TCS@UCSB Summer School on Spectral Independence **August 8-12, 2022**

Lecture 3: *August 8th, 2022*

Lecturer: *Nima Anari*

Intro to HDX

These notes were prepared by Yuzhou Gu based on the lecture by Nima Anari. This is part of the 2022 Summer School on *New tools for optimal mixing of Markov chains: Spectral independence and entropy decay*, which was held at the University of California, Santa Barbara (UCSB) from August 8, 2022 to August 12, 2022. More information on the summer school is available at: https://sites.cs.ucsb.edu/∼vigoda/School/

3.1 Introduction

The lecture will be in two parts. In the first part we talk about sampling from matroids in polynomial time, following [ALOV19]. In the second part we discuss the polynomial point of view, which gives insights that are otherwise difficult to obtain.

3.2 Matroids: Definitions

Before we introduce the definition of matroids, we give a few examples.

Example 3.1 (Graphical matroid). Given a graph $G = (V, E)$, the set $\{S \subseteq E : S \text{ contains no cycle}\}$ is a matroid. This is also called the spanning tree matroid.

Example 3.2 (Linear matroid). Let $V = \{v_1, \ldots, v_n\}$ be vectors in some vector space. Then $\{S \subseteq V : S \text{ is linear independent}\}$ is a matroid.

Definition 3.3 (Matroid, definition using independent sets). *A collection $\mathcal{I}$ of subsets of $[n]$ is a matroid if the following hold:*

- *$\mathcal{I}$ is downward closed ("simplicial complex" in Kaufman's talk), i.e., $S \in \mathcal{I}, T \subseteq S \Rightarrow T \in \mathcal{I}$.*
- *(Exchange Axiom) $S, T \in \mathcal{I}, |T| > |S| \Rightarrow \exists i \in T$ such that $S \cup \{i\} \in \mathcal{I}$.*

One immediate consequence of the exchange axiom is that all maximal (maximal means not contained in another independent set) independent sets are of the same size.

Definition 3.4 (Basis). *A basis is a maximal independent set. The* rank *$r(M)$ of a matroid M is defined as the common size of all bases.*

Note that the set of bases uniquely determines a matroid. We will use $\mathcal{F}$ to denote the collection of basis of a matroid.

Definition 3.5 (Matroid, definition using bases). *A collection $\mathcal{F}$ of subsets of $[n]$ is a matroid if the following hold:*

- *(Exchange Axiom) $S \neq T \in \mathcal{F}, \Rightarrow \exists i \in T \setminus S, j \in S \setminus T$ such that $(S \setminus \{j\}) \cup \{i\} \in \mathcal{F}$.*

Definition 3.6 (Truncation)**.** *Let M be a matroid. Let k be a positive integer. The set $\{S \in \mathcal{I} : |S| \leq k\}$ is called truncation of M. Note that truncation of a matroid is also a matroid.*

Exercise 3.7. *Let $\mathcal{F}$ be a set of basis of a matroid M. Let $\mathcal{F}'$ be the set of complements, that is, $\{[n] - B : B \in \mathcal{F}\}$. Show that $\mathcal{F}'$ are the basis of a matroid (called the dual of the matroid M).*

Exercise 3.8. *Let $\mathcal{I}$ be a set of independent sets of a matroid M. Let $S \subseteq [n]$. Let $\mathcal{I}''$ be the set of independent sets contained in S, that is, $\{T : T \in \mathcal{I}$ and $T \subseteq S\}$. Show that $\mathcal{I}''$ is a matroid (called the restriction of the matroid M; denoted $M|S$).*

Exercise 3.9. *Let $\mathcal{I}$ be a set of independent sets of a matroid M. Let $S \subseteq [n]$. Let $\mathcal{I}'$ be the set of independent sets containing S with S removed, that is, $\{T \setminus S : T \in \mathcal{I}$ and $S \subseteq T\}$. Show that $\mathcal{I}'$ is a matroid (called the contraction of the matroid M; denoted M/S).*

Remark 3.10. A matroid M' that can be obtained from M by contraction and restriction is called a minor of M.

3.2.1 Matroids: Geometric Perspective

We offer an equivalent (geometric) definition of a matroid. Let $\mathcal{F}$ be a collection of sets. Let us consider the polytope $\text{conv}(\{\mathbb{1}_B : B \in \mathcal{F}\})$ where

- $\mathbb{1}_B$ is the indicator vector, i.e., i-th coordinate is 1 if $i \in B$, and is 0 otherwise,
- conv means convex hull.

Theorem 3.11. *$\mathcal{F}$ is the basis of some matroid if and only if all edges of the polytope are parallel to $e_i - e_j$ for some $i \neq j$.*

3.2.2 Application of counting/sampling bases: Reliability

We will begin with some applications for sampling/approximate counting bases of a matroid.

An interesting application is to compute reliability. Let $X \subseteq [n]$ be a random set where every element is included independently with probability p. The *reliability* is defined as the $\mathbb{P}[X \supseteq \text{basis}]$, and the *unreliability* is defined as $\mathbb{P}[X \not\supseteq \text{basis}]$.

For a graphical matroid these problems are known as network reliability/unreliability as the problem corresponds to whether X is connected/disconnected. Karger [Kar99] presented an FPRAS for network unreliability, and Guo and Jerrum [GJ19] presented an FPRAS for network reliability.

The main result in this lecture yields an FPRAS for reliability of a matroid. In particular, rapid mixing of the bases-exchange walk yields an efficient sampling algorithm for generating a basis of a matroid (almost) uniformly at random; the classical result of [JVV86] then yields an FPRAS for computing the number of bases.

Note,

$$\mathbb{P}[X \subseteq \text{basis}] = \sum_k p^k (1-p)^{n-k} \#\{\text{indep set of size } k\},$$

and we can obtain an FPRAS for the number of independent sets (of a matroid M) of size k since the truncation of a matroid is a matroid.

Note, for linear matroids, we consider a linear code defined as $\{x \in \mathbb{F}^n : Hx = 0\}$ where H is a matrix. Under i.i.d. erasure noise, a codeword is reconstructible if and only if the columns erases are linearly independent. So $\mathbb{P}[X \subseteq \text{basis}] = \mathbb{P}[\text{can recover}]$.

3.2.3 Expansion of polytope: Mihail-Vazirani conjecture

Another motivation is the Mihail-Vazirani conjecture (see [FM92]), which roughly states that the random walk on skeleton of any $0-1$ polytope (with polynomial max degree) mixes in polynomial time. The concrete statement of the conjecture is the following statement about the conductance (or normalized edge-expansion). Let $G = (V, E)$ be the skeleton of a polytope. Then $\forall S \subset V$, $|E(S, V - S)| \geq \min\{|S|, n - |S|\}$.

This is known to imply that the mixing time of the random walk on the skeleton is poly(max deg). For matroids, we always have max deg = poly(n).

(The vanilla random walk does not converge to the uniform distribution but this issue can easily be solved when max degree is poly(n).)

3.3 Markov Chain for Sampling Bases

3.3.1 Bases Exchange Walk

The bases exchange walk is the following Markov chain (B_t) on the set of bases of a given matroid M. Let Ω denote the collection of bases of a matroid M. From $B_t \in \Omega$, the transitions $B_t \to B_{t+1}$ are defined as follows:

- Choose an element $e \in B_t$ uniformly at random.
- Let $F = \{f \in E : B_t \cup \{f\} \setminus \{e\} \in \mathcal{I}\}$ denote the set of edges that we can add to $B_t \setminus e$ while remaining independent.
- Choose an element $f \in F$ uniformly at random. Let $B_{t+1} = B_t \cup \{f\} \setminus \{e\}$.

Note the edge e is in F and hence $P(B_t, B_t) > 0$; thus, the chain is aperiodic.

Exercise 3.12. *Using the exchange-axiom, prove by induction that the chain is irreducible.*

Therefore, the bases-exchange walk is an ergodic Markov chain. Since the chain is symmetric then the unique stationary distribution is uniform over the set Ω of bases of the matroid M. In the terminology from Tali Kaufman's lecture, the bases-exchange walk is equivalent to the down-up walk on the bases of a matroid.

3.3.2 Main Theorem

The main result is a bound on the spectral gap of the bases-exchange walk.

Theorem 3.13 ([ALOV19]). *For any matroid M, the spectral gap of the bases-exchange walk satisfies $\geq \frac{1}{r(M)}$, where $r(M)$ is the rank of the matroid.*

Remark 3.14. The above result implies the Mihail-Vazirani conjecture for matroid polytopes. This utilizes the easy side of Cheeger's inequality that the conductance (or normalized edge-expansion) is at least one-half the spectral gap, see [ALOV19, Section 4.1] for more details.

Using the upper bound on the mixing time in terms of the spectral gap (see Pietro Caputo's lecture) we have the following.

Corollary 3.15. *The mixing time of the bases-exchange walk is $O(r(M)\log(|\Omega|)) = O(r(M)^2 \log n)$.*

By proving entropy contraction, [CGM21] improves the mixing time bound to $O(r(M)\log\log(|\Omega|)) = O(r(M)\log r(M) + r(M)\log\log n)$; this is presented in Heng Guo's first lecture. Furthermore, the $\log\log n$ term was subsequently removed in [ALO+21], achieving $O(r(M)\log r(M))$ mixing time. This is tight because we need $\Omega(r(M)\log r(M))$ time to replace every element due to coupon collector.

3.4 Proof of Main Theorem: Rapid Mixing

The proof of Theorem 3.13 uses [Opp18]'s Trickle Down Theorem and [KO20]'s Random Walk Theorem (local to global theorem), as presented in Tali Kaufman's lecture. Our final step to bound the mixing time of the bases-exchange walk is to use the Random Walk Theorem. To that end we need to bound the spectral gap of the local walk (this is the up-down walk from level 1) for every link. The links are defined for every independent set S. In particular, for an independent set S the link is defined as $\{B \setminus S : S \subseteq B, B \in \mathcal{I}\}$, which is a contraction of the matroid. Exercise 3.8 establishes that the contraction of a matroid is also a matroid.

These local walks for a particular link are weighted as these are the projections in the original simplicial complex. To handle these weighted local walks we utilize the Trickle Down Theorem. The Trickle Down Theorem says that if we have a good bound on the spectral gaps for all local walks on $(r-2)$-dimensional links then we obtain a good bound on the spectral gap for the local walks for all links, and then we can apply the Random Walk Theorem. This greatly simplifies matters because the local walks on $(r-2)$-dimensional links are unweighted, and hence they simply correspond to (unweighted) rank 2 matroids.

Theorem 3.16. *Rank 2 matroids are equivalent to a complete multipartite graph plus isolated vertices (we take the graph whose edges are the basis of the matroid and the vertices are the elements of the matroid).*

Proof. Isolated vertices are non-independent elements. After removing isolated vertices, the graph has an edge $\{i,j\} \in E$ if and only if $\{i,j\}$ is independent. By the Exchange Axiom, for any three vertices i,j,k with $\{j,k\} \in E$, either $\{i,j\} \in E$ or $\{i,k\} \in E$. Equivalently, if $\{i,j\} \notin E$ and $\{i,k\} \notin E$ then $\{j,k\} \notin E$. This means that we have a multipartite graph (where the partitions contain dependent elements). □

Theorem 3.17. *The second eigenvalue of the bases-exchange walk on rank 2 matroids is* ≤ 0.

Proof. The adjacency matrix A of a complete multipartite graph is the all ones matrix minus a block diagonal matrix, so it has the second eigenvalue ≤ 0. The random walk matrix $D^{-1}A$ is similar to $D^{-1/2}AD^{-1/2}$. It is easy to prove that $\lambda_2(A) \leq 0 \iff \lambda_2(D^{-1/2}AD^{-1/2}) \leq 0$ (this is a consequence of Sylvester's law of inertia). This finishes the proof. □

Theorem 3.18. *For any matroid M we have that M is 0-local expander.*

Before proving this theorem we restate Oppenheim's trickle-down theorem from Tali Kaufman's lecture.

Theorem 3.19 (Oppenheim [Opp18]). *If X is a pure simplicial complex such that*

(i) Its 1-skeleton is connected,

(ii) $\forall v \in X(0)$, $\lambda_2(X_v(0), X_v(1)) \leq \lambda$,

then, X is a $\frac{\lambda}{1-\lambda}$-local spectral expander.

We will utilize the following corollary.

Corollary 3.20 (Trickle without loss)**.** *If X is d-dimensional, all $(d-2)$-links are 0-local expanders, and all 1-skeletons are connected then X is a 0-local spectral expander.*

Proof of Theorem 3.18. Follows from Theorem 3.17 and Corollary 3.20 (since the $(d-2)$-links are rank 2 matroids). □

We can now prove the main theorem establishing rapid mixing of the bases-exchange walk.

Proof of Theorem 3.13. Hence the random walk theorem implies that the spectral gap γ for the transition matrix P of the bases exchange walk satisfies:

$$\gamma(P) \geq \frac{1}{r} \prod_{k=0}^{r-2} (1 - \lambda_k), \tag{3.1}$$

where

$$\lambda_k = \max_{S:|S|=k} \lambda_2(Q_S)$$

is the second largest eigenvalue for the local walk for the link defined by an independent set S, which by Theorem 3.18 we have that $\lambda_k \leq 0$ for every k. □

3.5 Discussion: Polynomials

We do not have time for polynomials today, so let us finish with a few remarks.

Let μ be a distribution on $\binom{[n]}{\text{rank}}$. Let $\lambda \in \mathbb{R}^n_{>0}$. Define $\lambda \star \mu$ as $(\lambda \star \mu)(S) = \mu(S) \prod_{i \in S} \lambda_i$. Then μ is 0-local expander if and only if $\lambda \star \mu$ is 0-local expander.

We proved that matroids are 0-local expanders. In fact the reverse is also true.

Proposition 3.21. *If M is a 0-local expander, then the support of M is a matroid.*

Proof. It is enough to show that edges are parallel to $e_i - e_j$. Let $\mathbb{1}_S - \mathbb{1}_T$ be an edge. Then there exists $w \in \mathbb{R}^n$ such that $\arg\max \langle w, x \rangle$ is the edge $\mathbb{1}_S - \mathbb{1}_T$.

Let $t \in \mathbb{R}$ be large, let $\lambda_i = \exp(tw_i)$. Then $(\lambda \star \mu)(R) = \mu(R) \exp(t\langle \mathbb{1}_R, w \rangle)$. As $t \to \infty$, we get μ conditioned on an edge. If it is a 0-local expander, then it must be of form $e_i - e_j$. □

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