Homework 2

Due: Wednesday, October 13, 2021 before the start of class at 11am.
Type your solutions using Latex and turn-in via Gradescope.
Let me know if you're having difficulties and I'll give you help/hints.
Zoom office hours are Tuesdays/Thursdays at 9pm.

Problem 1:

Consider the following Markov chain for shuffling a deck of n cards. Let Ω denote the set of all $n!$ permutations of $\{1, 2, \ldots, n\}$. We will choose a random position i and a random card c, then we will swap card c with the card in position i. Formally it is the following.

Notation: For a state $X \in \Omega$, let $X = (x_1, \ldots, x_n)$ denote the permutation where x_i is the card in the i-th position and let $X^{-1}(c)$ denote the position of card $c \in \{1, \ldots, n\}$. For example, for $X = (3, 4, 2, 1, 5)$ then card 3 is in the first position and $X^{-1}(2) = 3$ as card 2 is in position 3.

Markov chain:. From $X_t \in \Omega$,

1. Choose a position i u.a.r. from $\{1, \ldots, n\}$ and a card c u.a.r. from $\{1, \ldots, n\}$.
2. Let $j = X_t^{-1}(c)$ denote the position of card c in the current permutation X_t.
3. Swap the cards in positions i and j, i.e., let

$$X_{t+1}(i) = X_t(j), X_{t+1}(j) = X_t(i), \text{ and for all } k \neq i, j, X_{t+1}(k) = X_t(k).$$

Part (a): Show that this chain is ergodic with uniform stationary distribution.
Part (b): Give a coupling argument to prove that the mixing time is $O(n^2)$.

Hint: you should end up with a coupon collector type argument at the end of your analysis but the probabilities of collecting the coupons will be on the order of $1/n^2$ (instead of $1/n$).

Note: It is OK to use path coupling, however it is a bit subtle to do so correctly. In particular, be careful with the set $S \subset \Omega \times \Omega$ that you use as "neighboring" pairs and with the definition of the distance metric, make sure it's a path metric (i.e., for every pair $(X, Y) \in \Omega \times \Omega$ the shortest path in (Ω, S) between X and Y is of length equal to the distance between (X, Y)).