TCS@UCSB Summer School on Spectral Independence	**August 8-12, 2022**

Practice Problems

These are practice problems accompanying the 2022 Summer School on *New tools for optimal mixing of Markov chains: Spectral independence and entropy decay*, which was held at the University of California, Santa Barbara (UCSB) from August 8, 2022 to August 12, 2022. More information on the summer school is available at: https://sites.cs.ucsb.edu/~vigoda/School/

1 Exercises

1.1 Linear Algebra

Definition 1. *A matrix norm is a non-negative function $\|\cdot\| \to \mathbb{R}$ that satisfies the following properties*

- $\|A\| = 0$ *implies that A is the zero matrix,*
- $\|cA\| = |c| \cdot \|A\|$,
- $\|A + B\| \leq \|A\| + \|B\|$,
- $\|AB\| \leq \|A\| \, \|B\|$.

Exercise 2. *Show that max-row sum norm*

$$\|A\|_\infty = \max_{i \in [n]} \sum_{j=1}^{n} |A_{ij}|$$

is a matrix norm.

Definition 3. *For a $n \times n$ matrix A the spectral radius $\rho(A)$ of A is*

$$\rho(A) = \max_{\lambda} |\lambda|,$$

where the maximum is over all eigenvalues of A.

Exercise 4. *For any matrix norm $\|\cdot\|$ we have*

$$\rho(A) \leq \|A\|.$$

Exercise 5. *Suppose that A is a symmetric matrix. Let v be a row of A. Show that $\|v\|_2$ is a lower bound on the spectral radius $\rho(A)$.*

Fact 1 (Courant-Fisher-Weyl Variational Characterization of Eigenvalues)**.** *For a symmetric $n \times n$ matrix A the k-th largest eigenvalue λ_k is given by the following Rayleigh quotient.*

$$\min_{L \leq \mathrm{R}^n} \max_{x \in L, x \neq 0} \frac{x^T A x}{\|x\|_2^2},$$

where the first minimization is over $n + 1 - k$-dimensional subspaces L of $\mathbb{R}^n$.

Exercise 6. *Let A, B be $n \times n$ symmetric matrices. Suppose that B is positive semidefinite. Then for every $k \in [n]$*

$$\lambda_k(A - B) \leq \lambda_k(A).$$

1.2 Markov Chain Fundamentals

In the below problems, unless otherwise specified assume that we are considering a finite, discrete, ergodic Markov chain on state space Ω with unique stationary distribution π and transition matrix P.

Exercise 7. *Suppose P is the transition matrix of an ergodic reversible Markov chain with stationary distribution π. Let λ_2 be the second largest eigenvalue of P. Let $Q = \mathrm{diag}(\pi)P$ be the so-called ergodic flow matrix (note that Q is symmetric, since the chain is reversible). Show that*

$$Q - \pi\pi^T \preceq \lambda_2 \mathrm{diag}(\pi).$$

Show that

$$Q - \pi\pi^T \preceq \lambda_2 \left(\mathrm{diag}(\pi) - \pi\pi^T\right).$$

Exercise 8. *Prove that for a lazy reversible Markov chain the spectral gap $1 - \lambda_2$ is given by*

$$1 - \lambda_2 = \min_f \frac{\mathcal{E}(f, f)}{\mathrm{Var}_\pi(f)},$$

where the minimization is over non-constant f.

Exercise 9. *Use Jensen's inequality to show that, for any $A \subset V$,*

$$\mathrm{Ent}_A(f) \leq \mathrm{Cov}_A(f, \log f),$$

where $\mathrm{Cov}(f, g) = \mu_A[fg] - \mu_A(f)\mu_A(g)$ and $\mathrm{Ent}_A(f) = \mu_A[f \log f] - \mu_A[f]\mu_A[\log f]$. Then, use this fact to show that approximate tensorization implies MLSI (modified log-Sobolev) is $\Omega(1/n)$ for the Glauber dynamics. (More generally, block factorization for a weighting α implies MLSI for the heat-bath block dynamics defined by α.)

Exercise 10. *Let μ_i be a distribution on a finite set Ω_i, $i = 1, 2$. Let $\mu = \mu_1 \times \mu_2$ be the product distribution on $\Omega = \Omega_1 \times \Omega_2$.*

(a) Prove that, for any $f : \Omega \to \mathbb{R}$ we have:

$$\mathrm{Var}_\mu(f) \leq \mu_2(\mathrm{Var}_{\mu_1}(f)) + \mu_1(\mathrm{Var}_{\mu_2}(f)).$$

Note that $\mathrm{Var}_{\mu_2}(f)$ is a function from $\Omega_1 \to \mathbb{R}$ where $\mathrm{Var}_{\mu_2}(f)(x) = \mathrm{Var}_{y \sim \mu_2} f(x, y)$ is the variance of $f(x, y)$ where y is picked from μ_2.

(b) Analogously for entropy show

$$\mathrm{Ent}_\mu(f) \leq \mu_2(\mathrm{Ent}_{\mu_1}(f)) + \mu_1(\mathrm{Ent}_{\mu_2}(f)).$$

(c) Use part (b) to prove approximate tensorization of entropy for the uniform distribution over the n-dimensional hypercube $\{0, 1\}^n$.

Exercise 11. *Let π be a distribution on Ω. For $f : \Omega \to \mathbb{R}_{>0}$ let*

$$\mathrm{Var}_\pi(f) = E_\pi(f^2) - E_\pi(f)^2,$$

and

$$\mathrm{Ent}_\pi(f) = E_\pi(f \log f) - E_\pi(f) \log(E_\pi(f)).$$

Show that

$$\lim_{c \to \infty} \mathrm{Ent}_\pi((c + f)^2) = 2\mathrm{Var}_\pi(f).$$

Exercise 12. *Let π be a distribution on Ω and $f : \Omega \to \mathbb{R}_{>0}$. Show that*

$$\mathrm{Ent}_\pi(1 + f/c) = \frac{1}{2}c^{-2}\Big(\mathrm{Var}_\pi(f) + o(1)\Big),$$

and

$$\mathcal{E}(1 + f/c, \log(1 + f/c)) = c^{-2}\Big(\mathcal{E}(f, f) + o(1)\Big).$$

Let α be the Poincaré constant

$$\alpha = \inf_{f;\mathrm{Var}_\pi(f)>0} \frac{\mathcal{E}(f, f)}{\mathrm{Var}_\pi(f)}$$

and ρ_0 be the modified log-Sobolev constant

$$\rho_0 = \inf_{f;\mathrm{Ent}_\pi(f)>0} \frac{\mathcal{E}(f, \log f)}{\mathrm{Ent}_\pi(f)}.$$

Show that $\rho_0 \le 2\alpha$.

1.3 Matroids

Exercise 13. *Show that the lazy matroid basis exchange walk is ergodic.*

Exercise 14. *Let $G = (U \cup V, E)$ be a bipartite graph. Let M_U be the family of subsets of E where edges in a subset are not allowed to share an endpoint in U (they are allowed to share an endpoint in V). Argue that (E, M_U) is a matroid. What is $M_U \cap M_V$ (with M_V defined analogously to M_U)?*

Exercise 15. *Let M be a matroid of rank 2 over set Ω. Let A be $|\Omega| \times |\Omega|$ matrix with zero diagonal entries and for off-diagonal entries $A_{a,b} = 1$ if $\{a, b\} \in M$ and $A_{a,b} = 0$ otherwise. Show that $\lambda_2(A) \le 0$.*

Exercise 16 (The Structure of Graphs with At Most One Positive Eigenvalue)**.** *Let $G = (V, E, c : E \to \mathbb{R}_{>0})$ be a weighted undirected loopless graph without isolated vertices. Let A be its weighted adjacency matrix, and assume A has at most one positive eigenvalue. Prove that G must be supported on a complete multipartite graph, in the sense that there exists a partition $V = V_1 \sqcup \cdots \sqcup V_k$ of the vertices such that $c(u, v) > 0$ if and only if u, v lie in different blocks $V_i \ne V_j$.*

1.4 Spectral Independence and Simplicial Complexes

Exercise 17 (Connectivity of Links and Global Walks)**.** *Fix a pure simplicial complex X. Prove or disprove the following statements.*

(a) If the one-skeleton of every link of X is connected, then the global walk at every level of X is connected.

(b) If the global walk at every level of X is connected, then the one-skeleton of every link of X is connected.

Exercise 18 (Hardcore Model on Complete Bipartite Graphs)**.** *Recall that for a fixed $\lambda > 0$ and a graph $G = (V, E)$, the Gibbs distribution $\mu = \mu_{G,\lambda}$ of the hardcore model on G with parameter λ is defined by*

$$\mu(\sigma) \propto \lambda^{\#\{v:\sigma(v)=1\}}, \quad \forall \sigma \in \{0, 1\}^V \text{ s.t. } \{v : \sigma(v) = 1\} \subseteq V \text{ is an independent set.}$$

Let $\lambda > 0$ be arbitrary, and let μ denote the Gibbs distribution of the hardcore model on the complete bipartite graph $K_{n,n}$.

(1) Give an explicit formula for the univariate partition function

$$\mathcal{Z}(\lambda) = \sum_{I \subseteq V \text{ independent}} \lambda^{|I|}$$

for $K_{n,n}$.

(2) For each level k, explain intuitively what is the structure of the pinning which yields the "worst" spectral independence. More specifically, for each $0 \leq k \leq 2n-2$, explain intuitively what choice of $S \subseteq V$ with $|S| = k$ and $\xi : S \to \{0,1\}$ yields the largest $\lambda_{\max}\left(\Psi_{\mu^\xi}\right)$? (Here, recall Ψ_{μ^ξ} denotes the two-sided influence matrix of the conditional distribution μ^ξ.)

(Hint: What happens if ξ maps some vertex to 1?)

(3) Show that for every $0 \leq k \leq 2n-2$, there exists a pinning $\xi : S \to \{0,1\}$ on a subset of vertices $S \subseteq V$ with $|S| = k$ such that $\lambda_{\max}\left(\Psi_{\mu^\xi}\right) \geq \Omega_\lambda(n-k)$.

(Hint: What is the spectral independence of the worst pinning you constructed in (2)?)

(4) Show that for any fixed $\lambda \geq \Omega(1)$ independent of n, the Glauber dynamics/down-up walk has spectral gap (and hence, mixing time) at least $\exp(\Omega_\lambda(n))$.

(Hint: What moves are required to get from an independent set contained in the left half of $K_{n,n}$ to an independent set contained in the right half of $K_{n,n}$? Based on this, can you find a subset of configurations with poor conductance?)

(5) Conclude that no "average-case local-to-global statement" of the following form can be true in general:

"Let μ be a probability distribution over $\binom{\mathcal{U}}{n}$ for a finite set $\mathcal{U}$ and positive integer n (i.e. the facets of some pure simplicial complex of dimension-$(n-1)$). If for every $\Omega(\log n) \leq k \leq n-2$, $\lambda_{\max}\left(\Psi_{\mu^\xi}\right) \leq O(1)$ with very high probability (e.g. $1 - \frac{1}{\mathsf{poly}(n)}$) over the choice of ξ drawn from the induced level-k distribution μ_k, then the down-up walk has spectral gap $\Omega(1/\mathsf{poly}(n))$."

(Hint: Take μ to be the Gibbs distribution of the hardcore model over $K_{n,n}$ with parameter $\lambda > 0$. Estimate the probability of sampling a partial pinning on k vertices where no vertex is mapped to 1. What happens when you take $\lambda \to +\infty$, or λ to be a large constant?)

Exercise 19 (Top-Level Local Eigenvalues for Proper Colorings)**.** *Fix a pair of vertices u, v connected by an edge, and assume that the set of colors available to each u, v is $[q]$ for some $q \geq 3$. We build a graph on vertex-color pairs satisfying the proper coloring constraint where there is no edge between the pairs (u, c) and (v, c) for any $c \in [q]$, nor (w, c) and (w, c') for any $w \in \{u, v\}, c, c' \in [q]$ distinct. Compute the second largest eigenvalue of this graph.*

Motivation: These graphs can be found as the top links of the following simplicial complex of proper colorings: Fix a graph $G = (V, E)$ and a number of colors q, take the ground set of the complex to consist of all vertex-color pairs, with maximal faces in one-to-one correspondence with complete proper colorings of G.