We saw in class:

Given a graph $G=(V,E)$, (arbitrary G)

let $\mathcal{M}$ = matchings of G

We saw a MC to sample uniformly ($\mathcal{M}$) in time poly(n).

Given the sampler, how to estimate $|\mathcal{M}|$?
# of matchings

Consider an edge $e=(v,w)\in E$.

Let $\mathcal{M}_1=\{M\in\mathcal{M} : e\in M\}$
& $\mathcal{M}_2=\{M\in\mathcal{M} : e\notin M\}$

Clearly, $|\mathcal{M}|=|\mathcal{M}_1|+|\mathcal{M}_2|$.

Let $G_1 = G\setminus v\setminus w$ = G with vertices v & w removed (& all edges incident v or w)

Let $G_2 = G\setminus e$ = G with edge e removed.

Then, $|\mathcal{M}_1| = |\mathcal{M}(G_1)|$ & $|\mathcal{M}_2| = |\mathcal{M}(G_2)|$.

$$\Pr_G(e \in M) = \frac{|M_1|}{|M|} = \frac{|M_1|}{|M_1| + |M_2|}$$

G over uniform(M)

Thus, $$|M| = \frac{|M_1|}{\Pr(e \in M)}$$

<u>Alg.:</u> Generate samples from uniform($M(G)$),

estimate $\max\{\Pr(e \in M), \Pr(e \notin M)\}$

Say $\Pr(e \notin M) \geq \Pr(e \in M)$ so $\Pr(e \notin M) \geq 1/2$,

Use $O(m)$ samples to estimate

$P = \Pr(e \notin M)$ within factor $\left(1 \pm \frac{\epsilon}{2m}\right)$

then recursively estimate $|M_2|$

on G_2 (one less edge),

and return$\left(\frac{|M_2|}{P}\right)$.

Running time: Say $T_{mix} = O(m^2 n \log n)$

Then $T(m) \leq O(m^3 n \log n) + T(m-1)$

$\leq O(m^4 n \log n) = \text{poly}(n)$.

What about perfect matchings?

Let $\mathcal{P}$ = all perfect matchings.

For very dense graphs (min degree $> \frac{n}{2}$)

can sample from uniform($\mathcal{P}$)

$\Rightarrow$ estimate $|\mathcal{P}|$.

We'll see next week, ~~can~~ for any bipartite graph,

can sample from uniform($\mathcal{P}$)

& estimate $|\mathcal{P}|$

in poly(n) time.

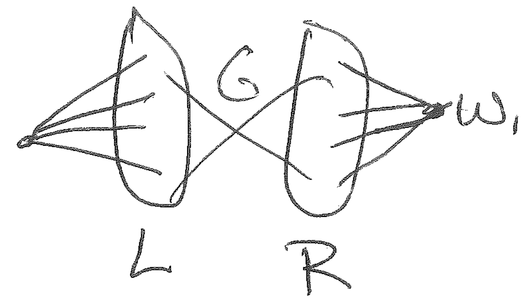
General graphs: open problem.

What about M_k = matchings of size k? (So $\mathcal{P} = M_{\frac{n}{2}}$)

Can we use this algorithm for perfect matchings to sample/count smaller matchings?

Given bipartite G,

add v_1 connected to L



& w_1 connected to R

Denote as G_1. Note G_1 is still bipartite.

Note, $|P(G_1)| = |M_{\frac{n}{2}-1}|$

(b/c v_1 matches with one vtx. in L
& w_1 " " in R.)

And add $v_1, \ldots, v_k$ connected to L
& add $w_1, \ldots, w_k$ " R.

Denote as G_k.

Then, $|P(G_k)| = (k!)^2 \, |M_{\frac{n}{2}-k}|$

equivalently, $|P(G_k)| = (n-k)!^2$ # matchings of size k in G.

What about planar graphs?

Given planar graph $G=(V,E)$ with 0-1 adjacency matrix A, we can form (in poly time) a directed version of G (orient each edge) so that for $\vec{A}$ (skew-symmetric adj. matrix)

$$\vec{A}(i,j) = \begin{cases} 1 & \text{if } \vec{ij} \in E \\ -1 & \text{if } \vec{ji} \in E \\ 0 & \text{if } (i,j) \notin E \end{cases}$$

then: $$|P|^2 = \det(\vec{A})$$

[Kasteleyn '61].

But can we also do $|M_k|$? matchings of size k?

The previous reduction doesn't preserve planarity.

[Jerrum '87] showed it's #P-hard

so no exact alg.

but Friday's talk shows an FPRAS.