. 
Since \( \mathcal{Y} \) is injective

\[ \Rightarrow |P_{m,m'}| \leq |\mathcal{Y}| \]

So \( \rho(\mathcal{M}, \mathcal{M}') \leq 2m \)

\[ \& \quad T_{\text{mix}} = O(\rho^2 \log(|\mathcal{Y}|)) = O(m^2 \log n) \]

Since \( |\mathcal{Y}| \leq n! \)

(cannot improve to \( O(mn^2 \log n) \)

Using \( T_{\text{mix}} = O(\rho \max \log(|\mathcal{Y}|)) \)

length of longest diff \( \leq n \).

More general, Parameter \( \lambda > 0 \),

Matching \( \mathcal{M} \in \mathcal{Z} \), has weight \( w(M) = \lambda^{|M|} \)

Goal: Sample from \( \pi(M) = \frac{w(M)}{Z} \)

where \( Z = \sum_{M' \in \mathcal{Z}} w(M') \).
Use same MC as before but change steps 5:
5. With prob. \( \frac{1}{2} \min \{ 1, \frac{w(x')}{w(x)} \} \) let \( X_{t+1} = x' \)
   else let \( X_{t+1} = x \).

Metropolis filter.

Note, \( \pi(m)p(m,m') = \pi(m')p(m',m) \)

So \( \pi \) is the stationary dist.

Same canonical paths as before,
for \( M \to M' \), congestion defined as:

\[
p(m,m') = \frac{1}{\pi(m)p(m,m')} \sum_{(i,f) \in \delta_m,m'} \pi(i)\pi(f)
\]

Since can consider \( M \to M' \) or \( M' \to M \),
assume \( \pi(m') \geq \pi(m) \) & hence

\[
p(m,m') = \frac{1}{2m}
\]

Then

\[
p(m,m') = \frac{2m\lambda^{\text{lim}}}{152} \sum_{(i,f) \in \delta_m,m'} \pi(i)\pi(f) = \frac{2m\lambda}{152} \sum_{(i,f) \in \delta_m,m'} \pi(i)\pi(f)
\]
So we have: \( \rho(m, m') = \frac{2m}{|E|} \sum_{(I,F) \in P_{m,m'}} \lambda^{(I|/|F|-1)} \)

In \( E \& M \rightarrow m' \), we have:

\[ |E| + |M| \geq |I| + |F| - 2 \]

\[ \Rightarrow \]

lose 1st edge in current component & extra edge for current slide.

Hence \( \rho(m, m') \leq 2m \hat{\lambda}^2 \)

where \( \hat{\lambda} = \max \{ \lambda, \hat{\lambda} \} \)

Thus, \( T_{\text{mix}} = O(\hat{\lambda}^2 m^2 n \log n) \).
$P =$ perfect matchings
$N =$ near-perfect matchings = exactly 2 unmatched vertices.

Define same chain as before but restrict to $P \cup N$, so $SZ = P \cup N$.

Claim 1: For a graph $G = (V,E)$, if the min degree is $\geq \frac{4}{n}$, then

$$\frac{|P|}{|V|} \geq \frac{1}{n^2}.$$ 

Thus, for $\lambda = 1$, $\Pi(P) \geq \frac{1}{n^2}$.

Claim: if $\frac{|P|}{|V|} \geq \frac{1}{n^2}$ then the MC with $T=1$

$T_{mix} = Poly(n)$. 