Homework 2
Due: Wednesday, October 13, 2021 before the start of class at 11am.
Type your solutions using Latex and turn-in via Gradescope.
Let me know if you’re having difficulties and I’ll give you help/hints.
Zoom office hours are Tuesdays/Thursdays at 9pm.

Problem 1:
Consider the following Markov chain for shuffling a deck of \( n \) cards. Let \( \Omega \) denote the set of all \( n! \) permutations of \( \{1, 2, \ldots, n\} \). We will choose a random position \( i \) and a random card \( c \), then we will swap card \( c \) with the card in position \( i \). Formally it is the following.

Notation: For a state \( X \in \Omega \), let \( X = (x_1, \ldots, x_n) \) denote the permutation where \( x_i \) is the card in the \( i \)-th position and let \( X^{-1}(c) \) denote the position of card \( c \in \{1, \ldots, n\} \). For example, for \( X = (3, 4, 2, 1, 5) \) then card 3 is in the first position and \( X^{-1}(2) = 3 \) as card 2 is in position 3.

Markov chain:. From \( X_t \in \Omega \),

1. Choose a position \( i \) u.a.r. from \( \{1, \ldots, n\} \) and a card \( c \) u.a.r. from \( \{1, \ldots, n\} \).
2. Let \( j = X_t^{-1}(c) \) denote the position of card \( c \) in the current permutation \( X_t \).
3. Swap the cards in positions \( i \) and \( j \), i.e., let

\[
X_{t+1}(i) = X_t(j), X_{t+1}(j) = X_t(i), \text{ and for all } k \neq i, j, X_{t+1}(k) = X_t(k).
\]

Part (a): Show that this chain is ergodic with uniform stationary distribution.

Part (b): Give a coupling argument to prove that the mixing time is \( O(n^2) \).

Hint: you should end up with a coupon collector type argument at the end of your analysis but the probabilities of collecting the coupons will be on the order of \( 1/n^2 \) (instead of \( 1/n \)).

Note: It is OK to use path coupling, however it is a bit subtle to do so correctly. In particular, be careful with the set \( S \subset \Omega \times \Omega \) that you use as “neighboring” pairs and with the definition of the distance metric, make sure it’s a path metric (i.e., for every pair \( (X, Y) \in \Omega \times \Omega \) the shortest path in \( (\Omega, S) \) between \( X \) and \( Y \) is of length equal to the distance between \( (X, Y) \)).