Optimal Mixing of MCMC via Spectral Independence

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Joint work with:

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- Independent Sets and Computational Phase Transition
- Spectral Independence Technique
- Proof approach: SI implies fast mixing of Glauber
- Conclusion

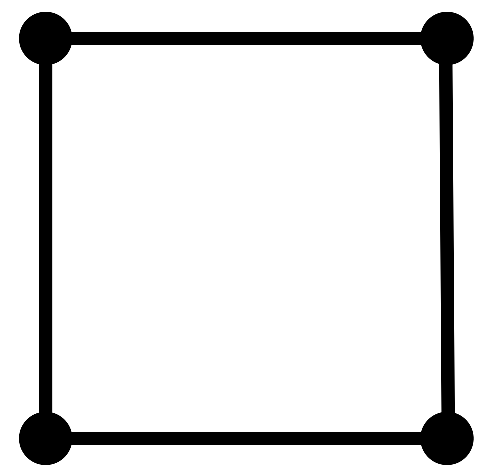
Sampling Independent Sets

- Given $G = (V, E)$, let $\Omega =$ set of all independent sets (IS) of G

  $I \subseteq V$ is an IS if I has no adjacent pairs

- Counting problem: Compute $|\Omega|$ = number of IS's of G

- Sampling problem: Sample IS from $\mu = \mathrm{uniform}(\Omega)$



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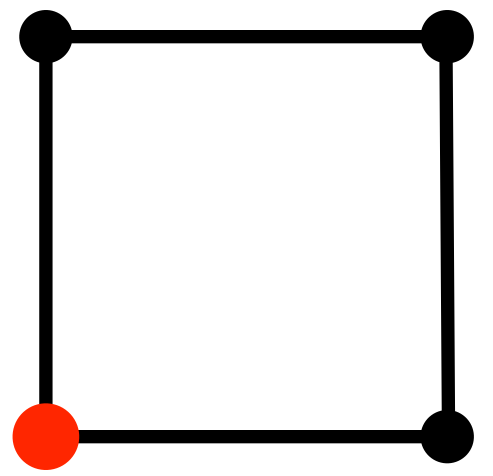
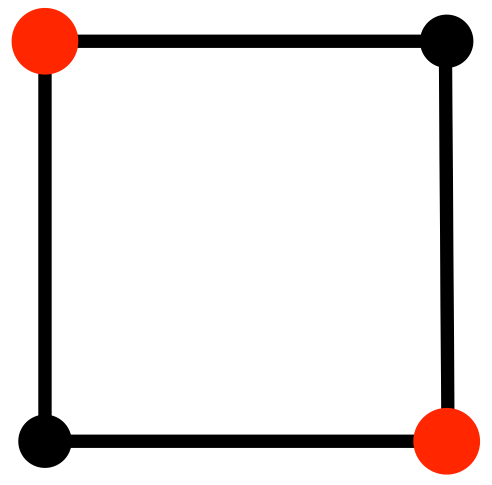


I_1



I_2

$|\Omega| = 7$

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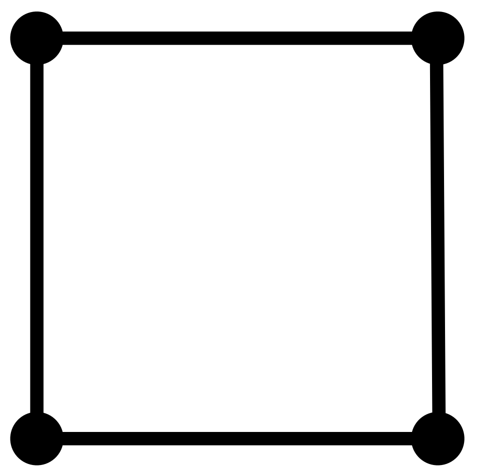
Typically, $|\Omega| \geq \exp(Cn)$ where $n = |V|$

Can we count/sample in time $poly(n)$?



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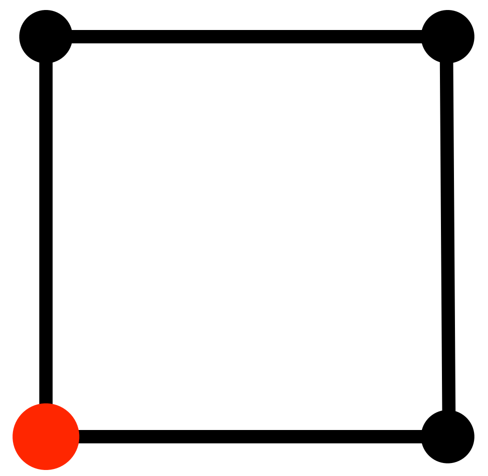

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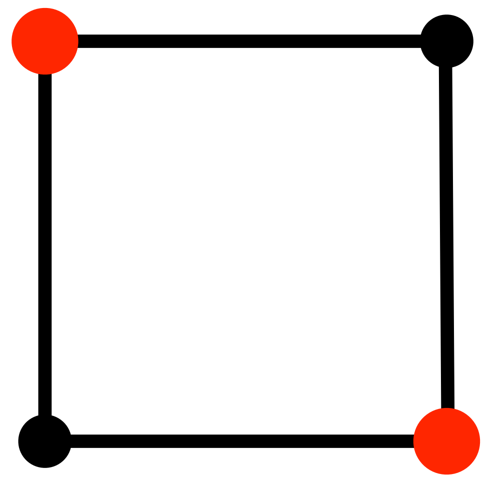

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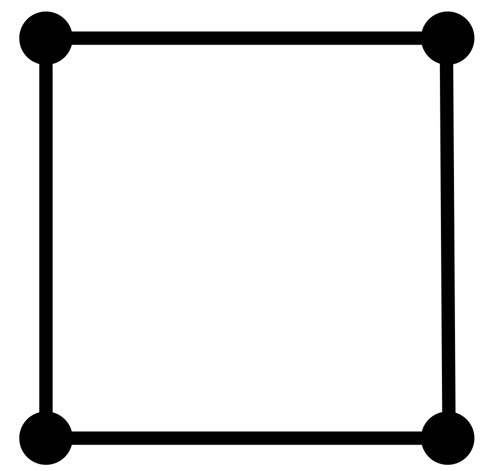
  Can we count/sample in time $poly(n)$?

- Exact counting is intractable (#P-complete)
- Can we approximate $|\Omega|$? Approx. Counting $\cong$ Approx. Sampling

[Jerrum-Valiant-Vazirani '86]



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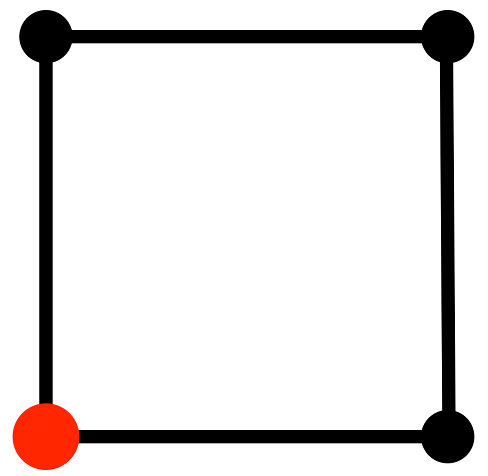

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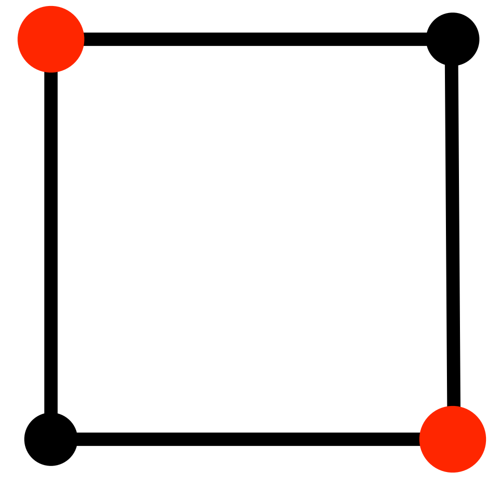

I_2

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Approx. Counting/Sampling

Counting: $|\Omega|$=# of IS's;   Sampling: $\mu = \mathrm{uniform}(\Omega)$

- *FPRAS* for Approx Counting: Given *G*, and $\epsilon, \delta > 0$, output *EST*:

$$\mathbf{Pr}\left(EST(1-\epsilon) \le |\Omega| \le EST(1+\epsilon)\right) \ge 1-\delta,$$

  in time $\mathrm{poly}(n, 1/\epsilon, \log(1/\delta))$.

- Approx. Sampler: Given *G*, and $\delta > 0$, samples from π where $||\pi - \mu||_{TV} \le \delta$

  in time $\mathrm{poly}(n, \log(1/\delta))$.

$$||\pi - \mu||_{TV} := \frac{1}{2}\sum_{x\in\Omega} |\pi(x) - \mu(x)|$$

- Approx Sampler in $O(n \log n)$ time then FPRAS in $O(n^2 \log^2 n)$ time.

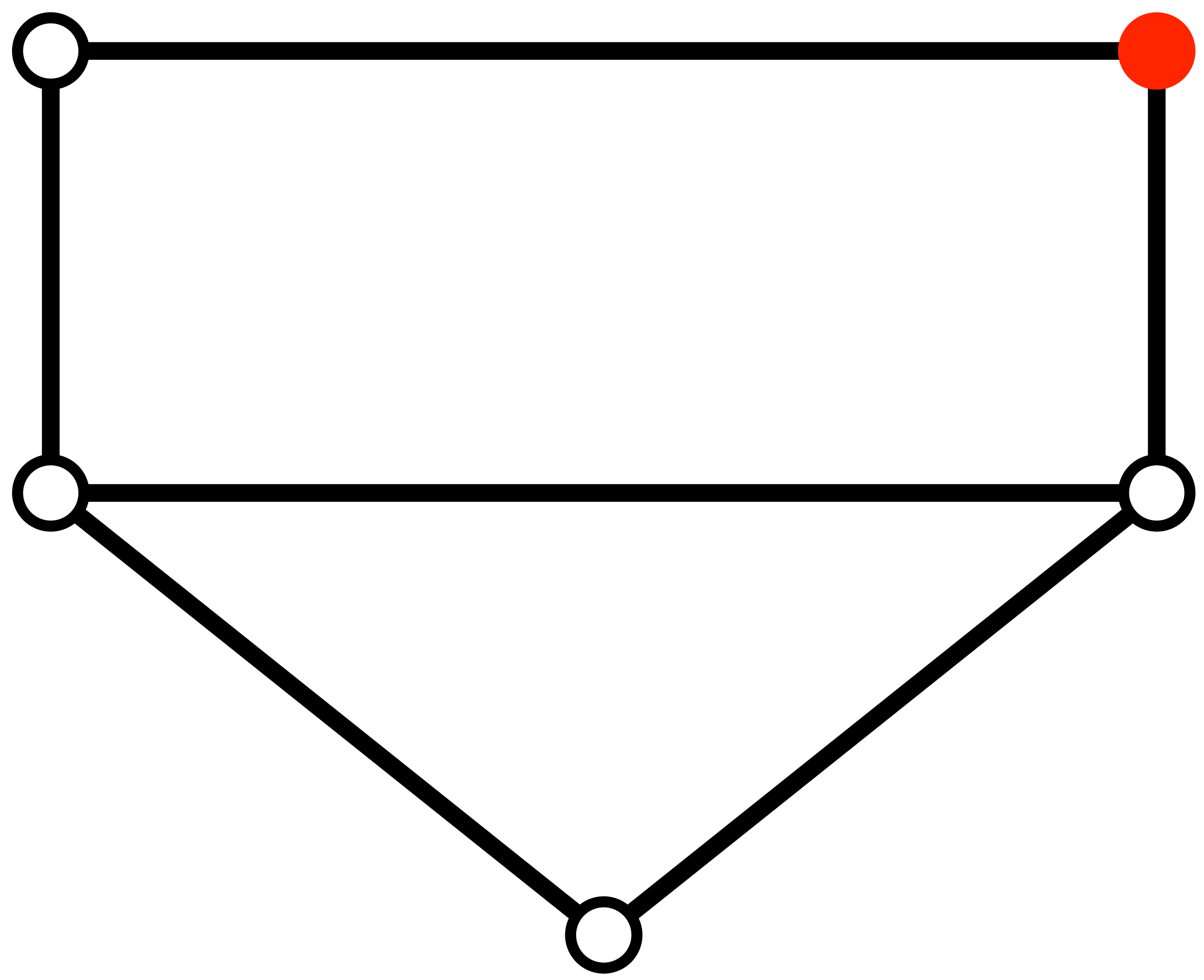
[Stefankovic-Vempala-Vigoda '09], [Huber'15], [Kolmogorov'18]

Glauber Dynamics / Gibbs Sampling

Goal: sample $\mu = \mathrm{uniform}(\Omega)$

From X_t :

1. Pick a vertex $v \in V$ u.a.r.
2. If $X_t \cap N(v) \neq \emptyset$, then $X_{t+1} = X_t$
3. Otherwise, $X_{t+1} = \begin{cases} X_t \cup \{v\}, & \text{with prob. } \frac{1}{2} \\ X_t \backslash \{v\}, & \text{with prob. } \frac{1}{2} \end{cases}$



Mixing time: $\max_{X_0} \min \left\{ t : \|X_t - \mu\|_{TV} \leq 1/4 \right\}$

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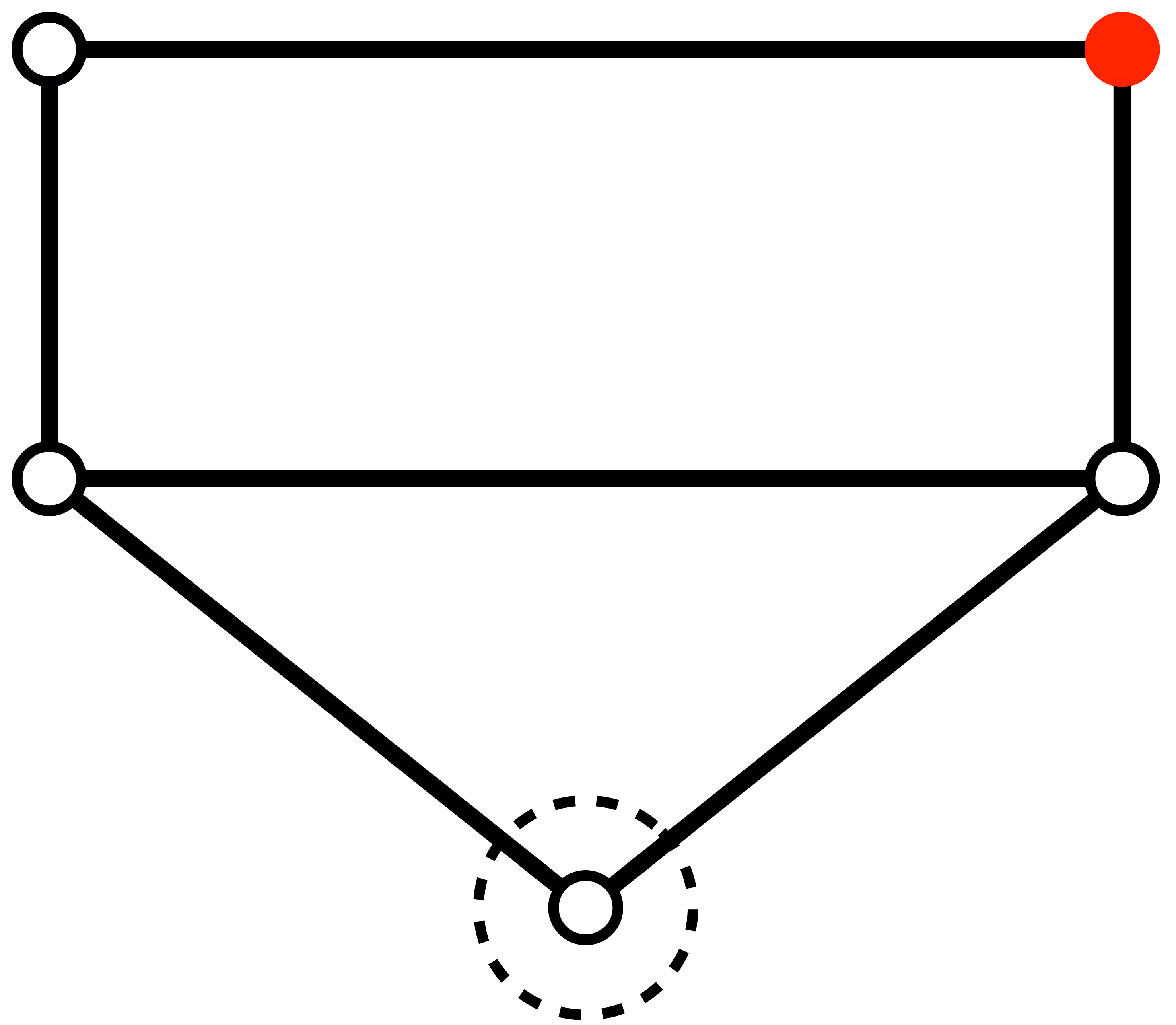

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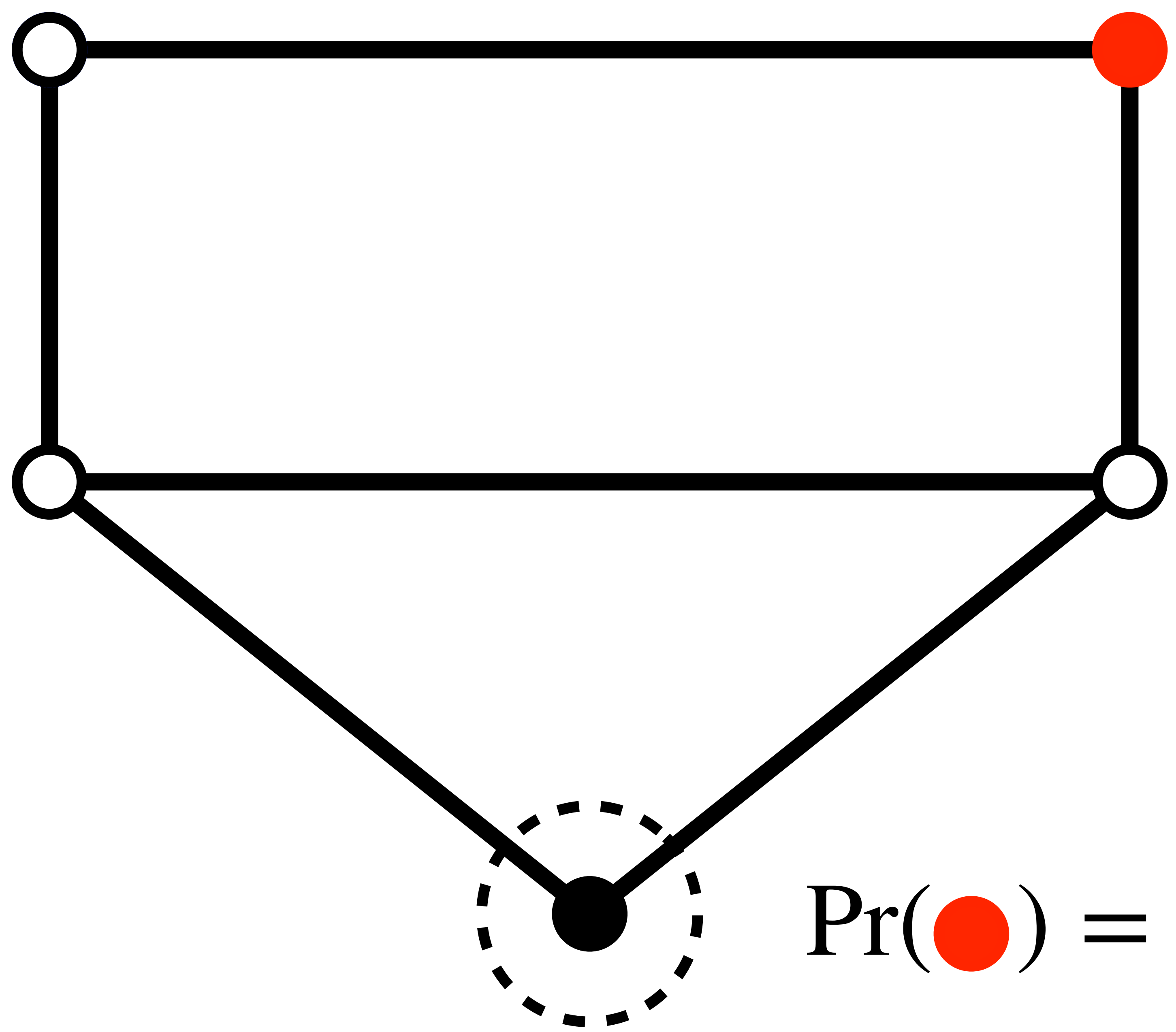

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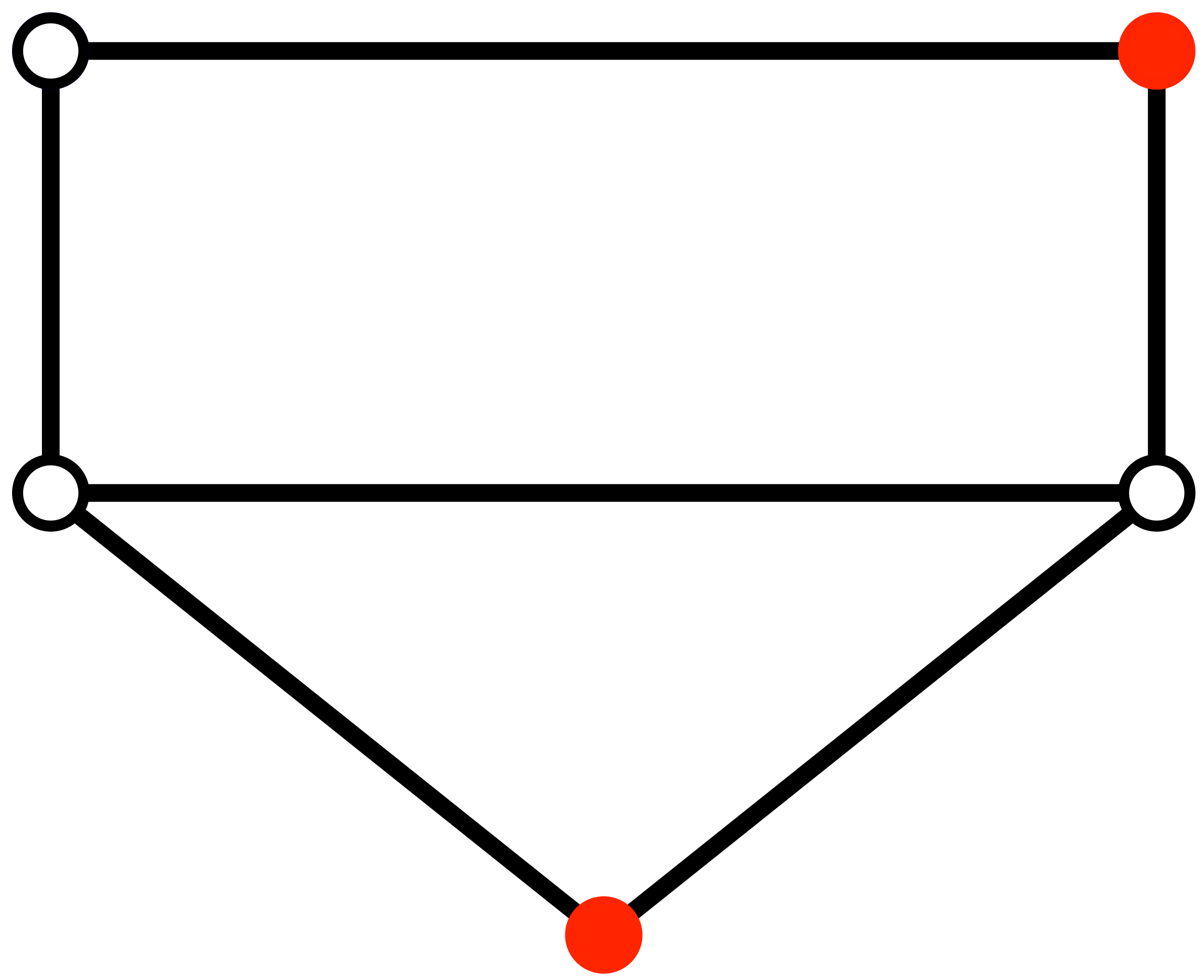

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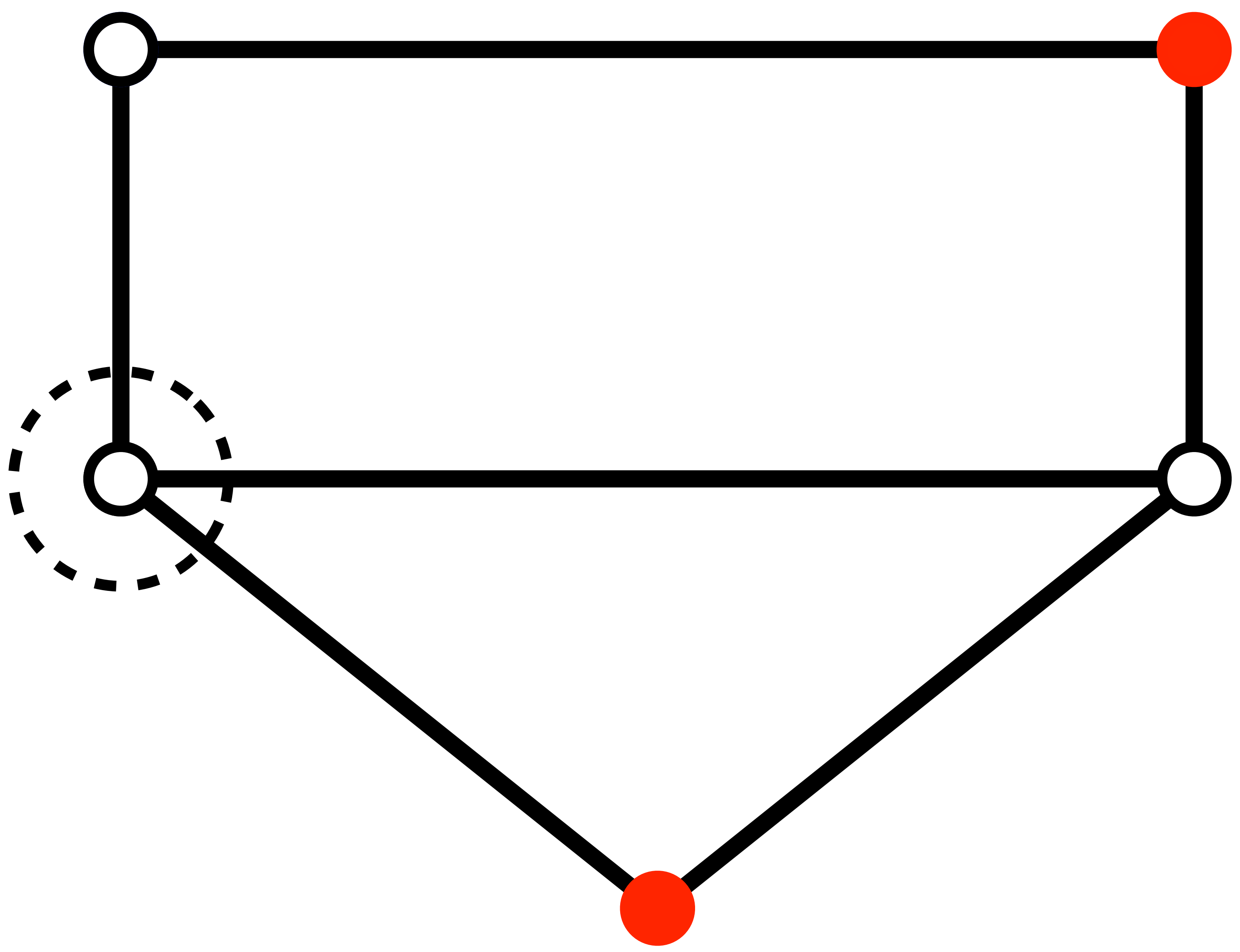

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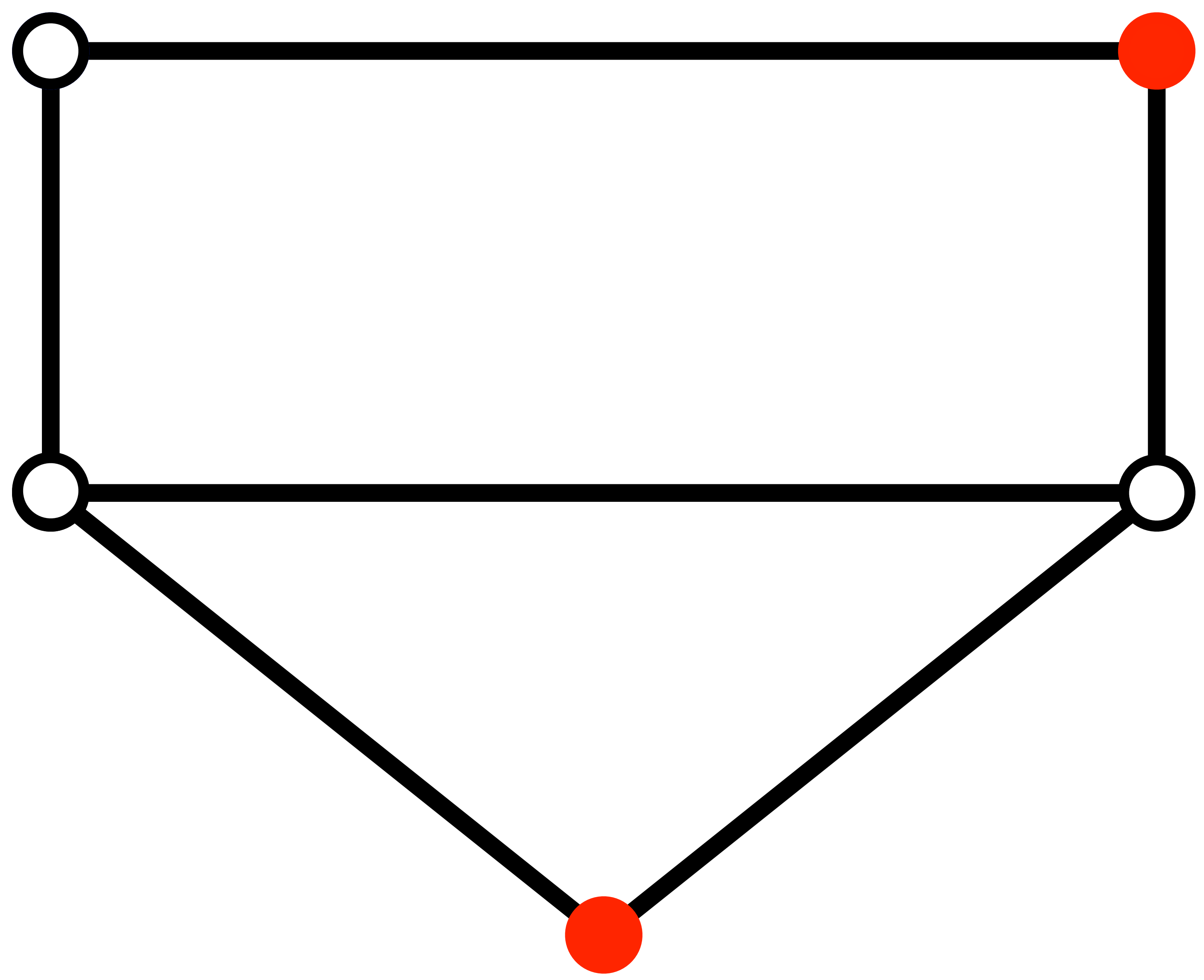

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Approx Counting Complexity

Approx counting # of IS's and sampling uniform(Ω)

- Input: For ***any*** $G = (V, E)$ with maximum degree Δ
  - $\Delta \leq 5$: $O(n \log n)$ mixing time for Glauber Dynamics. [CLV'21]

    $\implies O(n^2 \log^2 n)$ to approx $|\Omega|$

  - $\Delta \geq 6$: NP-hard to approx $|\Omega|$ within factor $\exp(Cn)$ some $C > 0$ [Sly'10]

    (even when restricted to *triangle-free* graphs of max deg. 6)

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    (even when restricted to *triangle-free* graphs of max deg. 6)

What happens between $5 \leftrightarrow 6$?

Hardcore (gas) Model

- Given $G = (V, E)$ and fugacity/activity $\lambda > 0$
- Ω = collection of all independent sets of G.

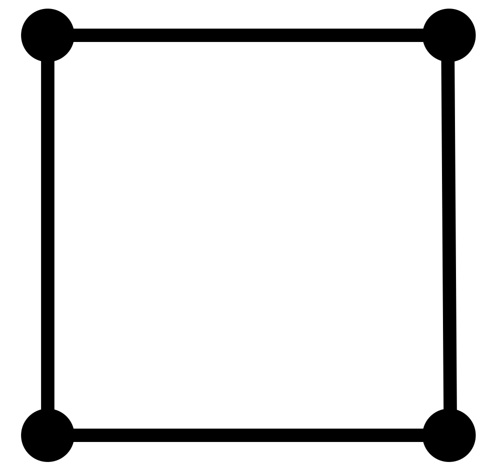
For $I \in \Omega$, $w(\sigma) = \lambda^{|I|}$

Gibbs distribution: $\mu(I) = \dfrac{w(I)}{Z_G(\lambda)}$

Partition function: $Z_G(\lambda) = \sum_{I \in \Omega} w(I)$

- $\lambda = 1$: $Z(1) = |\Omega|$ = number of IS's, and μ = uniform(Ω)

$Z(\lambda) = 1 + 4\lambda + 2\lambda^2$



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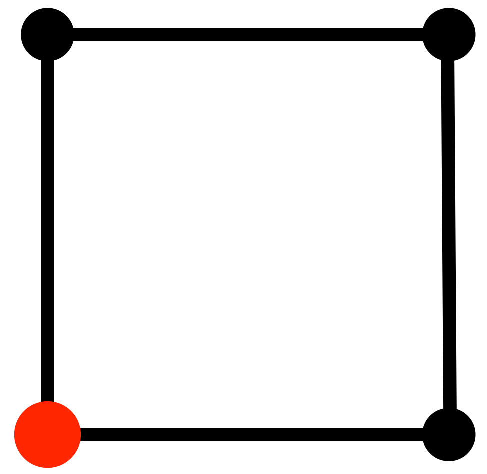
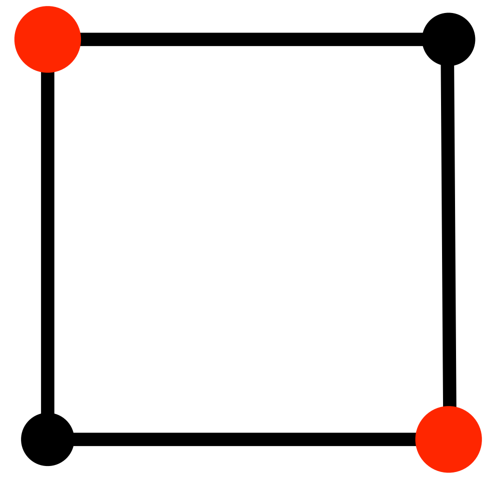


$\mu(I_1) = \lambda / Z$



$\mu(I_2) = \lambda^2 / Z$

Hardcore (gas) Model

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Gibbs distribution: $\mu(I) = \dfrac{w(I)}{Z_G(\lambda)}$

- Easy for small λ: see small IS's most of the time
- Hard for large λ: will see large IS's

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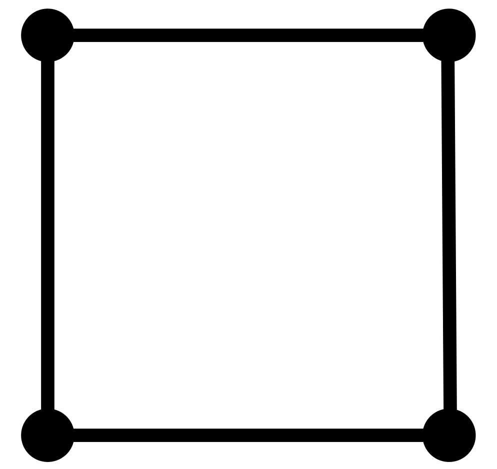
$Z(\lambda) = 1 + 4\lambda + 2\lambda^2$



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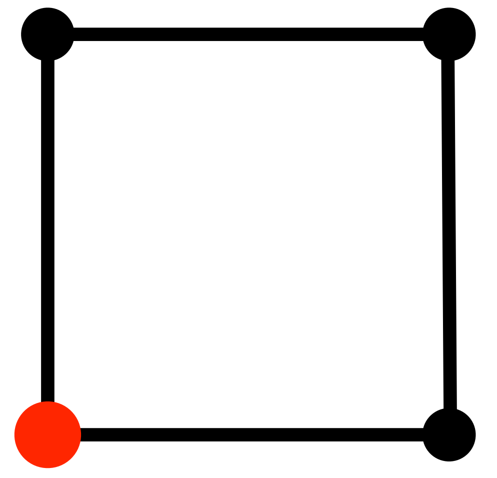

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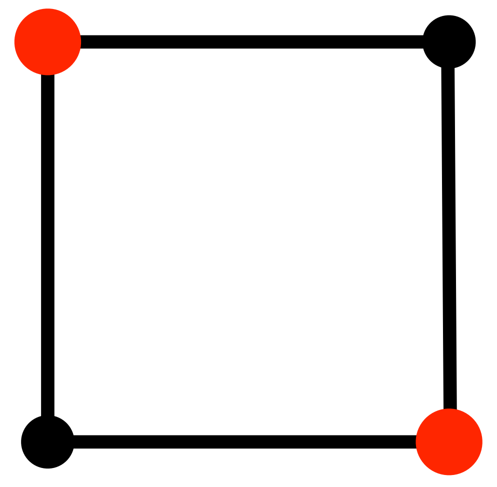

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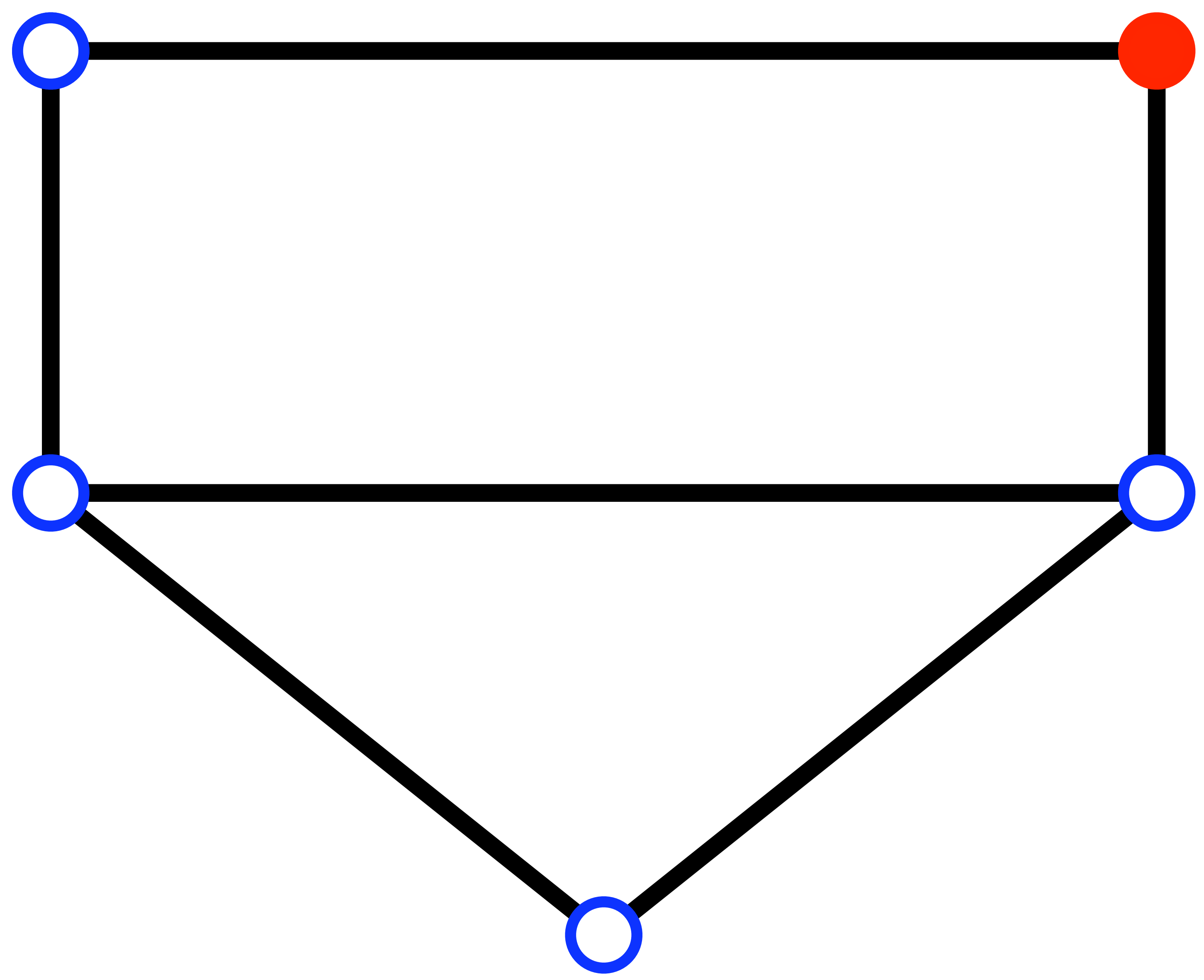
Glauber Dynamics / Gibbs Sampling

Goal: sample an IS approximately from μ (general λ)



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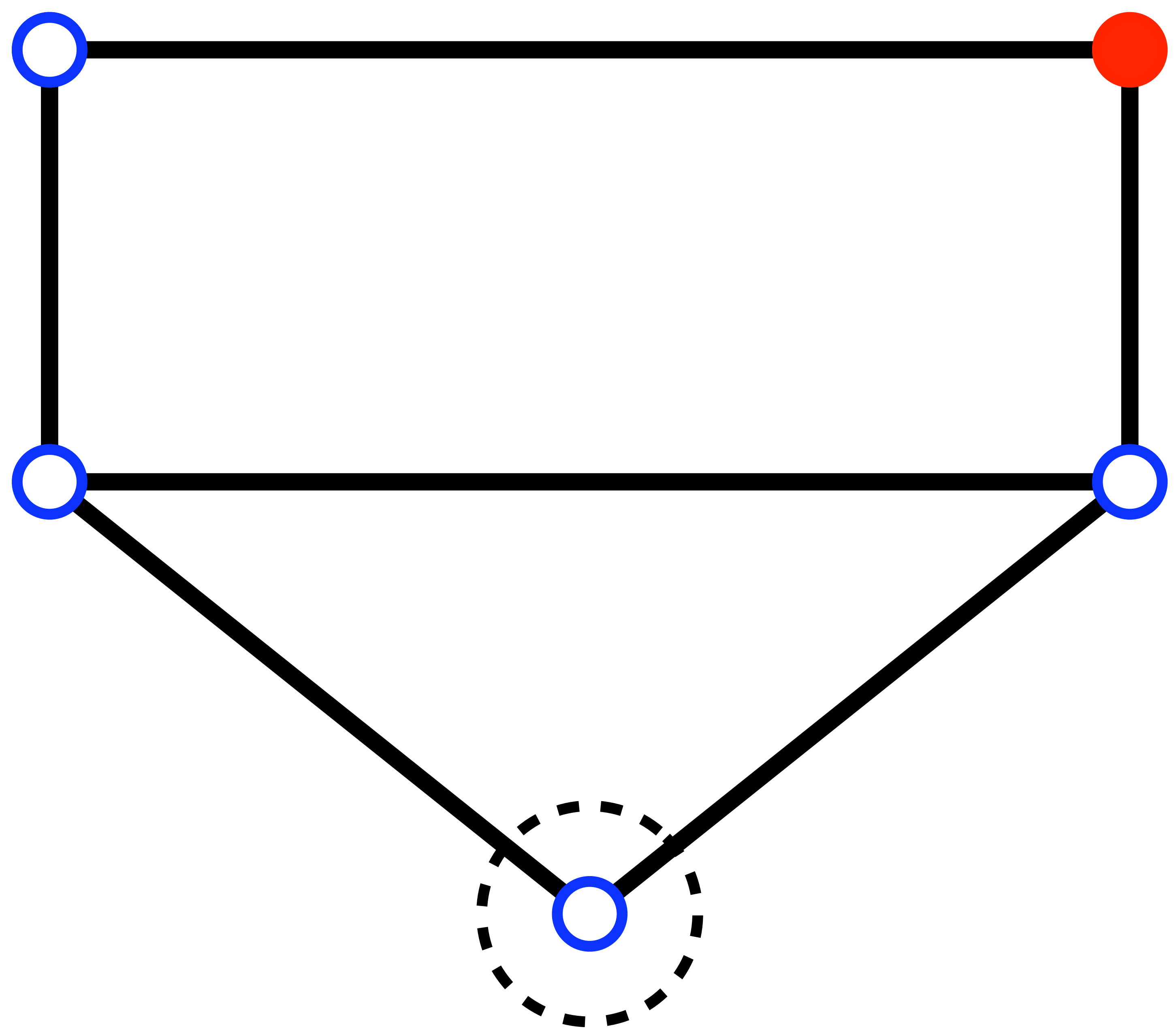

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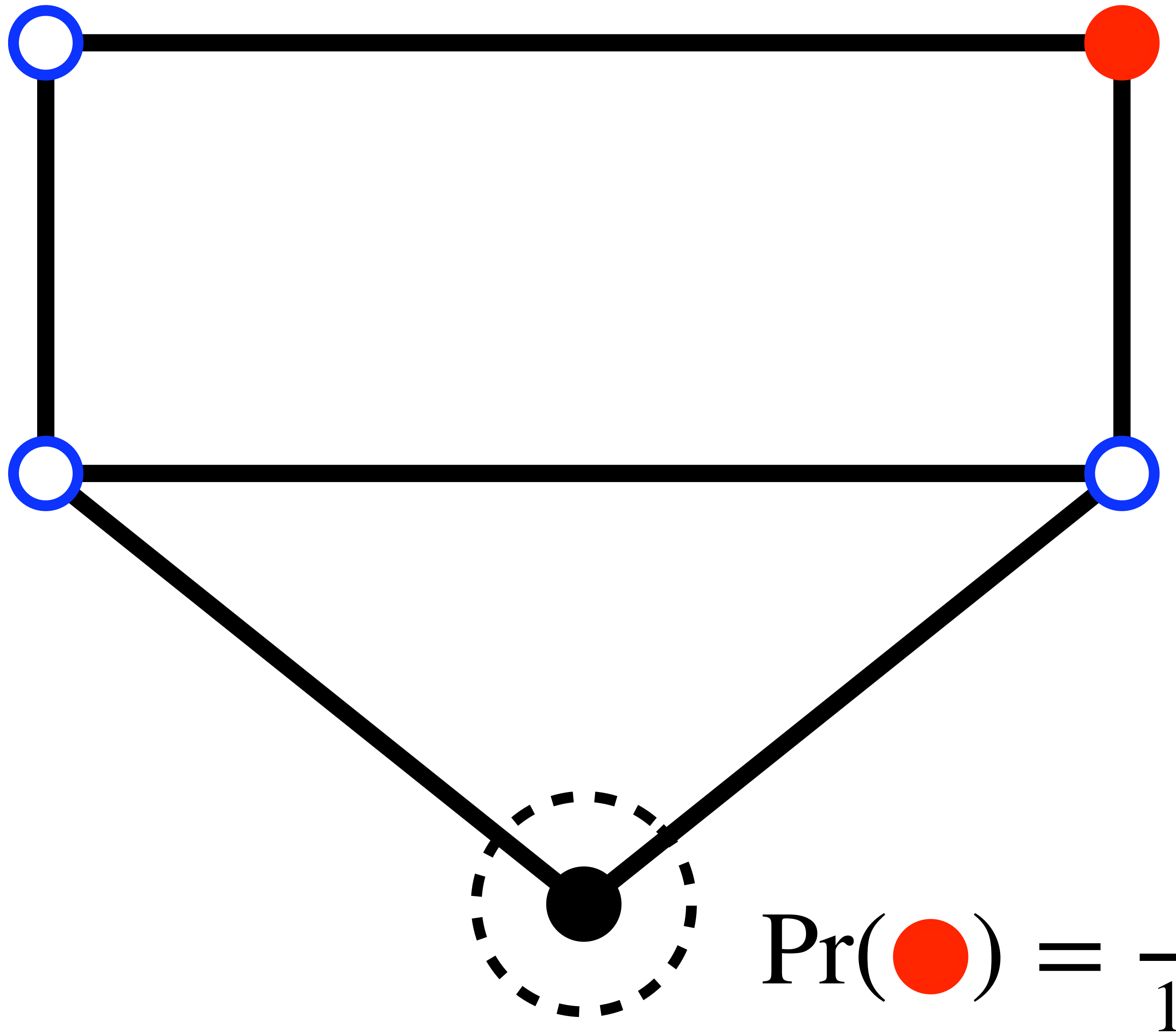

$\Pr(\bullet) = \dfrac{\lambda}{1+\lambda}$

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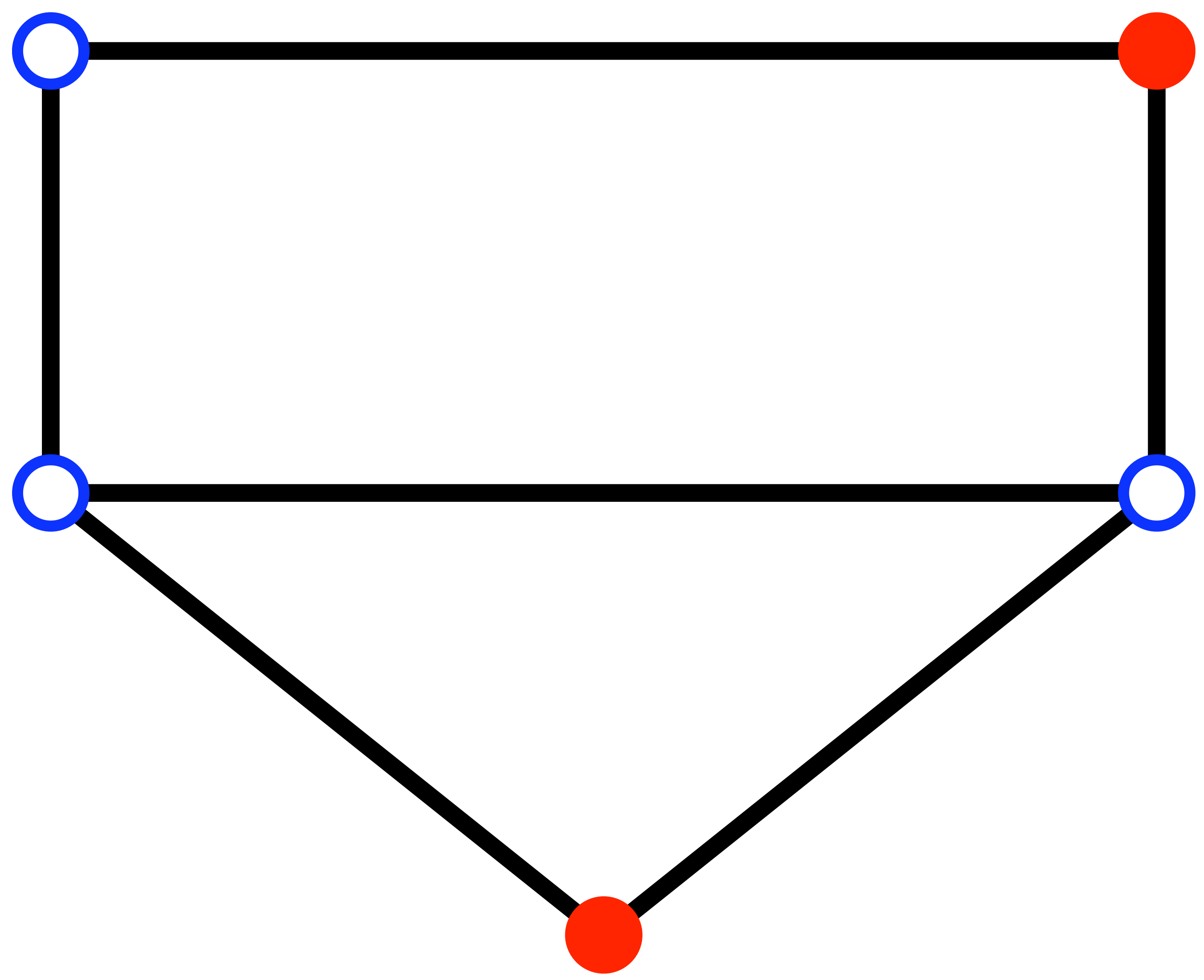

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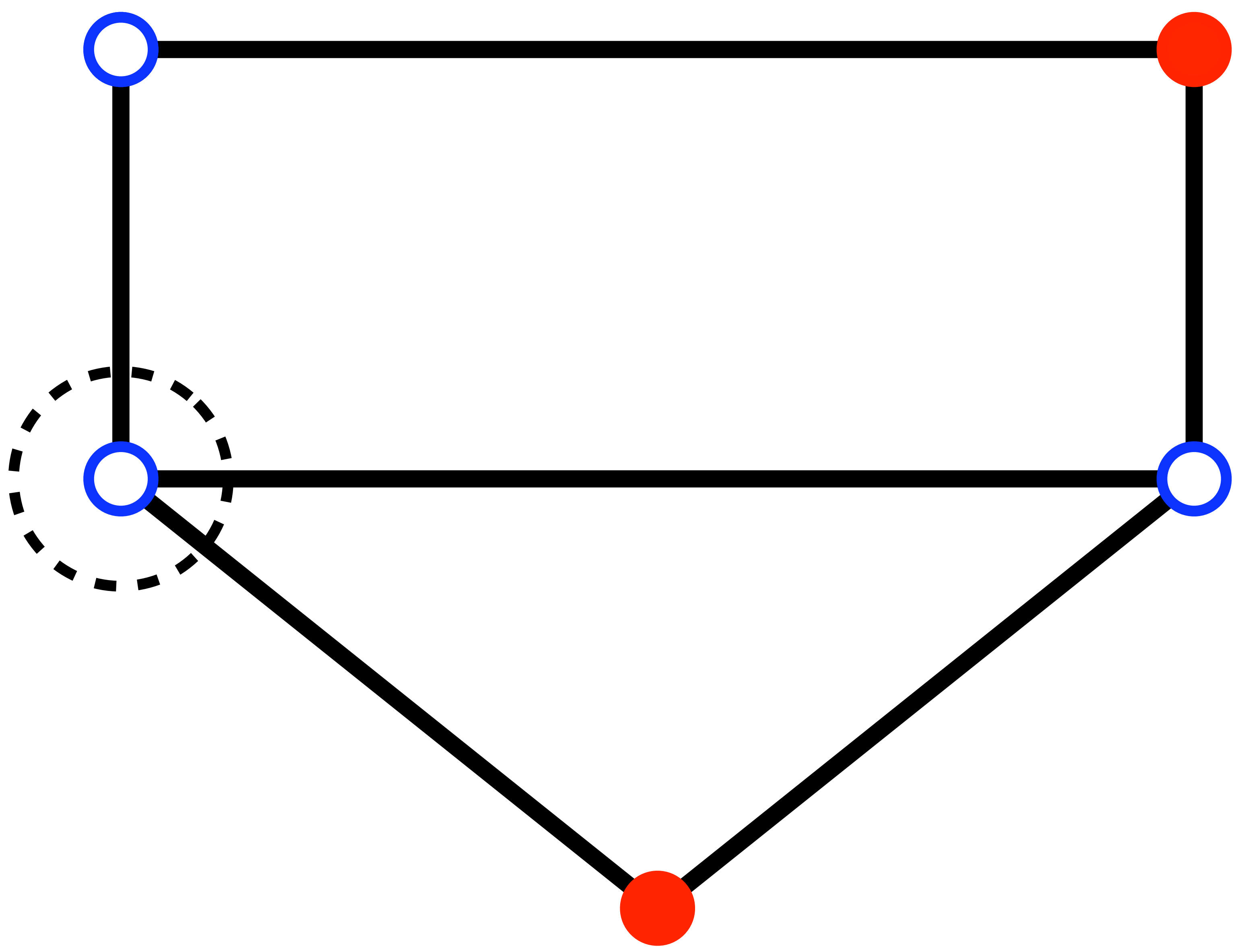

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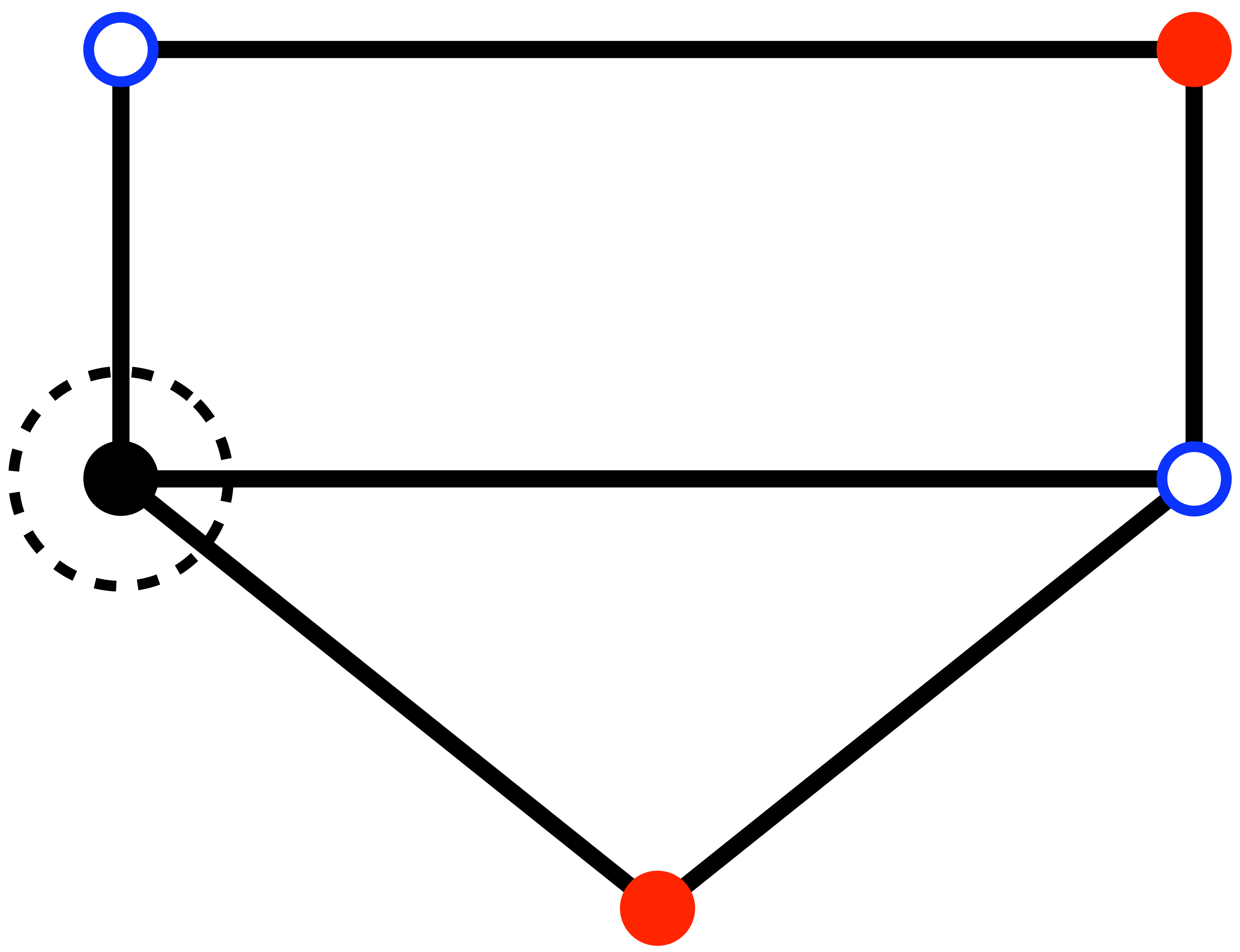

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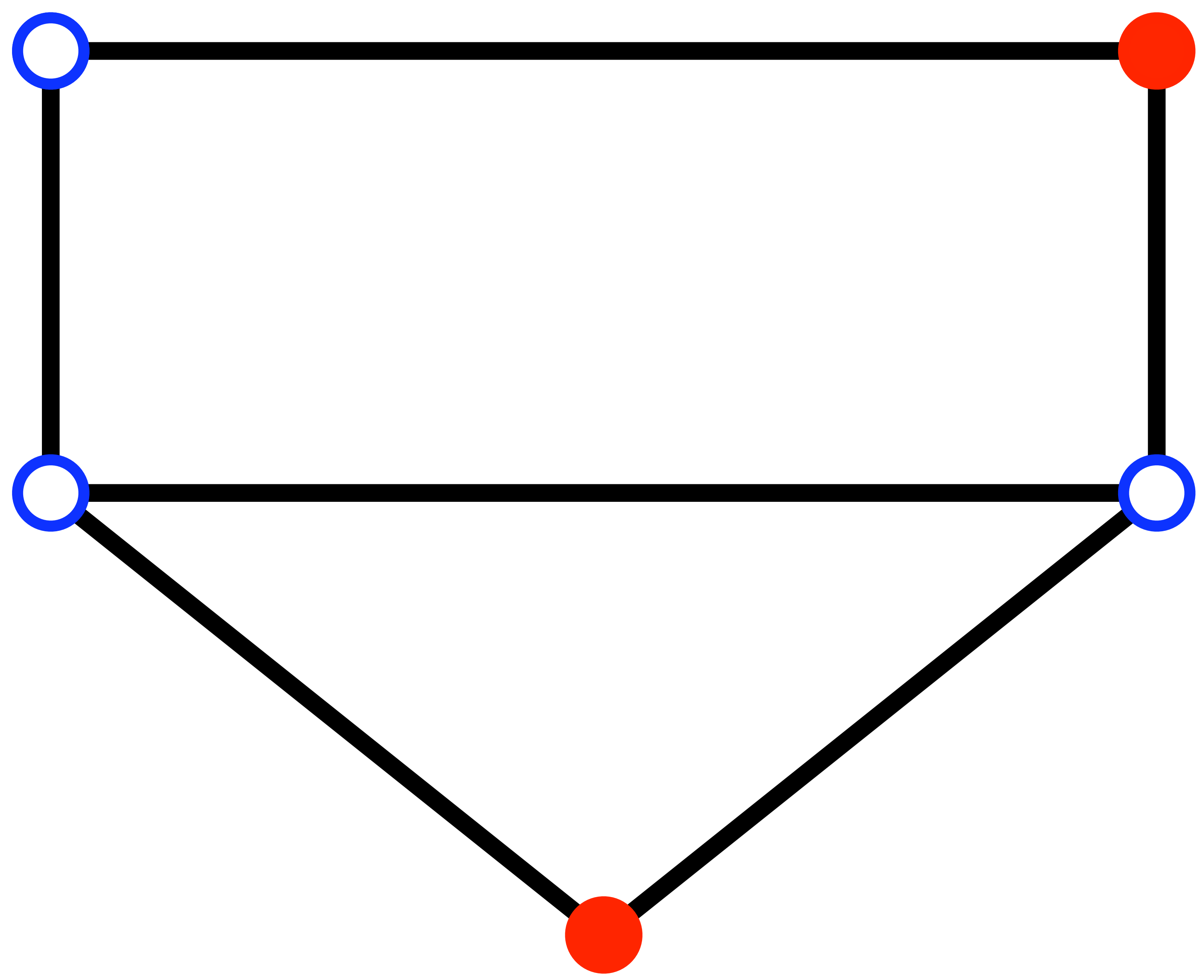

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$\lambda = 1$ case: Approx counting # of IS's and sampling uniform(Ω)

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    $\implies O(n^2 \log^2 n)$ FPRAS to approx $|\Omega|$
  - $\Delta \geq 6$: NP-hard to approx $|\Omega|$ within $\exp(Cn)$ some $C > 0$ [Sly'10]

What happens between $5 \leftrightarrow 6$?

Hard-core Model Complexity

Counting: compute $Z(\lambda) = \sum_{I \in \Omega} \lambda^{|I|}$; Sampling: sample from $\mu(I) = \frac{\lambda^{|I|}}{Z_G(\lambda)}$

- Input: For ***any*** $G = (V, E)$ with max deg Δ and $\lambda > 0$:
- For all constant Δ, there exists $\lambda_c = \lambda_c(\Delta)$:
  - All $\lambda < \lambda_c(\Delta)$: $O(n \log n)$ mixing time for Glauber Dynamics [CLV'21]
  - All $\lambda > \lambda_c(\Delta)$: NP-hard to approx $Z(\lambda)$ within $\exp(Cn)$

[Sly'10,Sly-Sun'14, Galanis-Štefankovič-V'16]

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What is $\lambda_c = \lambda_c(\Delta)$?

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- Input: For ***any*** $G = (V, E)$ with max deg Δ and $\lambda > 0$:

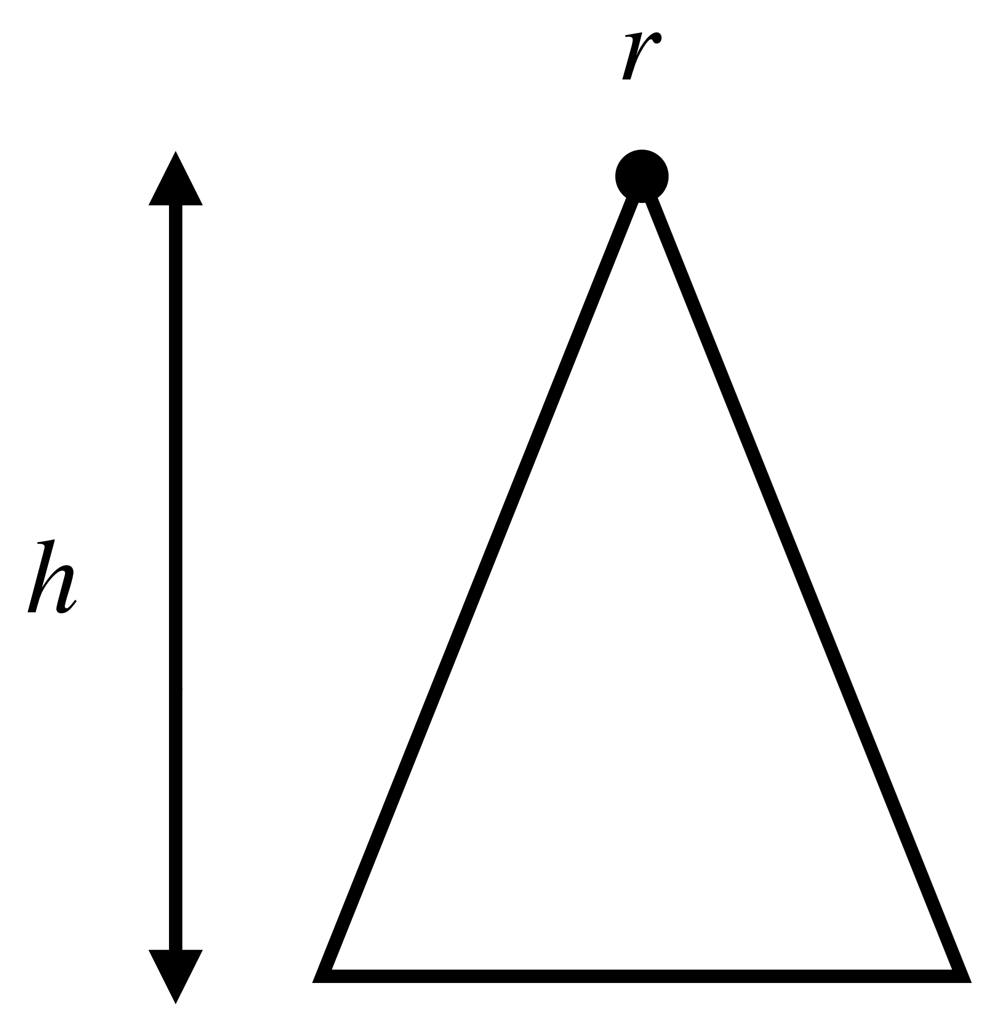
$$\lambda_c(5) = 4^4/3^5 \approx 1.053$$
$$\lambda_c(6) = 5^5/4^6 \approx 0.763$$

- For all constant Δ, there exists $\lambda_c = \lambda_c(\Delta)$: $\lambda_c(\Delta) = \frac{(\Delta - 1)^{\Delta - 1}}{(\Delta - 2)^{\Delta}} \sim \frac{e}{\Delta - 2}$
  - All $\lambda < \lambda_c(\Delta)$: $O(n \log n)$ mixing time for Glauber Dynamics [CLV'21]
  - All $\lambda > \lambda_c(\Delta)$: NP-hard to approx $Z(\lambda)$ within $\exp(Cn)$

[Sly'10,Sly-Sun'14, Galanis-Štefankovič-V'16]

What is $\lambda_c = \lambda_c(\Delta)$?

Phase Transition on Regular Trees



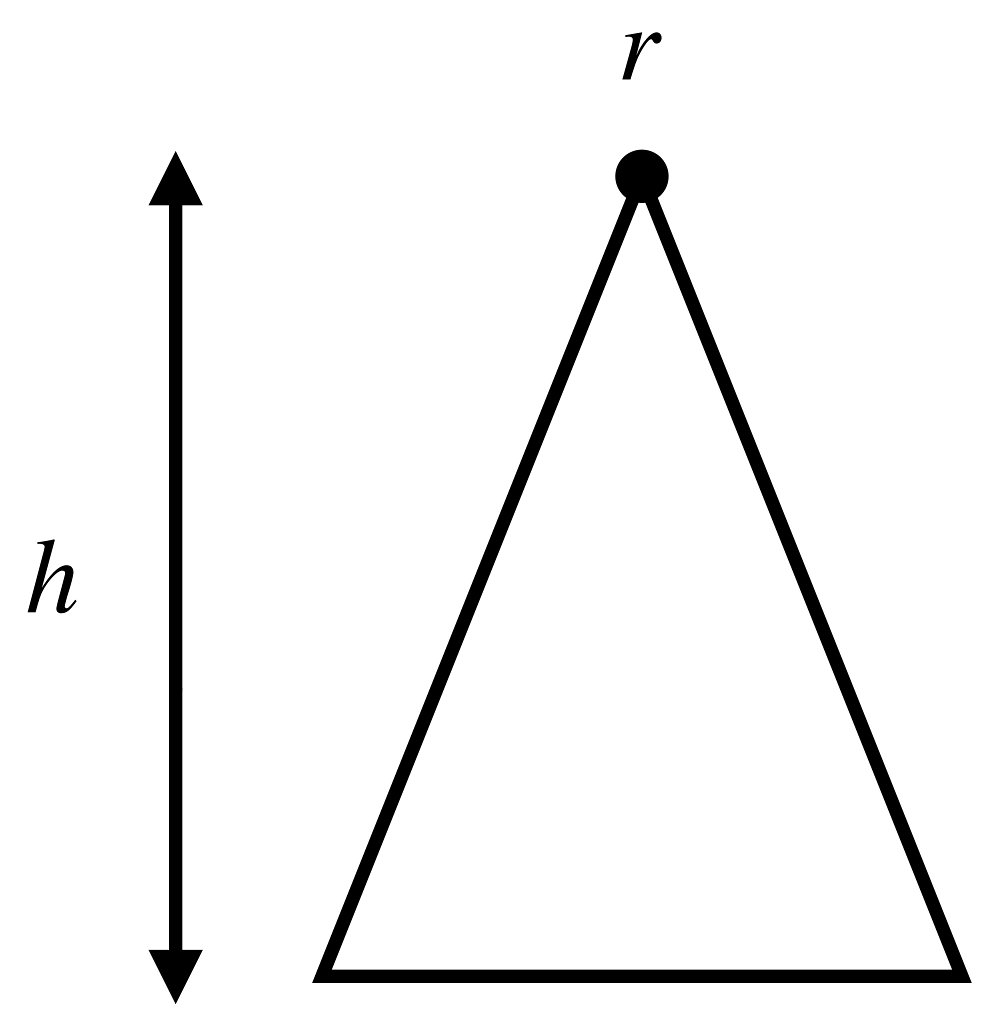
μ_h: Δ-regular tree of height h

Let μ_h = Gibbs measure for Δ-regular tree of height h.

Does configuration at leaves influence root?

Even height vs. odd height

Phase Transition on Regular Trees



μ_h: Δ-regular tree of height h

$$\lambda_c(\Delta) = \frac{(\Delta - 1)^{\Delta-1}}{(\Delta - 2)^{\Delta}}$$

Compare marginal at root for even vs odd height

$$\lambda < \lambda_c(\Delta) : \quad \lim_{h\to\infty} \mu_{2h}(r \in I) = \lim_{h\to\infty} \mu_{2h+1}(r \in I)$$

$$\lambda > \lambda_c(\Delta) : \quad \lim_{h\to\infty} \mu_{2h}(r \in I) < \lim_{h\to\infty} \mu_{2h+1}(r \in I)$$

Computational Phase Transition

Theorem (Hardcore Model) [Chen-Liu-V'21]:
For every $\delta \in (0,1)$ and $\lambda \leq (1-\delta)\lambda_c(\Delta)$, for any G of max degree $\leq \Delta$,
Glauber dynamics mixes in $\leq C(\Delta, \delta) \times n \log n$ steps.

$O(n \log n)$ mixing time of Glauber on any graph of max degree Δ
[Chen-Liu-V'21]

NP-hard to approx partition function within $\exp(\epsilon n)$ factor
[Sly-Sun'14, Galanis-Štefankovič-V'16]

0 $\quad$ $\lambda < \lambda_c(\Delta)$ $\quad$ $\lambda_c(\Delta)$ $\quad$ $\lambda > \lambda_c(\Delta)$

Root-leaf correlations decay for infinite Δ-regular tree $\mathbb{T}_\Delta$

Root-leaf correlations persist on $\mathbb{T}_\Delta$

$$\lambda_c(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}} \sim \frac{e}{\Delta-2}$$

Previous Results

- $\lambda \le (1-\delta)\lambda_c(\Delta)$: Exists poly-time approximate sampler for μ

Methods	Running Time	References
Correlation Decay	$n^{f(\delta)\times O(\log \Delta)}$	Weitz'06
Polynomial Interpolation	$O\left(n^{f(\Delta,\delta)}\right)$	Barvinok'16, Patel-Regts'17, Peters-Regts'19
Glauber Dynamics	$O\left(n^{f(\delta)}\right)$	Anari-Liu-Oveis Gharan'20
Glauber Dynamics	$O\left(n^{O(1/\delta)}\right)$	Z. Chen-Liu-Vigoda'20
Glauber Dynamics	$\le C(\Delta,\delta) n \log n$	Z. Chen-Liu-Vigoda'21
Glauber Dynamics	$\le C(\delta) n \log n$	X. Chen-Feng-Yin-Zhang'22, Y. Chen-Eldan'22
Glauber Dynamics	$\ge C'(\Delta) n \log n$	Hayes-Sinclair'07

- $\lambda > \lambda_c(\Delta)$: No poly-time approx sampler, assuming $\mathrm{RP} \neq \mathrm{NP}$ [Sly'10, Sly-Sun'14, Galanis-Štefankovič-V'16]

Contents

- Results for Hardcore Model and Computational Phase Transition
- Spectral Independence Technique
- Proof approach: SI implies fast mixing of Glauber
- Conclusion

Spectral Independence

- The (pairwise) influence of vertex u on vertex v is

$$\Psi(u,v) = \begin{cases} \mu(v \in I \mid u \in I) - \mu(v \in I \mid u \notin I), & u \neq v \\ 0, & u = v \end{cases}$$

- μ is α-spectrally independent if for any induced subgraph H: $\quad \lambda_1(\Psi_H) \leq \alpha$

$\uparrow$ Maximum Eigenvalue

[Anari-Liu-Oveis Gharan'20]

Ψ is not symmetric and can have negative entries,
but all eigenvalues real and non-negative.

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Maximum Eigenvalue

[Anari-Liu-Oveis Gharan'20]

- Small $\eta \implies$ Vertices are nearly independent

  empty graph: $\Psi(u,v) = 0,\ \alpha_{\mathrm{SI}} = 0$

- Large $\eta \implies$ Vertices are strongly correlated

  biclique: $\Psi(u,v) \approx -1/2$ for $u \in L, v \in R,\ \alpha_{\mathrm{SI}} = \Theta(n)$

Ψ is not symmetric and can have negative entries,
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Spectral Independence

- *General setting*: for $G = (V, E)$, for μ on $\{0,1\}^V$:
- A pinning τ is an assignment $\tau : S \to \{0,1\}$ for some $S \subset V$.
- The (pairwise) influence of vertex w on vertex v is

$$\Psi_\tau(w, v) = \begin{cases} \mu_\tau(\sigma(v) = 1 \mid \sigma(w) = 1) - \mu_\tau(\sigma(v) = 0 \mid \sigma(w) = 0), & w \neq v \\ 0, & w = v \end{cases}$$

where μ_τ is the distribution μ conditional on partial assignment τ

- μ is α-spectrally independent if for any pinning τ: $\lambda_1(\Psi_\tau) \leq \alpha$ [Anari-Liu-Oveis Gharan'20]

Ψ is not symmetric and can have negative entries,
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Main Theorems [CLV'21]

Let μ be Gibbs dist. of spin system on $G = (V, E)$ of constant max degree $\leq \Delta$.

Theorem 1: For all Δ, all $\alpha > 0$, there exists $C(\Delta, \alpha)$.

If μ is α-spectrally independent, then Glauber Dynamics has relaxation time:

$$T_{\text{relax}} \leq C(\Delta, \alpha) \times n \text{ steps.}$$

Theorem 2: All $\Delta, \alpha > 0, b > 0$, there exists $C(\Delta, \alpha, b)$.

If α-spectrally independent and b-marginally bounded then mixing time:

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Previously, $n^{C(\alpha)}$ mixing. [Anari-Liu-Oveis Gharan'20]

Contents

- Introduction: Hardcore Model and Phase Transition
- Glauber Dynamics and Spectral Independence
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Goal: Proof Sketch [CLV'21]

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Step 1: Local to Global: Random Walk theorem [Alev-Lau'20]

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Step 1: Local to Global: Random Walk theorem [Alev-Lau'20]

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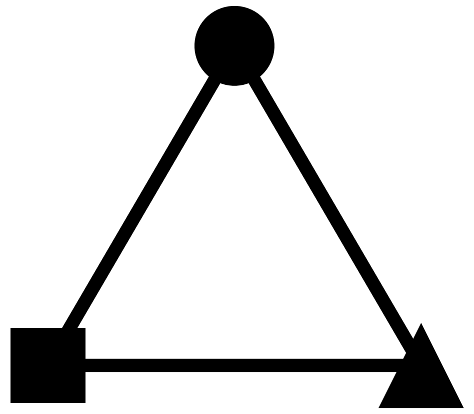
Step 3: $C(\alpha, \Delta) \times n$ relaxation time [Chen-Liu-V'21]

Glauber Dynamics: Alt. View



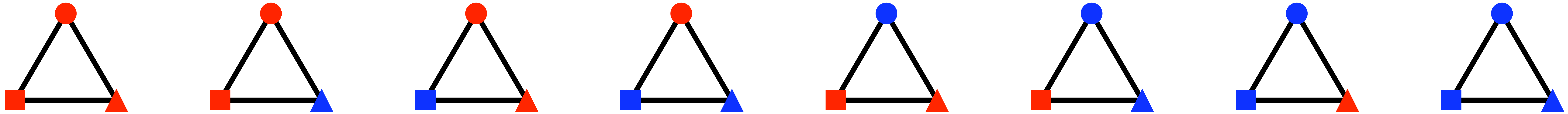
$G = (V, E)$

$Q = \{ \bullet , \bullet \}$

Ω_3: set of all full configurations

$\pi_3(V, \sigma) = \mu(\sigma), \forall \sigma : V \to \{ \bullet , \bullet \}$

[Anari-Liu-Oveis Gharan'20]



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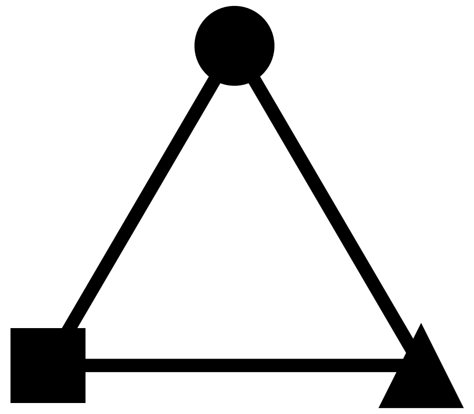

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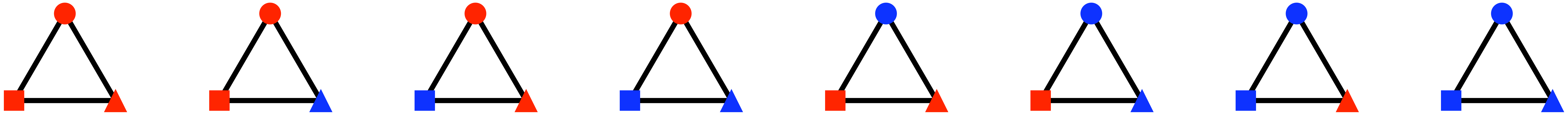

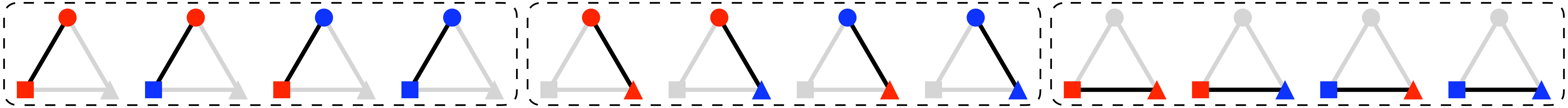

Ω_2: set of all partial configurations on two vertices

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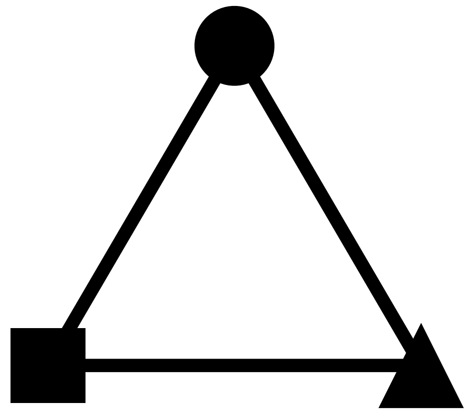

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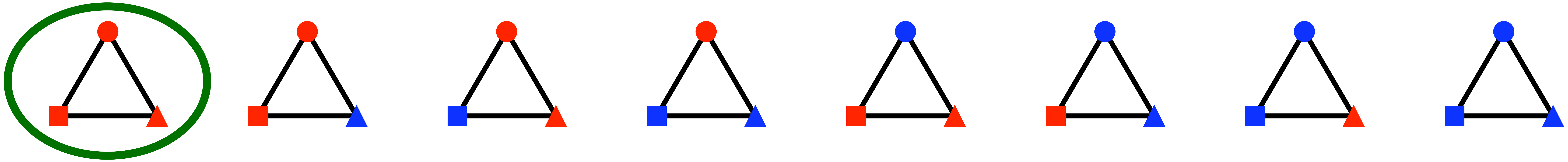

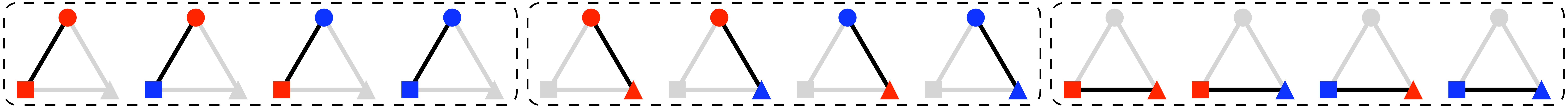

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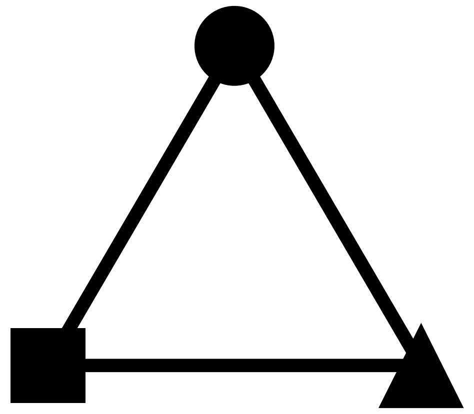

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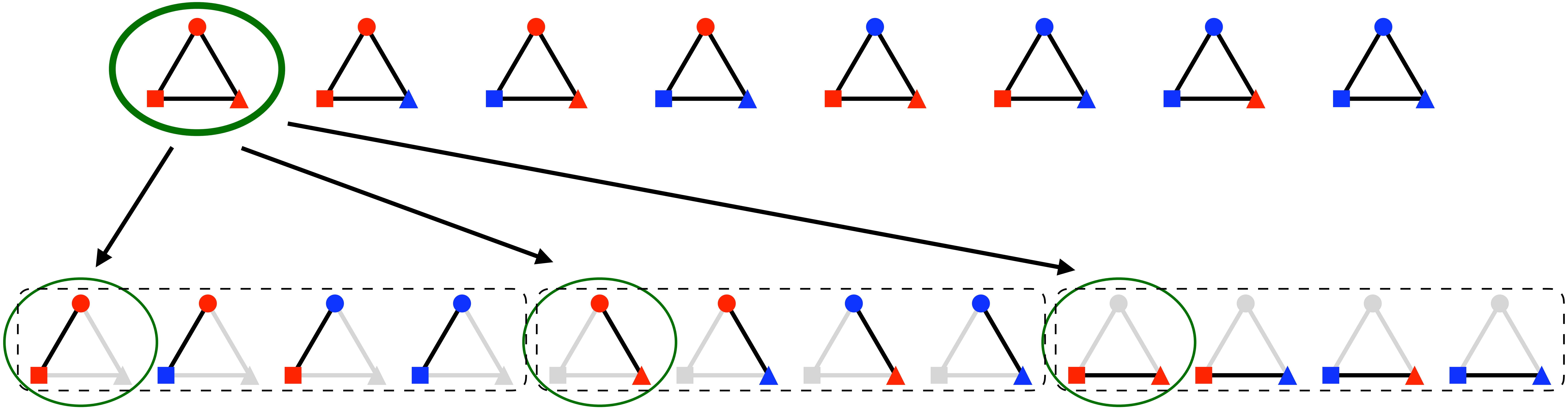

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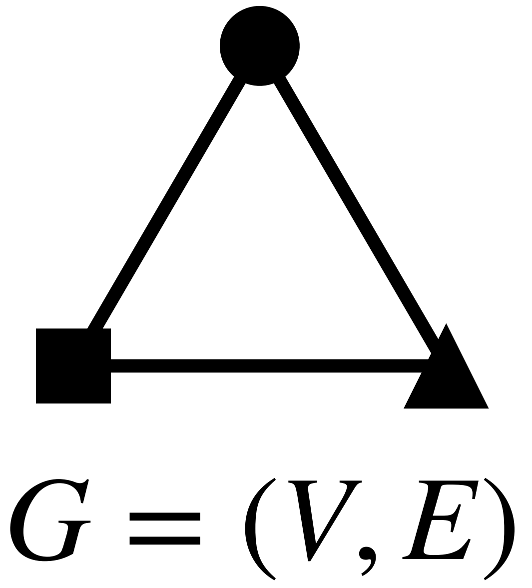

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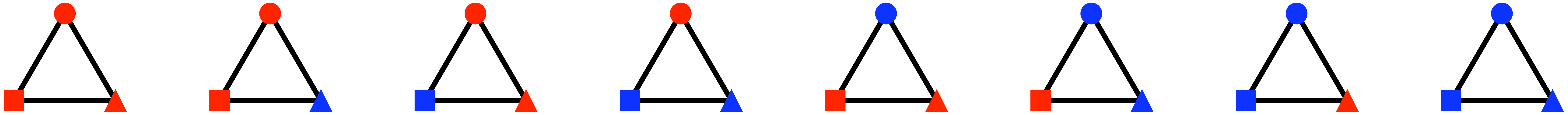

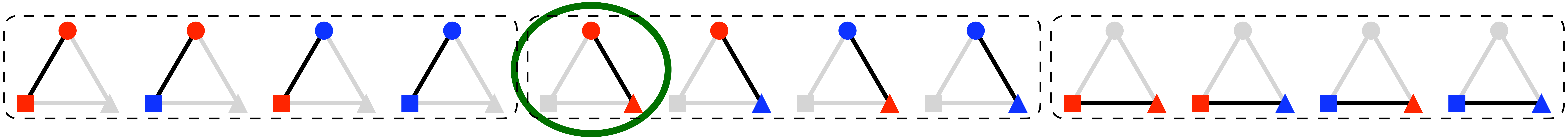

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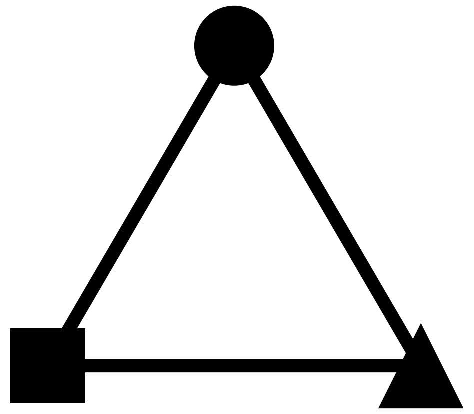

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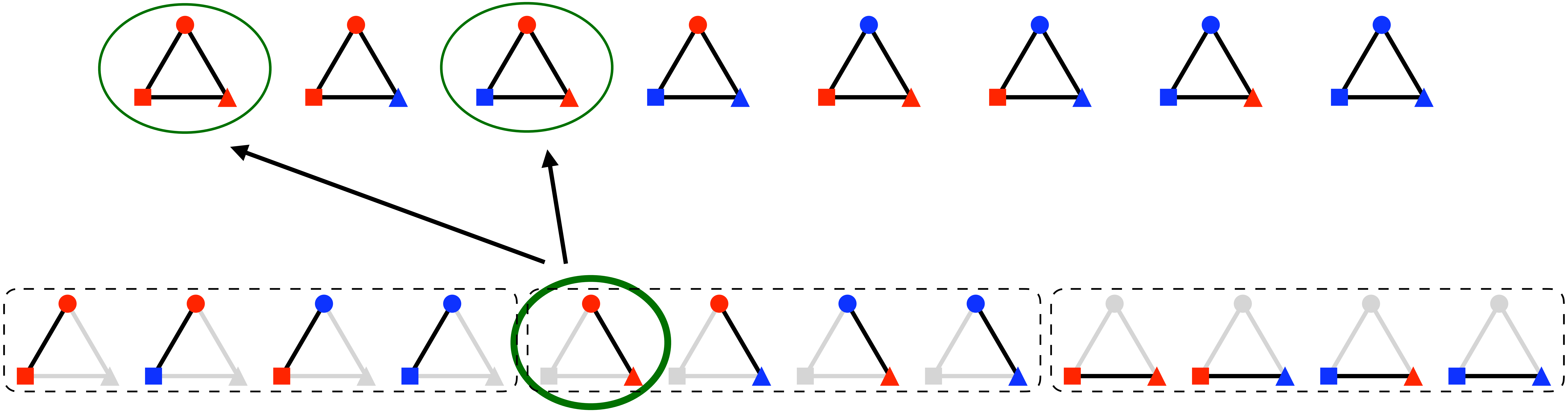

Ω_2: set of all <u>partial</u> configurations on <u>two</u> vertices

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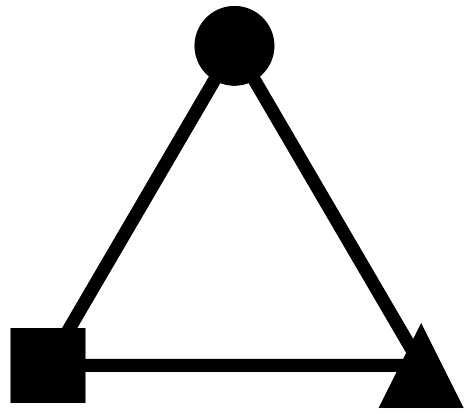

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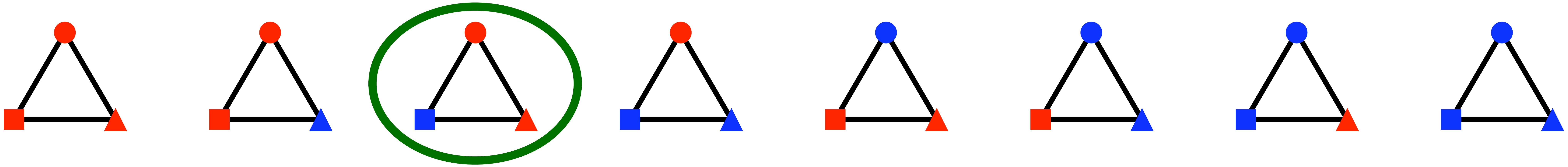

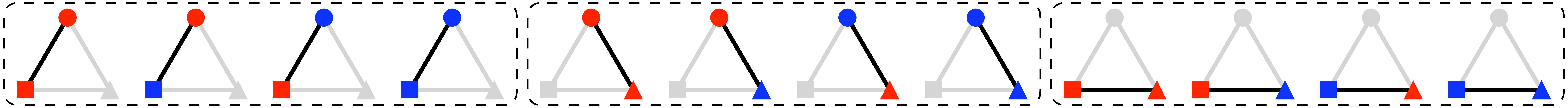

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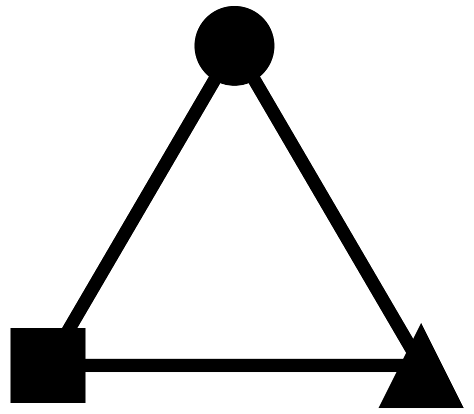

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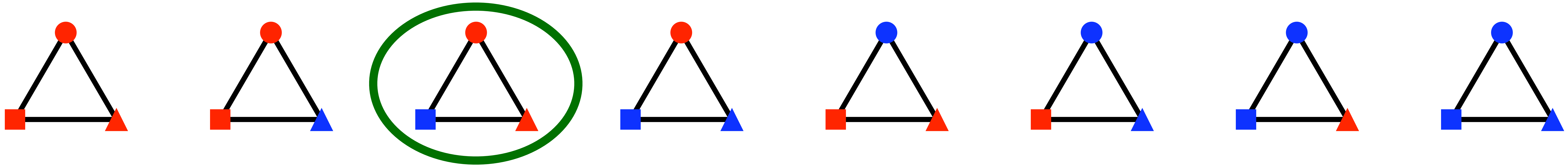
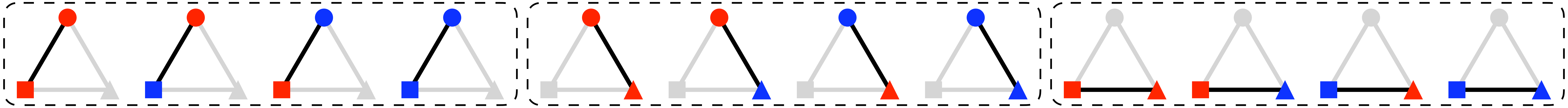


Glauber Dynamics is down-up walk $P_n^\vee$ from Ω_n to Ω_{n-1} to Ω_n



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Global Walks in Simplicial Complexes

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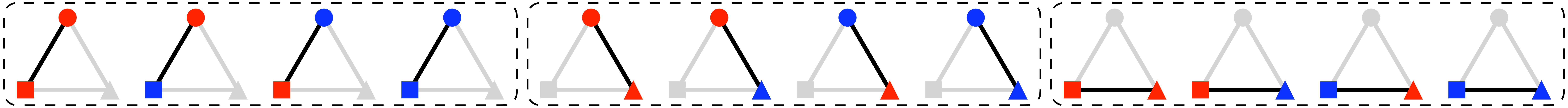

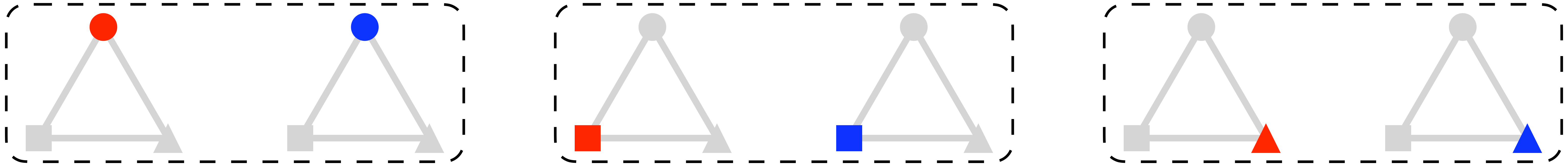

Ω_1: set of all partial configurations on one vertex

$$\pi_1(v, i) = \frac{1}{3}\mu(\sigma_v = i), \quad \forall v \in V, \, i \in \{ \textcolor{red}{\bullet}, \textcolor{blue}{\bullet} \}$$

Global Walks in Simplicial Complexes

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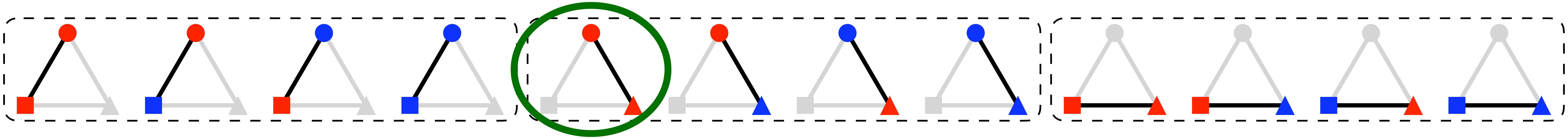

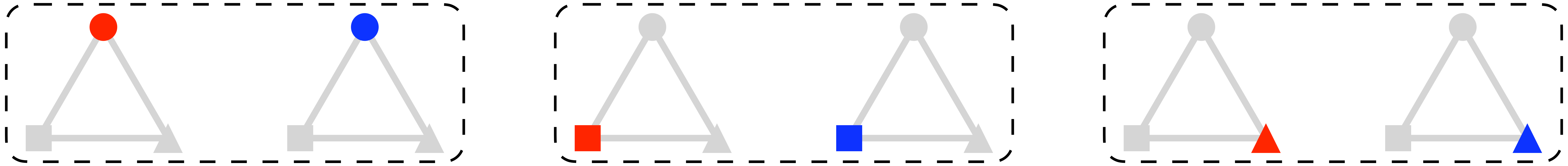

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Global Walks in Simplicial Complexes

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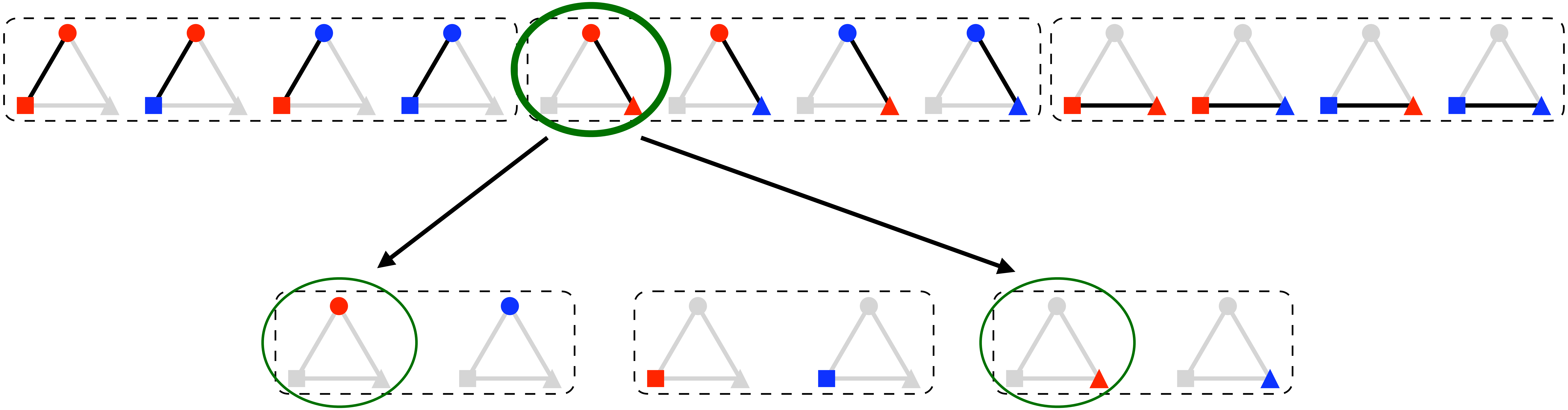

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Global Walks in Simplicial Complexes

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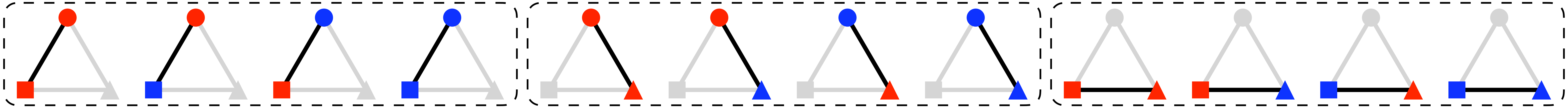

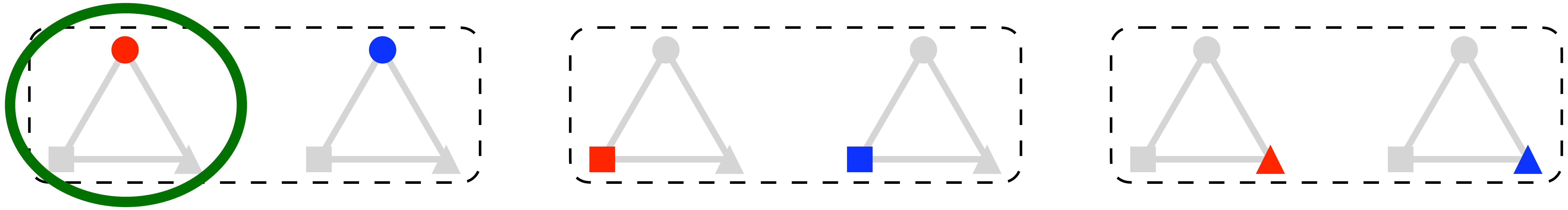

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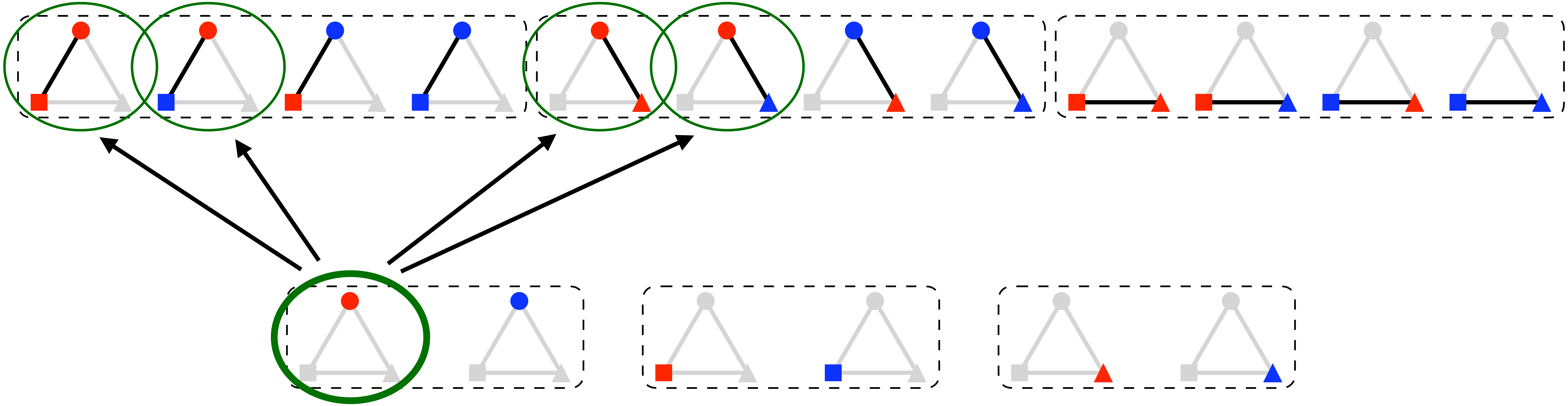

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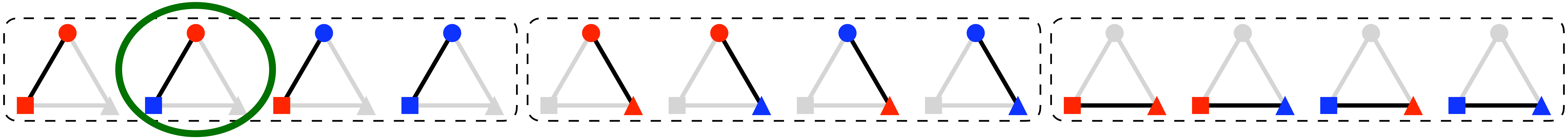

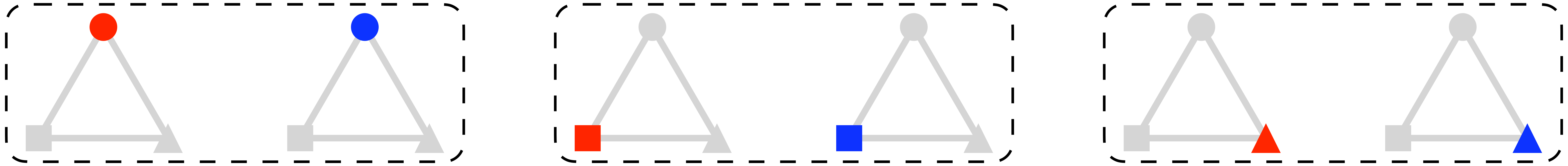

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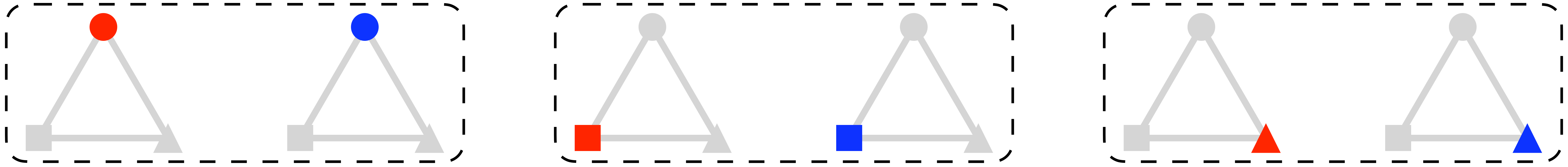

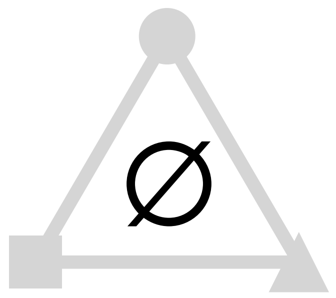

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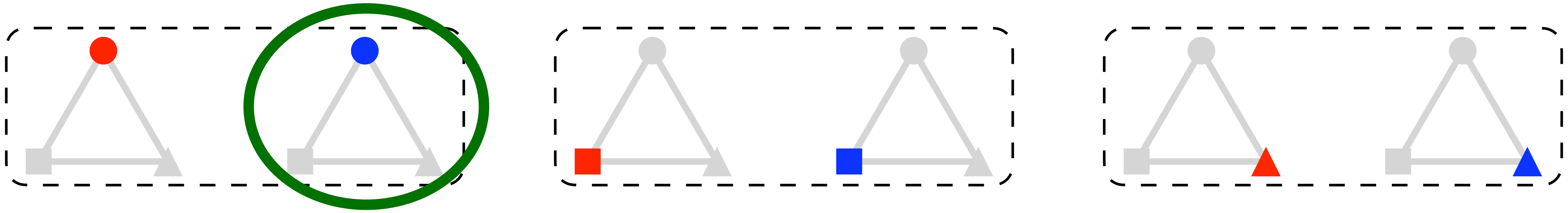

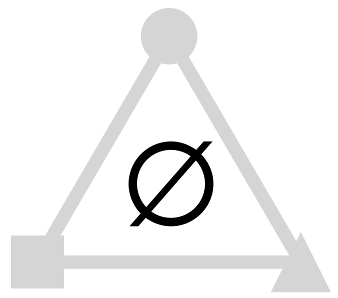

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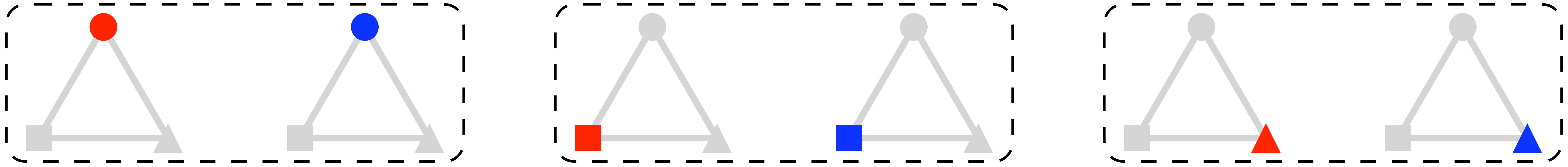

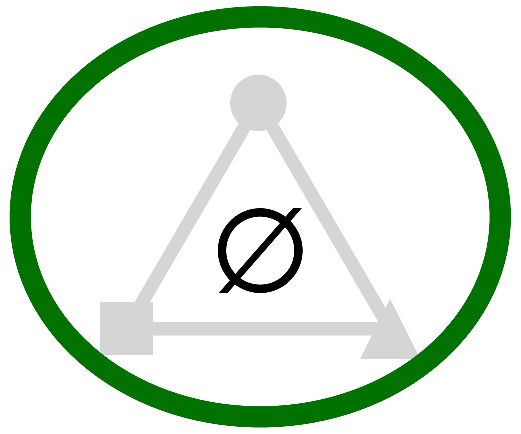

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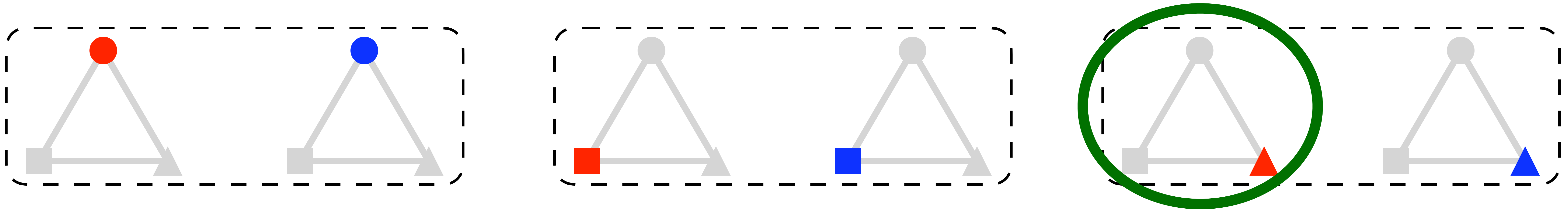

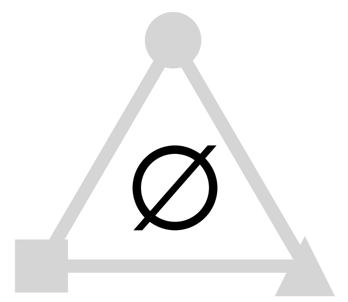

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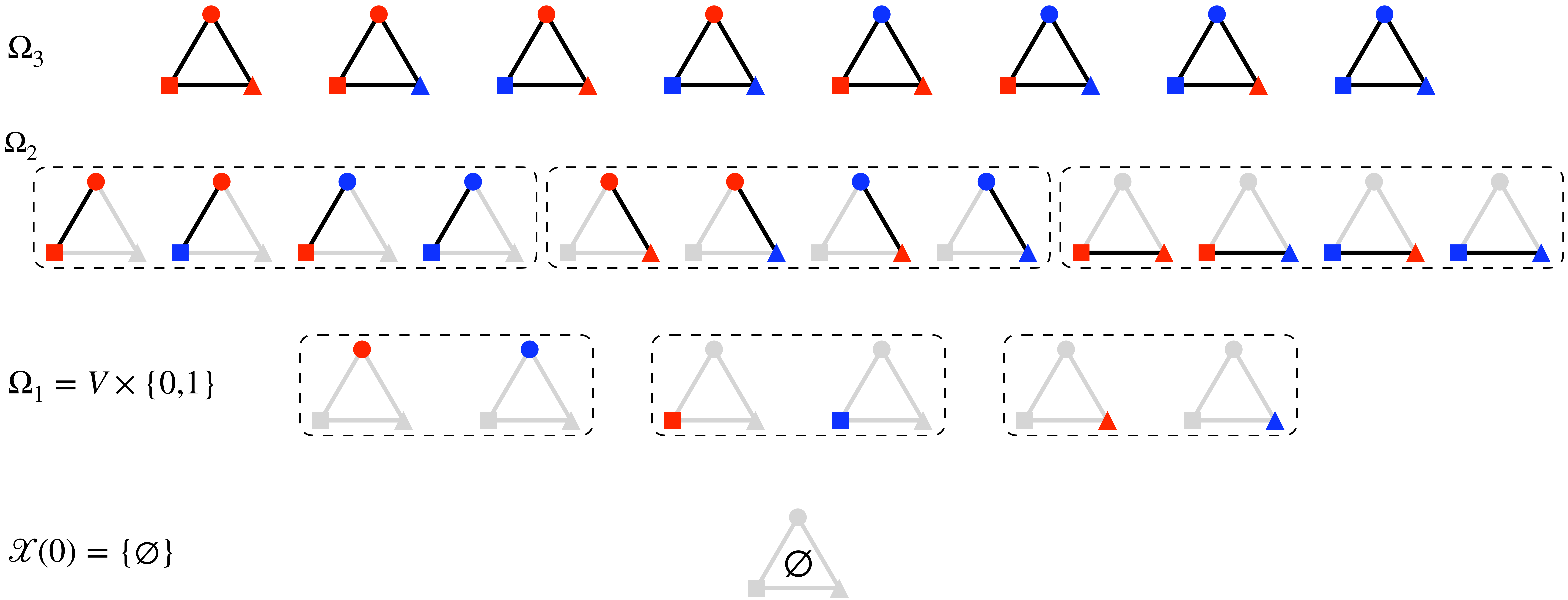

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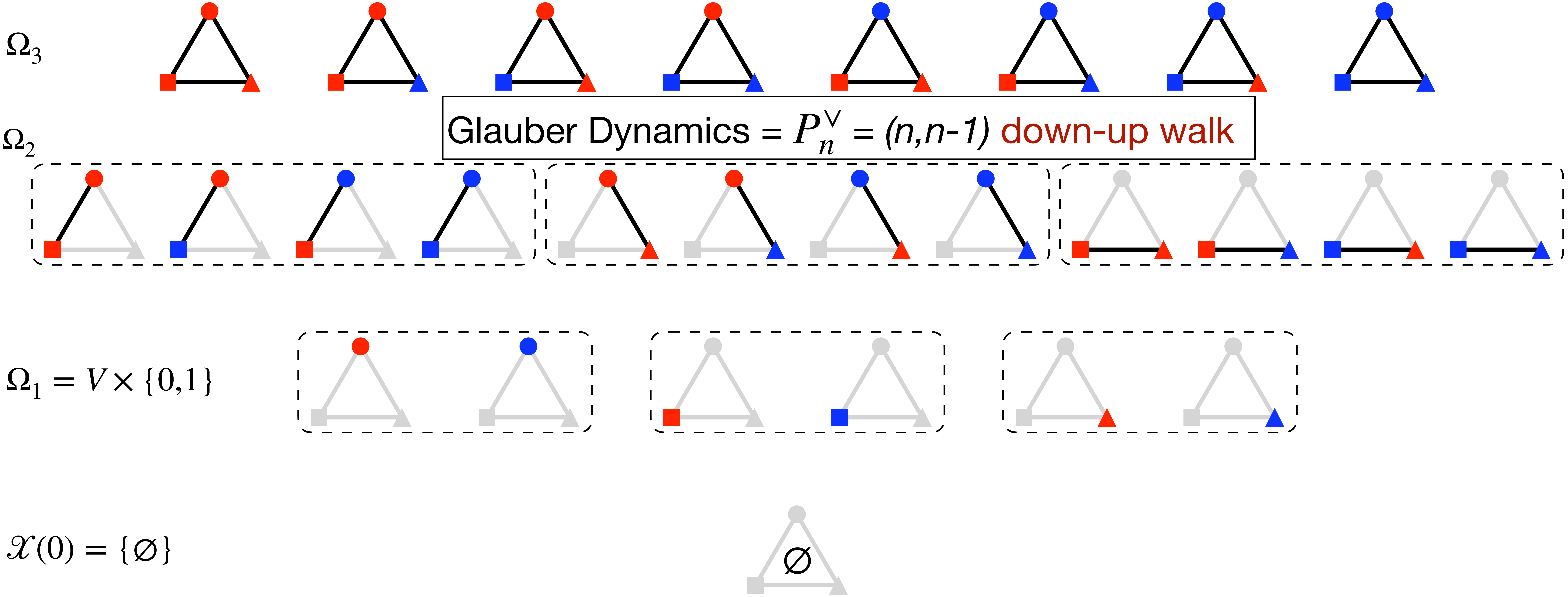

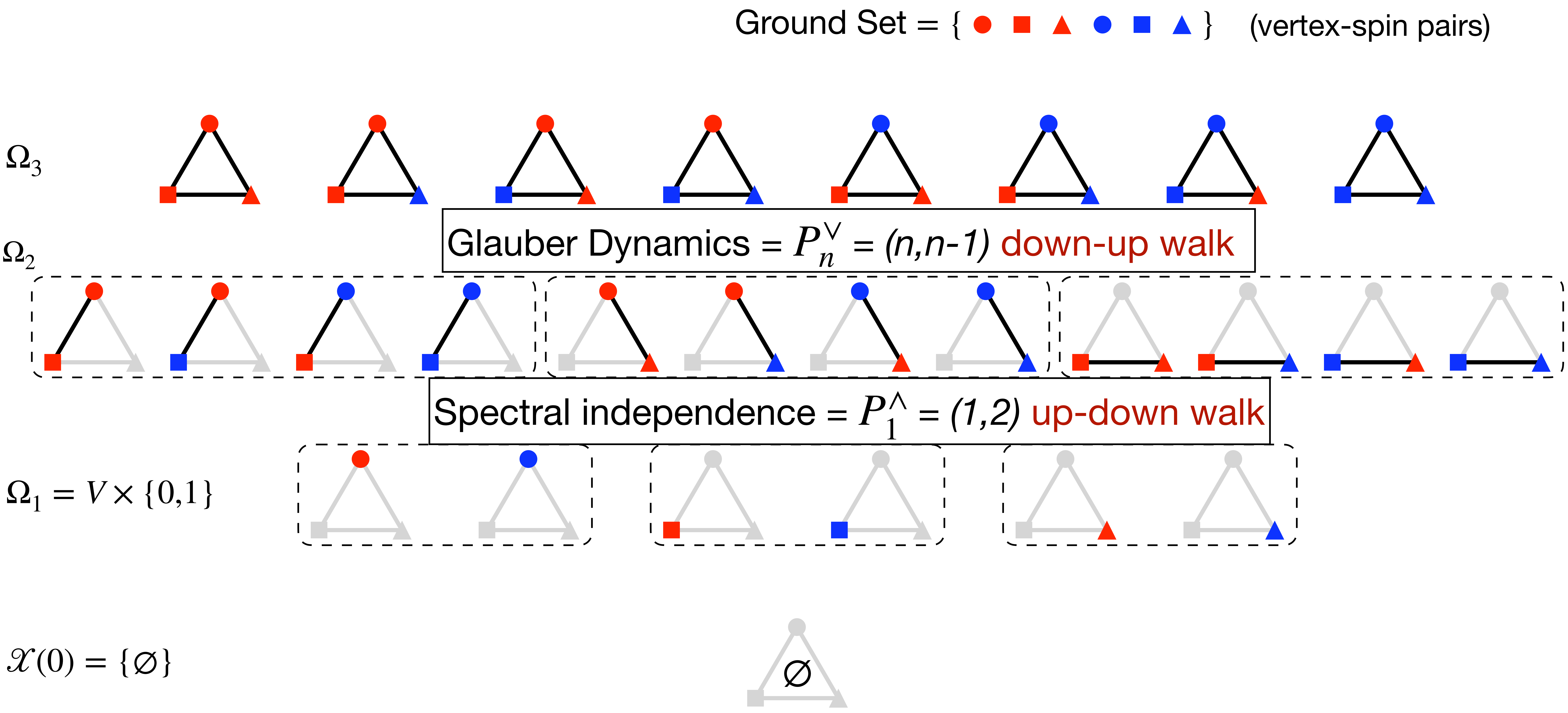
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$\Omega_1 = V \times \{0,1\}$
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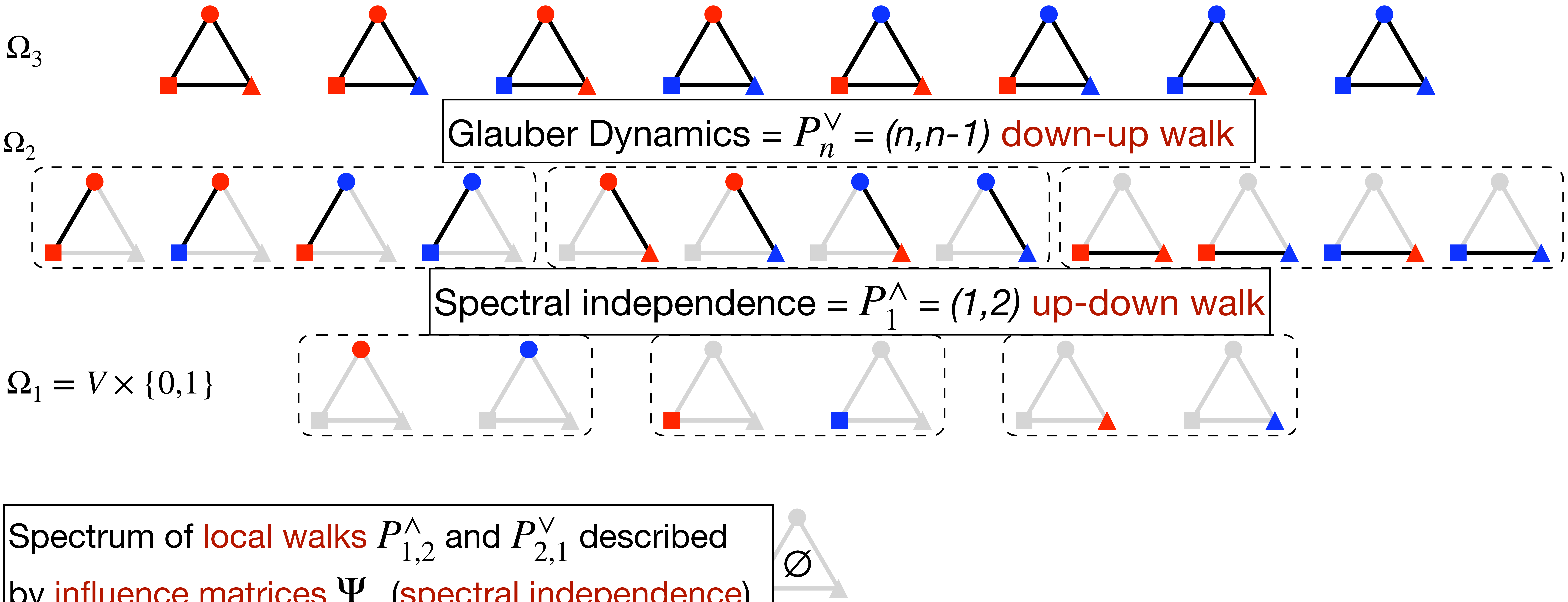

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Spectral independence = $P_1^{\wedge}$ = *(1,2)* up-down walk

Spectrum of local walks $P_{1,2}^{\wedge}$ and $P_{2,1}^{\vee}$ described by influence matrices Ψ_μ (spectral independence) [Anari-Liu-Oveis Gharan'20]

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up-down: $P_{k-1}^{\wedge} = P_{k-1}^{\uparrow} P_k^{\downarrow}$ down-up: $P_k^{\vee} = P_k^{\downarrow} P_{k-1}^{\uparrow}$

Useful fact: $gap(P_{k-1}^{\wedge}) = gap(P_k^{\vee})$

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Follows that: $\lambda_2(Q) = \dfrac{\lambda_{\max}(\Psi)}{n-1}$

Fix configuration τ on k vertices, then: $\lambda_2(Q_\tau) = \dfrac{\lambda_{\max}\left(\Psi_\tau\right)}{n-k-1} \leq \dfrac{\alpha}{n-k-1}$

Local to Global Theorem

- [Alev-Lau'20]: If α-Spectrally Independent, then:

$$gap(P_{\text{Glauber}}) = gap(P_n^{\vee}) \geq \frac{1}{n}\prod_{i=0}^{n-2}\gamma_i \geq \frac{1}{n}\prod_{i=0}^{n-2}\left(1 - \frac{\alpha}{n-i-1}\right) \geq \frac{C}{n^{\alpha+1}}$$

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How to prove Local to Global Theorem of [Alev-Lau'20]?

Spectral gap = Variance decay

- μ: Gibbs distribution over $\Omega \subset \{0,1\}^V$
- For a function $f : \Omega \to \mathbb{R}_{\geq 0}$, expectation: $\mathbb{E}_\mu f = \sum_{\sigma: V \to Q} \mu(\sigma) f(\sigma)$
  - Variance of f is $\mathrm{Var}_\mu(f) = \frac{1}{2} \sum_{\sigma, \eta \in \Omega} \mu(\sigma)\mu(\eta)(f(\sigma) - f(\eta))^2$
  - Local Variance = Dirichlet Form $D_P(f) = \frac{1}{2} \sum_{\sigma, \eta \in \Omega} \mu(\sigma) P(\sigma, \eta)(f(\sigma) - f(\eta))^2$
- Spectral gap γ equivalent to: $\min_f D_P(f) \geq \gamma \mathrm{Var}_\mu(f)$

then $\mathrm{Var}_\mu(Pf) \leq (1 - \gamma)\mathrm{Var}_\mu(f)$,

Local-to-Global Key Fact: $D_{P_k^\wedge}(f) \geq \gamma_{k-1} \frac{k}{k+1} D_{P_k^\vee}(f)$

Recall, $gap(P_k^\wedge) = gap(P_{k+1}^\vee)$

$$D_{P_k^\wedge}(f) := \frac{1}{2} \sum_{\sigma,\eta \in \Omega_k} \pi_k(\sigma) P_k^\wedge(\sigma, \eta)(f(\sigma) - f(\eta))^2$$

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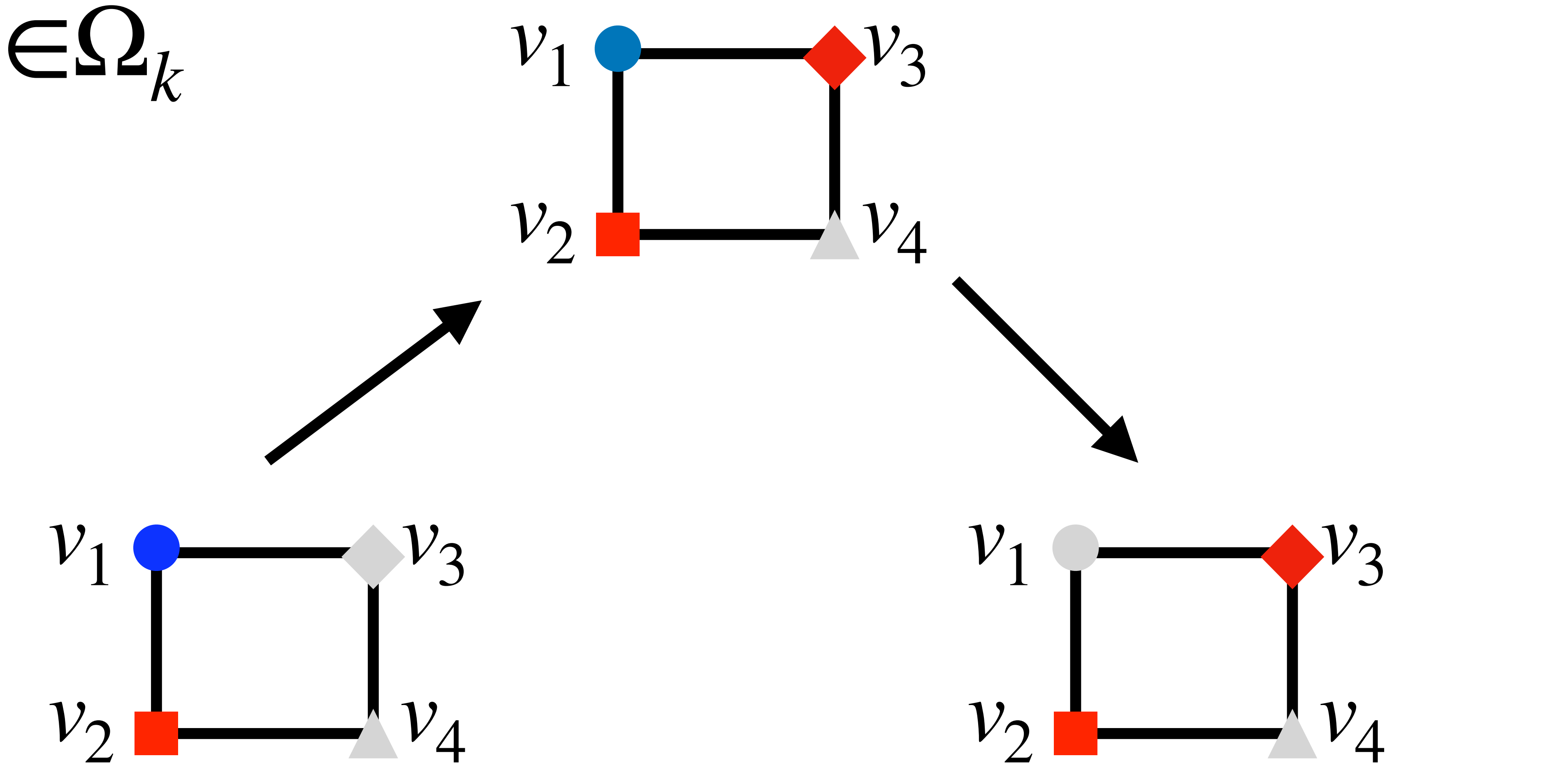

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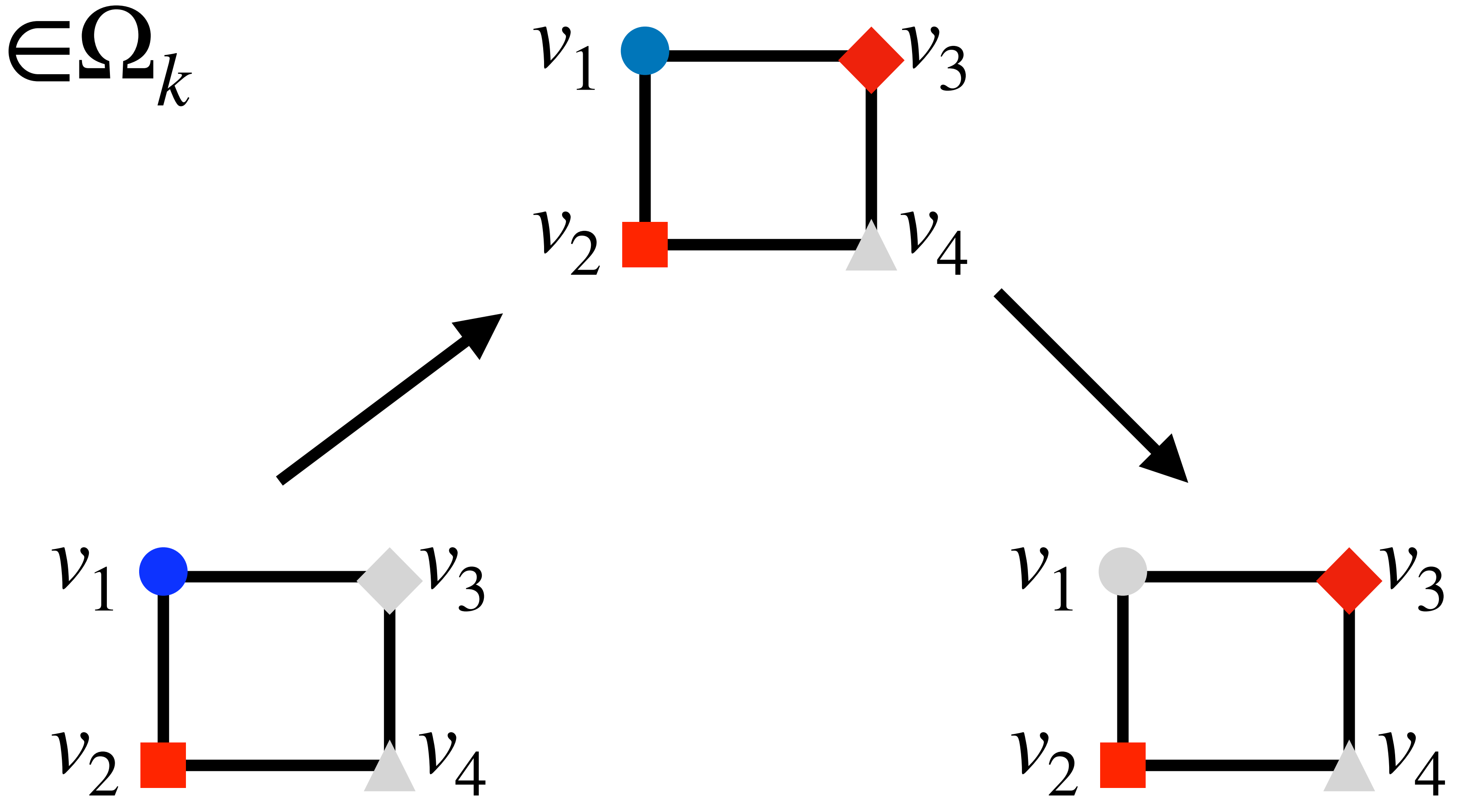

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Inductive proof of Local to Global Theorem

Goal: $gap(P_k^\vee) \geq \frac{1}{k}\prod_{i=0}^{k-2}\gamma_i$ equivalent to: $D_{P_k^\vee}(f) \geq \frac{1}{k}\prod_{i=0}^{k-2}\gamma_i \times \mathrm{Var}_{\pi_k}(f)$

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Key Fact
on previous slide

by induction

Goal: Proof Sketch [CLV'21]

Let μ be Gibbs dist. of spin system on $G = (V, E)$ of constant max degree $\leq \Delta$.

Theorem 1: For all Δ, all $\alpha > 0$, there exists $C(\Delta, \alpha)$.

If μ is α-spectrally independent, then Glauber Dynamics has relaxation time:

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Local to Global Theorem

- [Alev-Lau'20]: If α-Spectrally Independent, then:

$$gap(P_{\text{Glauber}}) = gap(P^{\vee}_{n,n-1}) \geq \frac{1}{n}\prod_{k=0}^{n-2}\gamma_k \geq \frac{1}{n}\prod_{k=0}^{n-2}\left(1 - \frac{\alpha}{n-k-1}\right) \geq n^{-\alpha-1}$$

where γ_k is spectral gap for local walk with worst assignment τ to k vertices.

Want $gap \geq \dfrac{C}{n}$ but losing too much when $k = o(n)$

Uniform Block Dynamics

βn-Uniform Block Dynamics where $\beta = O(1/\Delta)$.

In each step:

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Improved Local-to-Global:

$$gap(P_k^{\vee}) \geq \frac{\Gamma_{k-1}}{\sum_{i=0}^{k-1} \Gamma_i} \geq \frac{\Gamma_{k-1}}{k} \quad \text{where } \Gamma_i = \prod_{j=0}^{i-1} \gamma_j$$

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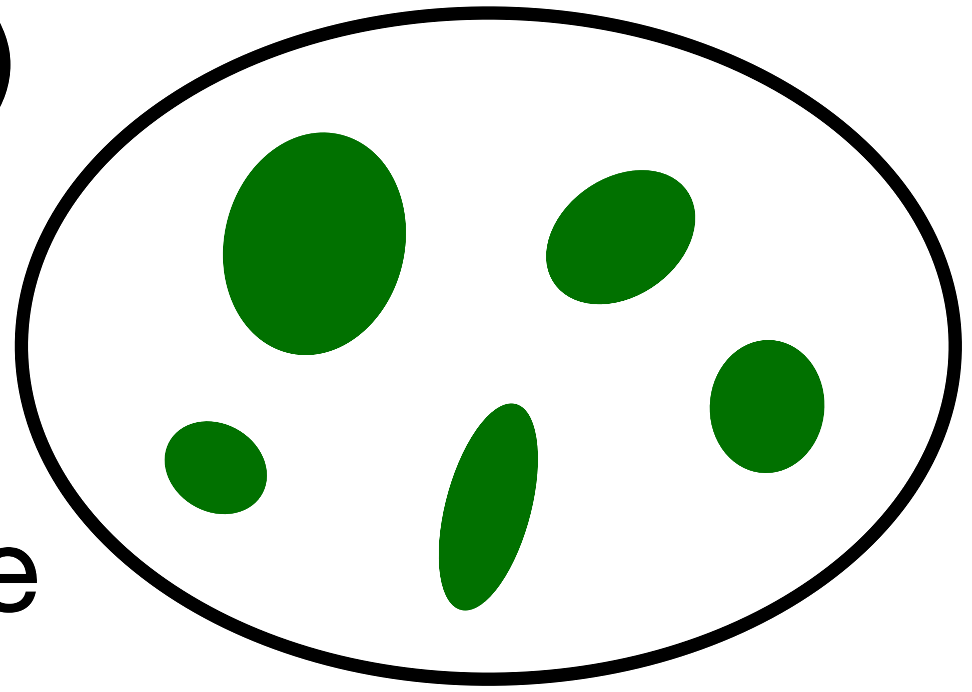
Yields $gap\left(P_{n,(1-\beta)n}^{\vee}\right) \geq C(\beta, \alpha)$

Shattering

- Pick $S \subseteq V$ of size $|S| = \beta n \approx \frac{n}{1000\Delta}$ randomly (Δ: max degree)

- Then S is "shattered" with high probability
  - Each connected component $T \in G[S]$ has $O(1)$ expected size



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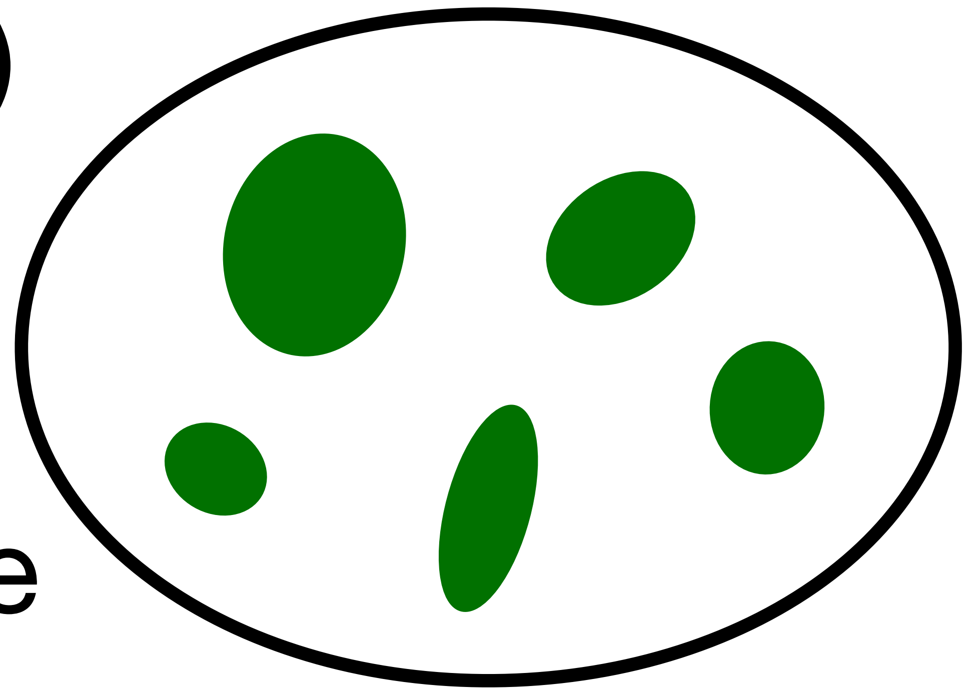

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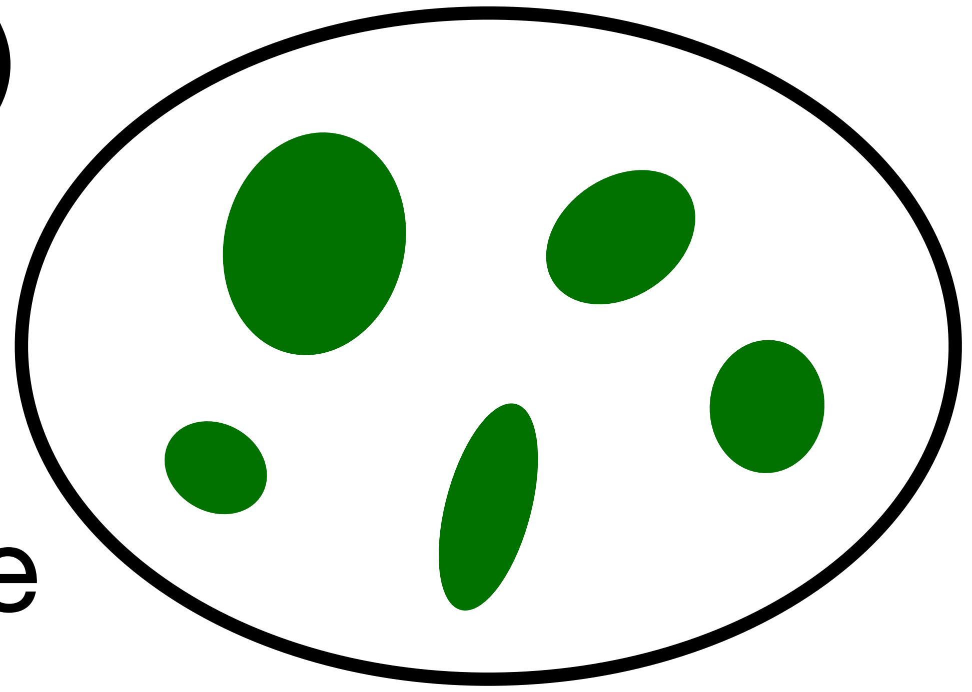

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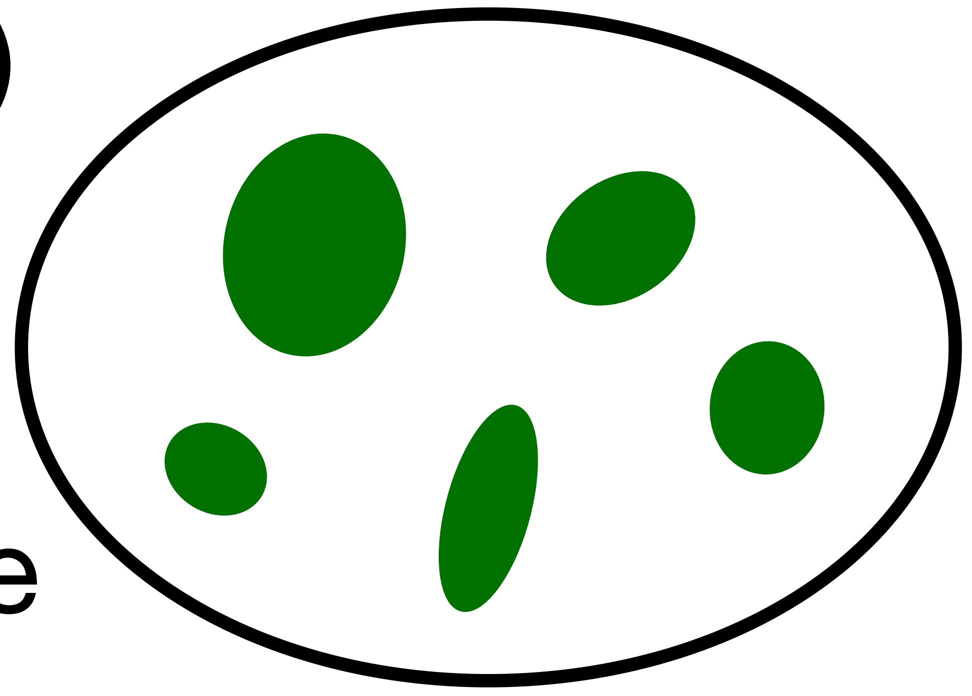

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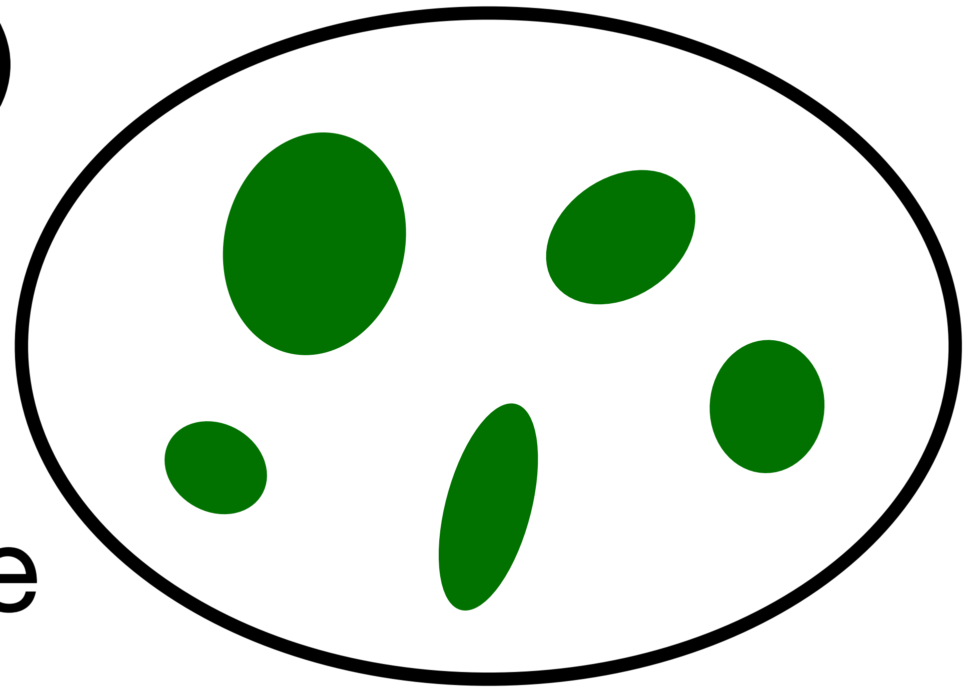

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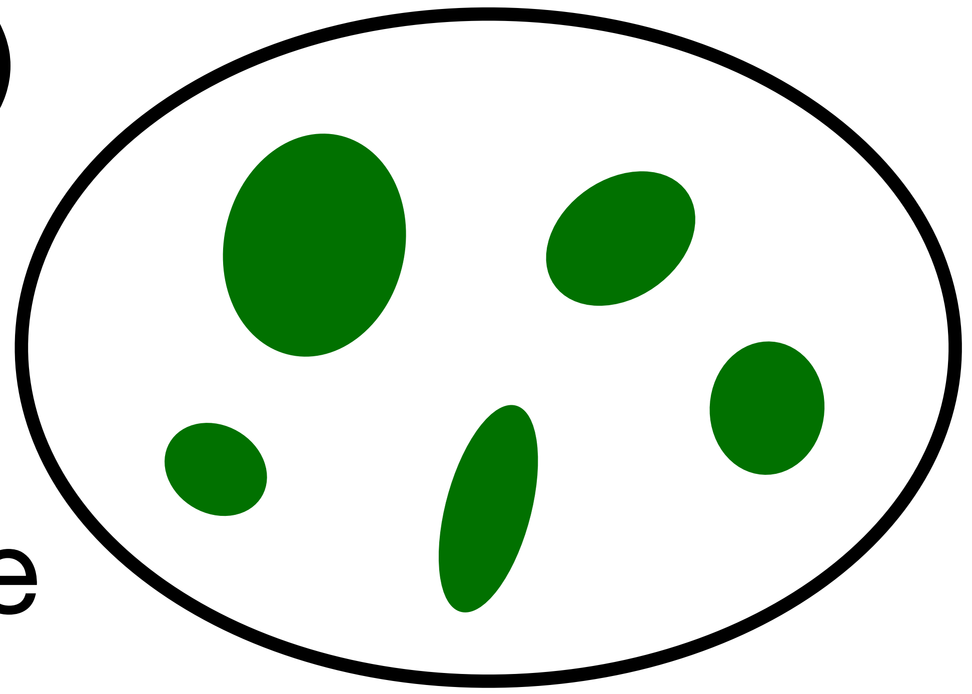

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$$= C'' n D_{P_{\mathrm{Glauber}}}(f)$$

(Relative) Entropy

- μ: Gibbs distribution over $\{0,1\}^V$
- For a function $f : \{0,1\}^V \to \mathbb{R}_{\geq 0}$:
  - The expectation of f is $\mathbb{E}_\mu f = \sum_{\sigma:V\to Q} \mu(\sigma) f(\sigma)$
  - The entropy of f is $\mathrm{Ent}_\mu(f) = \mathbb{E}_\mu \left[f \log\left(\frac{f}{\mathbb{E}_\mu f} \right) \right]$
- Entropy describes "fluctuation" of f w.r.t. μ ($\mathrm{Ent}_\mu(f) = 0 \Leftrightarrow f \equiv \mathbb{E}_\mu f$)
- Spectral gap γ then $\mathrm{Var}_\mu(Pf) \leq (1-\gamma)\mathrm{Var}_\mu(f)$
- Entropy decay (modified log-Sobolev constant) ρ then $\mathrm{Ent}_\mu(Pf) \leq (1-\rho)\mathrm{Ent}_\mu(f)$

Approximate Tensorization of Entropy

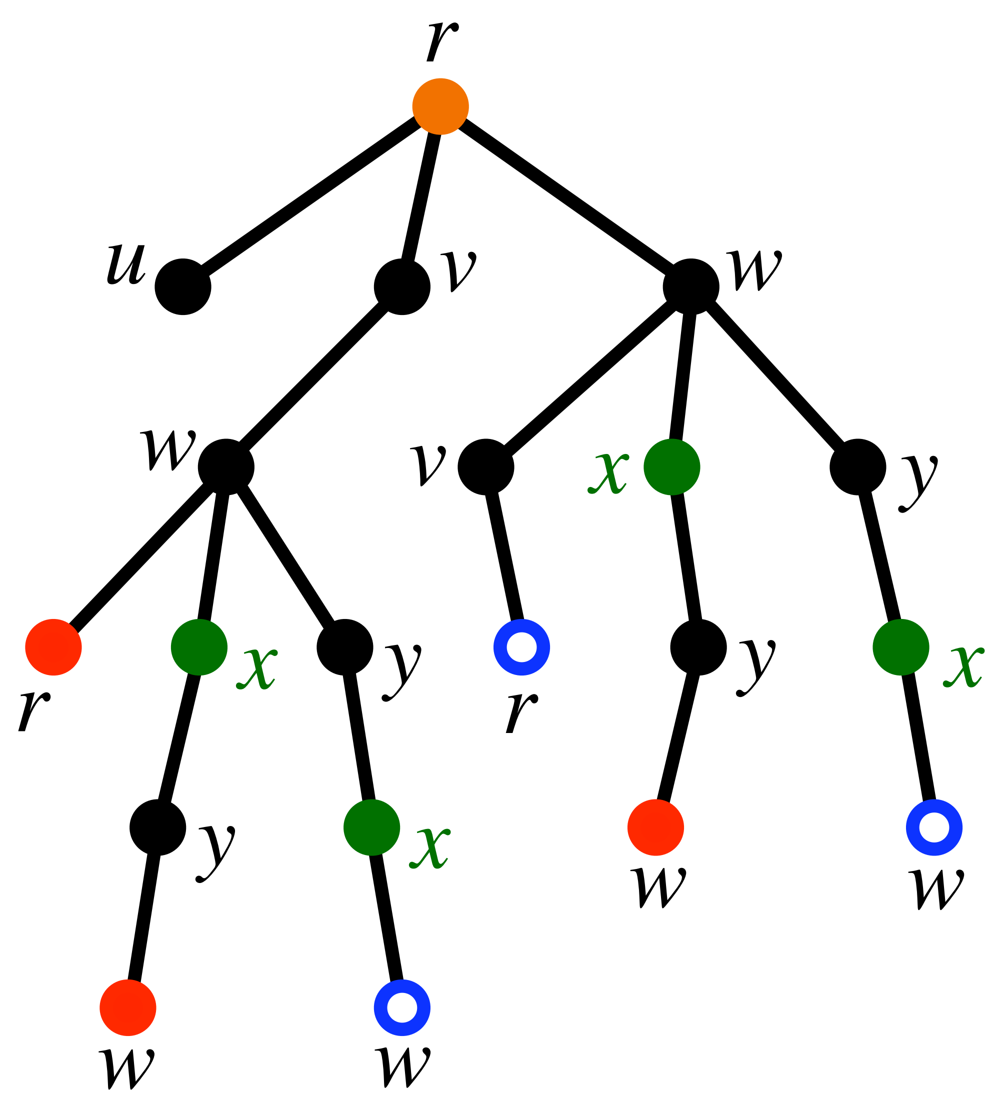
- μ satisfies approximate tensorization (AT) of entropy with constant C if

$$\mathrm{Ent}(f) \le C \sum_{v \in V} \mathbb{E}[\mathrm{Ent}_v(f)], \qquad \forall f : \{0,1\}^V \to \mathbb{R}_{\ge 0}$$

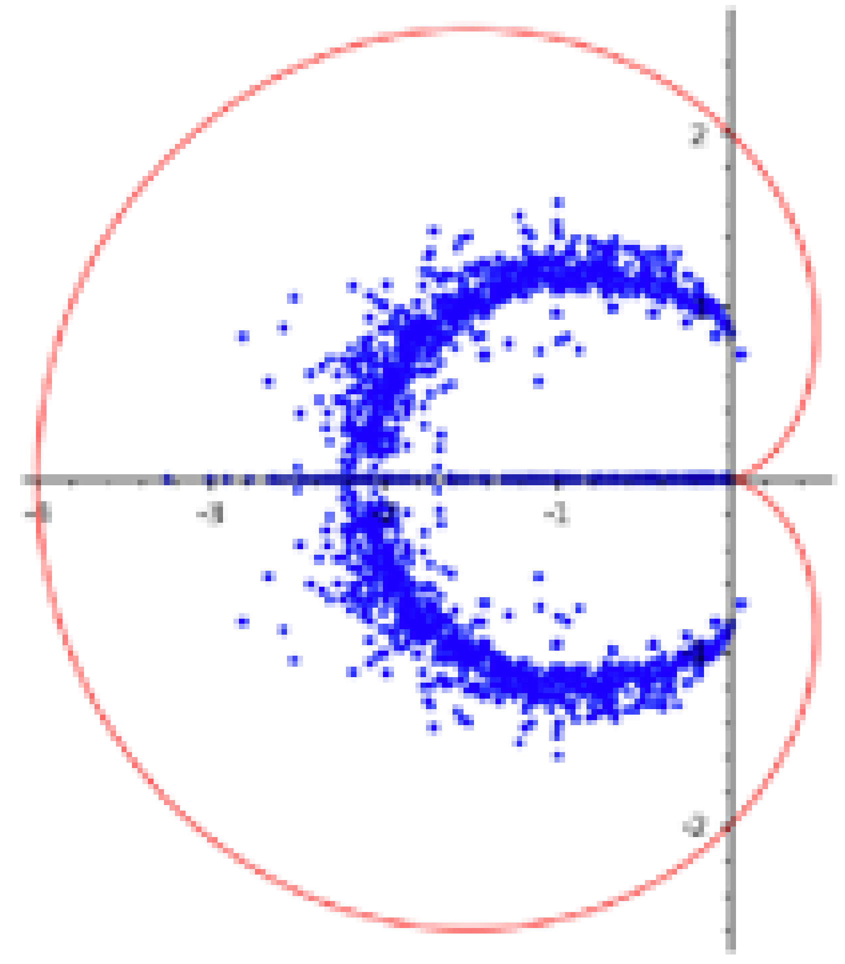
"Fluctuation" of f is attributed to the sum of "average local fluctuation" at individual vertices

Fact: AT holds for a product distribution μ with constant $C = 1$

Connection to Other Methods



Correlation decay
[Weitz'06, Sinclair-Srivastava-Thurley'14, Li-Lu-Yin'13]



Polynomial interpolation
[Barvinok'16, Patel-Regts'17, Peters-Regts'19]

- Large running time & hard to implement
- We can transform these results to optimal MCMC via Spectral Independence

Picture for Spectral Independence

[Chen-Liu-V’20]

Correlation Decay Proof Approach

[Blanca-Caputo-Chen-Parisi-Stefankovic-V ’22, Liu’21]

(Path) Coupling for any Local MCMC

[Chen-Liu-V’21b]

Zero-Free Region of Partition Function

Spectral Independence

Optimal Mixing and Optimal Entropy Decay

Glauber Dynamics

[Chen-Liu-V’21]

Any Block Dynamics

[BCCPSV’22]

Swendsen-Wang (SW) (Global dynamics for Ising)

Conclusion

We showed: $T_{\text{mix}} \leq C(\delta, \Delta) n \log n$ where $\lambda \leq (1-\delta)\lambda_c(\Delta)$ for any $\delta > 0$

Subsequent work:

$T_{\text{mix}} \leq f(\delta) n \log n$ for any G with max deg Δ when $\lambda < \lambda_c(\Delta)$ [CFYZ'22],[CE'22]

$T_{\text{mix}} = O(n^3)$ for any bipartite G with max deg Δ on LHS when $\lambda < \lambda_c(\Delta)$ [Chen-Liu-Yin'23]

Open problem: k-Colorings when $k \geq \Delta + 2$?

- General graphs: $k > (11/6 - 10^{-5})\Delta$ [BCCPSV'21, L'21]
- Triangle-free graphs: $k > 1.764\Delta$ [CGSV'21,FGYZ'21]
- Correlation Decay on Trees: $k \geq \Delta + 3$ [Chen-Liu-Mani-Moitra'23]
  $T_{\text{mix}} \leq C(\Delta) n \log n$ when $k \geq \Delta + 3$ and girth $\geq g(\Delta)$

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