1

Plan:

- Polynomials
- Domain sparsification + Avg local-to-Global

• μ on $\binom{[n]}{k}$, generating poly

$$g_\mu(z_1, \dots, z_n) = \sum_S \mu(S) \prod_{i \in S} z_i$$

• THM: μ is a 0-local expander iff $\log g_\mu$ is concave over $\mathbb{R}^n_{>0}$. (log-concavity)

• Strongly log-concave (Gurvits), Lorentzian polynomials (Brändén-Huh)

• Ways to prove log-concavity:
  - Trickle-down
  - Hodge Theory [AHK '18]
  - Real-Stability: Whenever $z_1, \dots, z_n \in \mathbb{C}$ are in the same half-plane $\Rightarrow g_\mu(z_1, \dots, z_n) \neq 0$ (For homogeneous polys or half planes gives the same theory)

• THM [Gårding 59] Real-stable $\Rightarrow$ log-concave over $\mathbb{R}^n_{>0}$

• Eg. Spanning trees give a real stable polynomial

THM: $\det\left(\sum_{i=1}^{n} z_i A_i\right)$ with $A_i \succeq 0$ is real-stable.

Take A_i = Laplacian of an edge

By the matrix-tree thm $\det\left(\sum z_i A_i\right) = \sum_T \prod_{e \in T} z_e$

- **Ex 2**: Determinantal point process: $L \in \mathbb{R}^{n \times n}$, $L \succeq 0$

$\mu(S) \propto \det(L_{S,S})$, these are real-stable

- **Prop**: Real stab $\Rightarrow$ 0-local-expansion

$\Downarrow$

Negative correlation $:= \mathbb{P}_{S \sim \mu}[i,j \in S] \le \mathbb{P}[i \in S]\,\mathbb{P}[j \in S]$ $\forall i \neq j$

- THM [Feder-Mihail] If all links of a matroid have neg. correlation

$\Rightarrow$ Mihail-Vazirani true for that matroid [Borcea-Brändén-Liggett]

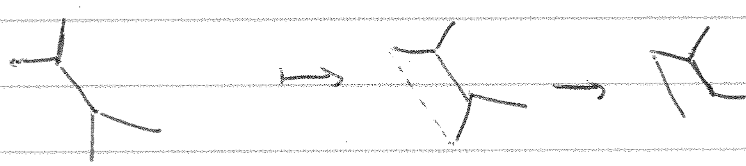
- [Jerrum-Son 02, Jerrum-Son-Tetali-Vigoda 05, ...]
showed for any distribution.

- Domain sparsification:

checking reconnecting edges.

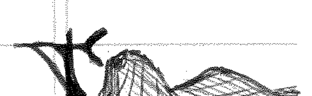
Cost of down up walk on spanning trees $O(|V| \log|V| \cdot |E|)$

Dual Down-Up walk



Mixing in $O(|E| \log|E|)$ each step costs $O(\log|V|)$

[Anari-Liu-Oveis-Gharan-Vinzant-Vuong 21]



2

- Log-concave poly $\equiv$ 0-local expander

μ on $\binom{[n]}{k}$

$$g_\mu(z_1, \dots, z_n) := \sum_S \mu(S) \prod_{i \in S} z_i$$

$D^2 g|_1$ = adj matrix of link of ϕ

$$(D^2 g)_{ij} = \partial_i \partial_j g_\mu |_1 = \sum_{S \ni i,j} \mu(S) = \mathbb{P}[i,j \in S]$$

- $\lambda_2(D^2 g) \le 0$ iff λ_2 (RW on $D^2 g$) ≤ 0

0-local expansion for link of ϕ

- **Lemma**: g_μ defined on $C = \mathbb{R}^n_{\ge 0}$ hom. of deg d. TFAE

i) μ is log-concave

ii) $g_\mu^{\frac{1}{d}}$ is concave

iii) $\lambda_2(D^2 g_\mu) \le 0$

iv) g_μ is quasi-concave i.e. $\{z \mid g(z) \ge t\}$ is convex for all t.

- **Lemma**: $\lambda_2(M) \le 0$ iff $\exists$ V with $\dim V = n-1$ s.t. $x^t M x \le 0$ for $x \in V$

iff $(x^t M x)(a^t M a) \le (a^t M x)^2$ for $a \in \mathbb{R}^n_{\ge 0}$

Proof of iii) $\Rightarrow$ i) $\quad D^2(\log g) = g D^2 g - D_g D_g^t \underset{\text{we want}}{\overset{?}{\le}} 0$

$$g(x^t D^2 g\, x) \le (x^t \nabla g)^2$$

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \qquad z\nabla g(z) = \sum_i z_i \partial_i g = \sum_i \sum_{S \ni i} \mu(S) z^S = k g$$

$$z^t D^2_g z = k(k-1) g$$

$$D(\partial_i g) \cdot z = (k-1)\partial_i g \implies D^2_g z = (k-1) D_g$$

$$(x^T D^2_g x)\underbrace{(z^t D^2_g z)}_{k(k-1)g} \le (z^t D^2_g x)^2 = (x^t D_g)^2 (k-1)^2$$

- λ - external field, $(\lambda * \mu)$ 0-local-expander

$\implies$ log-concavity at λ

Choose $c = 1+\eta$

- THM: λ_2 (R.W. on link of empty set) $\le \frac{\eta}{k-1} \le$

$$\Updownarrow$$

$g_\mu(z_1^{1/c}, \ldots, z_n^{1/c})$ is log-concave at $z = 1$

$$D_{\chi^2}(\nu D_{k \to 1} \| \mu D_{k \to 1}) \le \frac{c}{k} D_{\chi^2}(\nu \| \mu)$$

- Def (Fractional log-concavity) g_μ c-FLC if $g_\mu(z_1^{\frac{1}{c}}, \ldots, z_n^{\frac{1}{c}})$ is log-concave

c-FLC $\implies$ D-FLC $\forall D \ge c$

Any poly of deg k is $\frac{1}{k}$-FLC

· PROP 1: ~~Poly...&~~ k-hom ⇒ k-FLC

· PROP 2: f, g C-FLC ⇒ fg is also C-FLC

· THM: C-FLC ⇒ C-step D.U. walk mixes in time $\hat{O}(k^c)$

· Example: $G = (V, E)$, $w: V \to \mathbb{R}_{\geq 0}$, $\lambda: E \to \mathbb{R}_{\geq 0}$

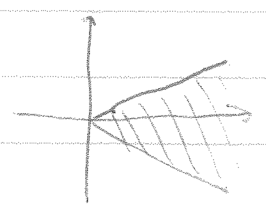
"matchings M" where $\text{weight}(M) = \prod_{e \in M} \lambda(e) \prod_{v \notin M} w(v)$ (distribution on matchings)

Heilmann-Lieb: $\sum_M \text{weight}(M) \prod_{v \notin M} z_v = g((z_v)_{v \in V})$

whenever $z_1, \dots, z_n$ has $Re(z_i) > 0 \Rightarrow g \neq 0$ (Hurwitz-Stable)

$g\left(\frac{z_1}{z_1'}, \dots, \frac{z_n}{z_n'}\right) \prod_{i=1}^{n} z_i'$

$\sum_M w(M) \prod_{v \notin M} z_v \prod_{v \in M} z_v'$ which is non-zero

on the region 

non-hom $\xrightarrow[\text{stable}]{}$ hom is also upper half stable and hence log-concave

$1 + z_1 z_2 \mapsto y^2 + z_1 z_2$

Real-stable $\xRightarrow{2\text{-FLC}}$ log-concave

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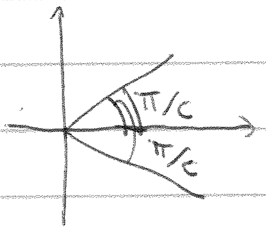


C-sector stability ⇒ C-FLC

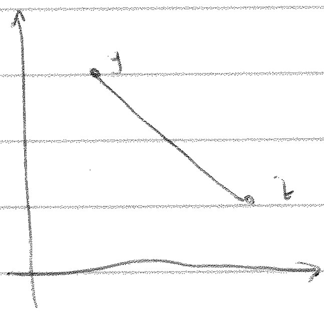
• μ is 1-SS, g_μ stable

want ⇒ g_μ is log-concave

wts $g_\mu(\alpha y + (1-\alpha) z) \geq g_\mu(y)^\alpha g_\mu(z)^{1-\alpha}$

$h(s,t) = g_\mu(sy + tz)$ enough to show

$h(s,t)$ is log-concave



h is a homogeneous poly

$h = c_0 s^0 t^k + c_1 s^1 t^{k-1} + \dots + c_k s^k t^0$

$f(r) = c_0 + c_1 r + \dots + c_k r^k$ $\quad f = h(1,r)$ $\quad r = \alpha + i\beta$

$f(r)$ is real-rooted

• Because $f(r)$ is r.r. ~~Π(r+~~ $f(r) = \prod_{i=1}^{k} (\gamma_i r + \lambda_i)$ positive coefficients

So $h(s,t) = \prod_i (\gamma_i + \lambda_i s)$

• [$A_{n,i}$, Ivvan]

6

- Gårding proved log-concavity above the roots for hyperbolic polynomials

- WTS log-concavity $g_\mu(z_1^{1/c}, \dots, z_n^{1/c})$

POTENTIAL THEORY

$\cancel{\text{THM}: g: \mathbb{C} \to \mathbb{R}, \text{ subharmonic}}$ (i.e. $f(a)$ average $> f(a)$)

$\log|g|$

$\frac{g'}{|g|}$

{equivalently $\Delta g \geq 0$ on all $\mathbb{C}$} (analog of convexity)

- THM: $g: \mathbb{C} \to \mathbb{C}$ holomorphic $\Longrightarrow$ $|g|\, g'$ is subharmonic

- Suppose h is subharmonic, $|h(z)| < O(\log|z|)$

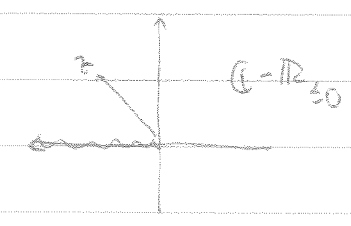
Riesz rep.
$\Longrightarrow$ $\cancel{h(z) = \int}$ $\exists\, \mu$ measure on $\mathbb{C}$ s.t.

$$h(z) = \int_{\mathbb{C}} \log|z - x|\, \mu(dx) + C$$

μ supported on sub-harmonic points (i.e. non-harmonic)

- ~~Say~~ $g(z) = c_0 + c_1 z + \dots + c_n z^n$ $\quad \frac{c}{2}$-SS $\to$ C-FLC

WTS $\frac{c}{2}$-SS $\Longrightarrow$ C-FLC



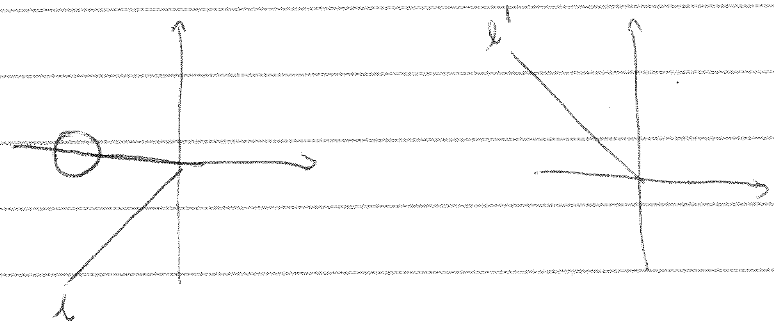
$g(z^{1/c})$ pretend is well-defined

"$= c_0 + c_1 z^{1/c} + \dots + c_n z^{n/c}$" (well defined on $\mathbb{C} - \mathbb{R}_{\leq 0}$)

$\overline{h(z)} = h(\bar{z})$, $\log|h(z)|$ on entire $\mathbb{C}$ and it

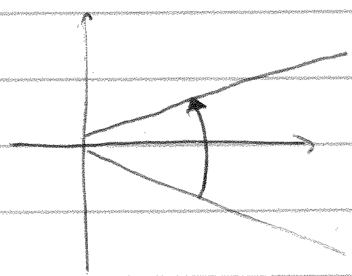
(i.e. it continuously extends to $\mathbb{R}_{\leq 0}$)

WTS subharmonicity on $\mathbb{R}_{\le 0}$



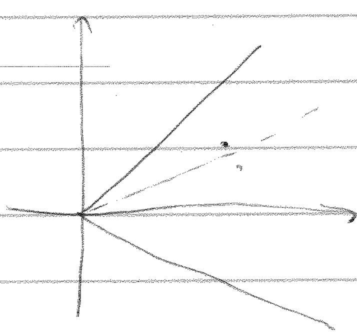
$h = \max(s, s')$ subharmonic

$|s(z_2)| \le |s(z_1)|$



$|g(z)|$ decreasing along the arc

Sahasuebulte -



$\log|g|$ is harmonic

$|g(z)| \le g(|z|)$