TCS@UCSB Summer School on Spectral Independence	**August 8-12, 2022**
Open Problems	

These are open problems presented at the 2022 Summer School on *New tools for optimal mixing of Markov chains: Spectral independence and entropy decay*, which was held at the University of California, Santa Barbara (UCSB) from August 8, 2022 to August 12, 2022. More information on the summer school is available at: https://sites.cs.ucsb.edu/∼vigoda/School/

1 Fixed-Size Independent Sets

(Ewan Davies is attending the summer school so feel free to talk with him to discuss the below further.)

[Davies-Perkins] presented an FPRAS for independent sets of a fixed size αn for all $\alpha < \alpha_c$ where $\alpha = \alpha(\lambda)$ and α_c relates to the critical point λ_c of the hard-core model. The running time of their algorithm was further improved by [Jain-Perkins-Sah-Sawhney] who presented a local CLT. However, the mixing time of the simple remove-add chain on independent sets of a fixed size remains open.

Problem 1 (Fixed-size independent sets)**.** *Does approximate tensorization for the hard-core model imply approximate tensorization for fixed-size independent sets?*
(This is related to Conjecture 5 in [Davies-Perkins].)

For context on zero-freeness and its relation to spectral independence see [Chen-Liu-Vigoda].

Problem 2. *Does a zero-free region of the (multivariate) independence polynomial $Z(\lambda_1, \dots, \lambda_n)$ around a point λ imply a zero-free region around λ for the degree-k homogeneous part of $Z(\lambda_1, \dots, \lambda_n)$ where $k = \alpha(\lambda)n$? This would imply spectral independence of the remove-add chain on independent sets of a fixed size k.*

2 Ising Model

[Chen-Liu-Vigoda] proved spectral independence for even subgraphs with constant external field λ; this is based on zero-free results of [Guo-Liao-Lu-Zhang]. It implies a fast MCMC algorithm for the Ising model with constant field λ. In contrast, [Jerrum-Sinclair] presented an FPRAS (with larger polynomial run-time) for the zero field case by proving rapid mixing for a Markov chain on even subgraphs with external field $\rho = \Omega(1/n)$; the chain is defined on all subgraphs where the weight of a subgraph $S \subseteq E$ is $w(S) = \lambda^{|S|}\rho^{\mathrm{odd}(S)}$ and $\mathrm{odd}(S)$ is the number of vertices with odd-degree in S.

Problem 3. *Prove spectral independence for the distribution on even subgraphs for external field $\rho = \Omega(1/n)$. This would yield a faster FPRAS for the Ising model.*

(Check whether spectral independence holds on the cycle of length n.)

3 Matchings

[Chen-Liu-Vigoda] established spectral independence for a Markov chain to generate a random matching for *constant degree graphs*. However, a classical result of [Jerrum-Sinclair] proved rapid mixing for a Markov chain for matchings for any graph.

Problem 4. *Prove spectral independence for matchings on graphs with unbounded degree.*

Conjecture: max eigenvalue of influence matrix is bounded (independent of degree) even though max absolute row sum is not. See Section 8 in [Chen-Liu-Vigoda].

Check whether the conjecture is true on a cycle of length n with every edge replaced by k parallel edges.

4 b-Matchings

A b-matching is a subset of edges with maximum degree $\leq b$; hence the case $b = 1$ are matchings. It was shown in [Huang-Lu-Zhang] that the Glauber dynamics for sampling b-matchings mixes rapidly when $b \leq 7$. The restriction on b is needed in their proof technique which uses canonical paths and the winding method of [McQuillan].

Problem 5. *Is $b \leq 7$ a real threshold, i.e., can one establish rapid mixing (any poly(n) mixing time) for $b = 8$?*

5 Bounded Tree-Width

(Daniel Frishberg is attending the summer school so feel free to talk with him to discuss the below further.)

For the Glauber dynamics on independent sets (i.e., the hard-core model with $\lambda = 1$), [Eppstein-Frishberg] prove mixing time $n^{f(\text{tw})}$ where tw is the tree-width.

Problem 6. *Prove mixing time $\exp(\text{tw}) \times \text{poly}(n)$, i.e., that the mixing time of the Glauber dynamics is fixed parameter tractable (a fixed-parameter tractable algorithm is known via dynamic programming, but the mixing time of the Markov chain is unclear).*

6 Hardcore Model on Random Regular Bipartite Graphs

For the hard-core model on random d-regular bipartite graphs, we know $O(n \log n)$ mixing time of the Glauber dynamics for $\lambda < \lambda_c(d)$ where $\lambda_c(d) = O(1/d)$, and we know an FPTAS via the polymer method (see [Jenssen-Keevash-Perkins]) for $\lambda = \Omega(\log d/d)$ (see [CGSV]).

Problem 7. *Obtain an* FPRAS *for the hard-core model on random d-regular bipartite graphs for all λ.*

Note, [Mossel-Weitz-Wormald] showed that the Glauber dynamics is exponentially slow mixing when $\lambda > \lambda_c(d)$ but one can hope to restrict the chain to one of the two "phases" of mostly even and mostly odd independent sets.

An interesting approach that worked for the antiferromagnetic/ferromagnetic Ising model is [Koehler-Lee-Risteski] which uses [Anari-Jain-Koehler-Pham-Vuong].

7 Avoiding Worst-Case Pinnings

(See related exercise for the hard-core model on the complete bipartite graph for a counterexample to a "first attempt" of the below.)

Can one avoid worst-case pinnings and only consider those which occur with high probability in the stationary distribution? In other words, can one prove a local-to-global result assuming fast mixing for almost all links instead of all links?

Examples where such a result might be useful include:

- *Ferromagnetic Potts model*: it is an open question to obtain an FPRAS for graphs of maximum degree d in the tree uniqueness region. A natural approach is to use the random-cluster model, which is an equivalent formulation of the ferromagnetic Potts model in terms of edge subgraphs. One of the big difficulties is that worst-case pinnings for the random-cluster model can push you into the non-uniqueness region (via wired boundary conditions).

- *Martinelli's problem:* Prove poly(n) mixing time for the Glauber dynamics for the ferromagnetic Ising model on $\sqrt{n} \times \sqrt{n}$ box of $\mathbb{Z}^2$ for low temperature with all-+ boundary condition. Relatively easy to do so for zero-temperature ($\beta = \infty$). Open problem to do so for non-zero temperature (any fixed, finite β).

8 Continuous Space

Extend spectral independence to distributions over continuous space, e.g., $\mathbb{R}^n$. See [Chen-Eldan]. What's the analog of a pinning?

Potential applications:

- Hard-spheres model, see [Helmuth-Perkins-Petti] and [Hayes-Moore].

- Can one sample from Gibbs distributions on a continuous space under some condition such as the PL* condition considered in [Liu-Zhu-Belkin], which is related to log-Sobolev inequalities.

9 Rainbow Spanning Trees

Given a graph where edges are colored with $n-1$ different colors, a rainbow spanning tree is a spanning tree where all edges have distinct colors. This corresponds to the bases in the intersection of two matroids, see Vondrák's notes.

Problem 8. *Is there an* FPRAS *for generating a uniformly random rainbow spanning tree?*

More generally, can one efficiently sample a random basis in the intersection of two matroids? This includes bipartite perfect matchings as a special case, see the [Jerrum-Sinclair-Vigoda] permanent algorithm.

10 Proper Colorings

Problem 9. *Is there an* FPRAS *for generating a uniformly random proper q-coloring of a graph of maximum degree Δ when $q \geq \Delta + 2$?*

Current best is $q \geq \left(\frac{11}{6} - \epsilon\right)\Delta$ due to [Chen-Delcourt-Moitra-Perarnau-Postle] building on [Vigoda] for general graphs; this implies spectral independence. A simple one-step contractive coupling argument for Glauber yields $q \geq 2\Delta + 1$. Lots of work on special classes of graphs (e.g., triangle-free, see [FGYZ] and [CGSV]), large-girth (e.g., [DFHV]), line graphs ([Abdolazimi-Liu-Oveis Gharan]). Any substantial improvement on the constant would be a significant breakthrough.

11 Acyclic Orientations

An orientation of an undirected graph G is an assignment of orientations for every edge of G. An orientation is acyclic if the resulting directed graph does not contain a cycle. The number of acyclic orientations of an undirected graph G is an evaluation of the Tutte polynomial at the point $(2, 0)$ (see, for example, [Goldberg-Jerrum]).

Problem 10. *Is there an* FPRAS *for generating a uniformly random acyclic orientation of an undirected graph?*

There is a bijection between acyclic orientations and the facets of a pure simplicial complex known as the *broken-circuit complex*, which we believe is a local spectral expander/spectrally independent (see Stanley's Lecture Notes and references therein).

12 Expansion of 0/1 Polytopes

A 0/1 polytope is the convex hull of a subset of the hypercube $F \subseteq \{0,1\}^n$. A well-known characterization of matroids is as follows: $F \subseteq \{0,1\}^n$ is the set of indicators of bases of a matroid iff every edge of the convex hull of F is parallel to $e_i - e_j$ for two standard basis elements e_i, e_j.

Motivated to show that basis exchange walks mix rapidly for matroids, Mihail and Vazirani conjectured that the graph $G = (V, E)$ formed by vertices (0-dimensional faces) and edges (1-dimensional faces) of any 0/1 polytope has expansion ≥ 1 (see [Kaibel] for some history):

$$|E(S, V - S)| \geq \min\{|S|, |V - S|\} \quad \forall S \subseteq V$$

We now know the conjecture holds for matroid polytopes [Anari-Liu-OveisGharan-Vinzant] but it remains open for general 0/1 polytopes. Note that this conjecture, even if true, would not immediately give a fast mixing random walk for arbitrary 0/1 polytopes, because degrees in this graph can be exponentially large and the expansion condition only guarantees mixing in poly(max degree) time. A special case of polytopes that are guaranteed to have maximum degree $\leq \text{poly}(n)$ are those with short edges.

Problem 11. *Suppose that the convex hull of $F \subseteq \{0,1\}^n$ has edges (1-dimensional faces) of length at most $O(1)$. Is the uniform distribution on F spectrally independent?*

Note that pinnings/conditionings of this distribution are distributions from the same family. More generally, if one can show spectral independence, it automatically translates to spectral independence under arbitrary external fields, a.k.a. fractional log-concavity [Alimohammadi-Anari-Shiragur-Vuong]. A precise and sharp version of the above conjecture, which nails down the constants, and yields some sharp mixing time bounds [Anari-Jain-Koehler-Pham-Vuong], is as follows:

Problem 12. *Suppose that $F \subseteq \{0,1\}^n$ and every edge of the convex hull of F that goes between $x, y \in F$ is orthogonal to $(1, \ldots, 1)$ and has $\|x - y\|_1 \leq 2c$. Is the uniform distribution on F*

$1/c$-fractionally log-concave? In other words, is the following function of $z_1, \ldots, z_n$ concave on $\mathbb{R}^n_{>0}$?

$$g(z_1, \ldots, z_n) := \log\left(\sum_{x \in F} \prod_{i=1}^{n} z_i^{x_i/c}\right)$$

13 Faster Local Walks

Down-up random walks always mix in polynomial time for spectrally independent distributions, but not necessarily in nearly-linear time. For example, the distribution of monomers in a monomer-dimer system is spectrally independent because of stability of the associated polynomial (Heilmann-Lieb], [Alimohammadi-Anari-Shiragur-Vuong]), but the natural down-up walk or the block dynamics for it mixes in nearly quadratic time in the worst case (see [Anari-Jain-Koehler-Pham-Vuong]). For the worst case examples, it is trivial to come up with other random walks that mix much faster.

Can we in general replace down-up random walks by other, "reasonable", random walks that always mix in nearly-linear time? A potential definition of "reasonable" is random walks that are only allowed to move locally, i.e., by changing at most $O(1)$ coordinates in product spaces, or changing at most $O(1)$ elements in set systems.

Problem 13. *Are there local random walks, perhaps specifically built for the distribution at hand, that mix in nearly-linear time?*