Lecture 1

Simplicial Complexes and Trickling Down Thm

In these two lectures we will introduce <u>simplicial complexes</u> which are higher dimensional analoges of graphs and will prove two main theorems about them:

1. <u>Trickling down Theorem:</u> Study spectral Gap of an underlying graph of a s.c via spectral Gap on its marginals. (local spectral gaps imply global spectral Gap)
2. <u>Random walk Theorem</u> Study RW's convergence on s.c via spectral properties of their marginals.

1. Simplicial Complexes

<u>Def:</u> a Simplicial Complex X is a set system over n elements that is downward closed $\forall\, s \in X,\ s' \subseteq s \Rightarrow s' \in X$.

<u>dimension</u> of $s \in X$ is $|s|-1$

<u>dimension of X</u> maximal dimension of a set in X

a set is <u>maximal</u> if there is no larger set in X that contains it.

X is <u>pure</u> if all its maximal sets are of same dimension.

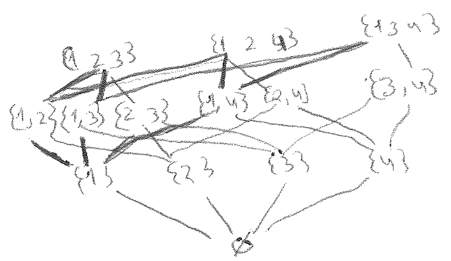
$X(i)$ - sets of X of dim i

$X(-1)$ - $\emptyset$ empty set set of dim -1, i.e with $-1+1=0$ elements.

<u>Links of a Simplicial Complex</u>

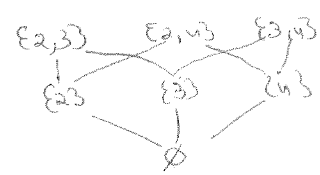
For $\delta \in X$ $\quad X_\delta := \{ s - \delta \mid s \in X,\ \delta \subseteq s \}$ The link of δ, marginal of X.

Example:



X

$X_{\{1\}}$



<u>Weights</u>

For a d dimensional s.c X and a distribution π on its top faces (weighting function that sum up to 1 on top faces $\pi: X(d) \to \mathbb{R}$ $\quad \sum_{\delta \in X(d)} \pi(\delta) = 1$ $\quad \sum_{\delta \in X(d)} \pi(\delta) = 1$

The weight of a $k \le d$ dimensional face τ

$$w(\tau) := \begin{cases} \pi(\tau) & \text{if } \tau \in X(d) \\ \frac{1}{|\tau|+1} \sum_{\substack{\delta \in X(k+1) \\ \tau \subseteq \delta}} w(\delta) & \text{otherwise} \end{cases}$$

averaging of the weights "above" τ.

if π is not specified it is assumed to be uniform.

<u>Weighted links</u>

$$X_\tau := \{ \delta \setminus \tau \mid \delta \in X,\ \tau \subseteq \delta \}$$

$$w_\tau(\beta) = \Pr_{\substack{\beta' \sim \pi \\ \beta' \cup \tau}} [\beta' = \beta \mid \tau \subseteq \beta'] = \frac{w(\beta \cup \tau)}{w(\tau) \binom{|\beta|+|\tau|}{|\tau|}}$$

<u>co-chains</u>

For X of dimension d $\quad -1 \le k \le d$ a <u>k-co-chain</u> is $F: X(k) \to \mathbb{R}$ $\quad C^k(X,\mathbb{R}) := \{ F: X(k) \to \mathbb{R} \}$

functions from k-faces to $\mathbb{R}$.

<u>Inner products</u> Let X be a pure d-dimensional s.c let $F, G \in C^k(X,\mathbb{R})$

Define the inner product $\langle F, G \rangle = \sum_{\delta \in X(k)} w(\delta) \cdot F(\delta)\, G(\delta)$ $\quad$ example for d-reg graphs $G(V,E)$ $|V| = n$ with uniform weights on edges: $\frac{1}{|E|} = \frac{1}{|V| \cdot d/2} = \frac{2}{|V| d}$

1-skeleton of X $\quad (X(0), X(1))$ $\quad (M_\delta^{\prime})^{\tau}(v,u) = \frac{w(\{v,u\})}{2w(v)}$ $\quad w(e)$ $\quad w(v) = \frac{1}{2}\sum w(e)$ $\quad \frac{1}{2} \cdot d \cdot \frac{2}{|V| d} = \frac{1}{|V|}$

Def: λ-local-spectral expanders [Spectral HDE] | $\lambda_2(X(0), X(1))$ Second e.v of weighted adjacency matrix; weighted adj matrix $A_{v,u} = \frac{w(\{v,u\})}{2w(v)}$ $\langle AF, g\rangle = \langle F, Ag\rangle$ (2)

Self adjoint operator; orthogonal basis; orthonormal of eigenvectors; diagonalized; $1 = \lambda_1 \geq \lambda_2 \geq \lambda_3 \dots \lambda_n \geq -1$

A pure d dim s.c. X is λ-local-spectral-expander if for every $\sigma \in X$ of dim at most $d-2$ $\lambda_2(X_\sigma(0), X_\sigma(1)) \leq \lambda$ where $(X_\sigma(0), X_\sigma(1))$ is a weighted graph called the 1-skeleton of X. [this include $\sigma = \emptyset$, i.e 1-skeleton of the whole complex]

In particular for $d=1$ X_0 is a graph and it mean that the second e.v of the weighted adjacency operator describing the graph is at most $\leq \lambda$. ~~for $F \in C^0(X,\mathbb{R})$ $F \perp \mathbb{1}$~~

~~i.e $\langle (M')^+_{X_0} F, F\rangle \leq \lambda \langle F, F\rangle$~~

$$A_g(v,u) = (M')^+_0(v,u) = \frac{w((v,u))}{2w(v)} = w_v(u) \quad \text{if } w(e) = \frac{2}{|V|\cdot d} \ \forall e \quad \frac{1}{2}\frac{2}{Vd}\frac{V}{d}$$

$$w(v) = \frac{1}{2}d\,\frac{2}{|V|d} = \frac{1}{|V|}$$

Trickling down Theorem (Oppenheim) – also local to global spectral expansion

Let X be a pure sc of dim d with connected links. [whose 1-skeleton is connected] if for every vertex v X_v is a λ-local spectral expander then X is a $\frac{\lambda}{1-\lambda}$ local spectral expander

[the non lazy RW matrix $(M')^{+}_0$ adj matrix has second e.v. $\leq \frac{\lambda}{1-\lambda}$]

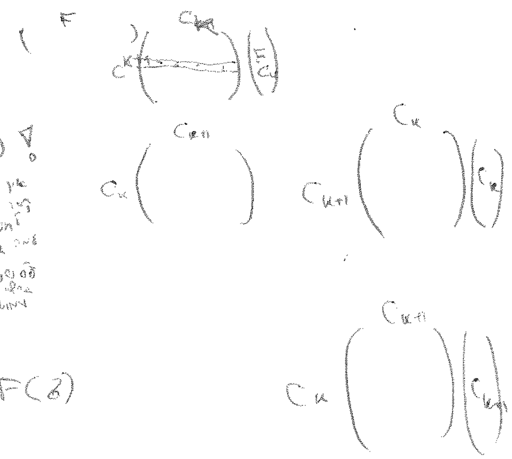
Corollary 1 if X is dim d SC whose 1-skeleton is connected and whose link of $(d-2)$ dim faces are λ spectral expanders then X is $\frac{\lambda}{1-(d-2)\lambda}$ local spectral expander.

Trickle down with loss

Corollary 2: if $d-2$ links are 0-local spec expanders (top links) then X is a 0-local-spec expander.

Trickle down no loss.

Now we move to define RWs on S.C



Signless-Differential Up Operator $\quad d_k : C^k(X,\mathbb{R}) \to C^{k+1}(X,\mathbb{R})$

(down averaging) $\quad d_k F(\sigma) := \underset{\substack{\tau \in X(k) \\ \tau \subseteq \sigma}}{E} F(\tau) = \sum_{\tau \in \binom{\sigma}{k+1}} \frac{1}{k+2} F(\tau)$ (?) [illegible]

Its adjoint operator down operator

[illegible]

$d^*_k : C^{k+1}(X,\mathbb{R}) \to C^k(X,\mathbb{R})$ is the following $F \in C^{k+1}(X,\mathbb{R})$

(up averages) $\quad d^*_k F(\tau) = \underset{\substack{\sigma \in X(k+1) \\ \tau \subseteq \sigma}}{E}[F(\sigma)] = \sum_{\substack{\sigma \in X(k+1) \\ \tau \subseteq \sigma}} w_\tau(\sigma) F(\sigma)$

[illegible]

~~Def Down-up walk $M^-_k = d_{k-1} d^*_{k-1}$~~

$F \in C^k(X,\mathbb{R})$, $G \in C^{k+1}(X,\mathbb{R})$

Claim, d, d^* are adjoint operators, namely $\langle d_k F, G\rangle = \langle F, d^*_k G\rangle$ $\boxed{\text{exercise!}}$

$\langle F, d^*_k G\rangle$

Def Down-up walk $M^-_k := d_{k-1} d^*_{k-1}$

have same eigenvalues with same multiplicities

Up-Down walk $M^+_{k-1} := d^*_{k-1} d_{k-1}$ true for AA^*, A^*A operators

Thm (Convergence of RW in local spec expanders) [KM, DK, KO, AL]

If X is a dim d S.C γ-local-spec exp $k < d$

$$\lambda_2(M^+_{k-1}) = \lambda_2(M^-_k) \leq 1 - \frac{1}{k+1} + \gamma\frac{k}{2} \leq 1 - \frac{1}{k+1}(1-\gamma)^k$$

if $\gamma_i := \max\{\underbrace{\lambda_2(X_\tau(0), X_\tau(1))}_{:=\lambda_\tau} \mid \tau \in X(i)\}$

$$\lambda_2(M^+_{k-1}) = \lambda_2(M^-_k) \leq 1 - \frac{1}{k+1}\prod_{i=-1}^{k-2}(1-\gamma_i)$$

Coro 3 (of RW + Trickling Down) For d dim S.C where links of $d-2$ dimensional faces (top links) are 0-local spectral expanders then $\lambda_2(M^-_k) = \lambda_2(M^+_{k-1}) = 1 - \frac{1}{k+1}$ as those are 0-local spec exp. (45)

This will be important for subsequent talk on matroids, for which the matroids axioms imply that their top links are 0-local-spec-exp and hence $\lambda_2(M^-_k) \leq 1 - \frac{1}{k+1}$: fast mixing for RWs on matroids!

Now we will focus on showing the Trickling down thm. ③

proof (of Trickling down Theorem)

Def Restriction

For $\tau \in X(i)$ (face) and $F \in C^k(X,R)$ (co-chain) Define the restriction of F to X_τ to be

$$F^{\tau}(\sigma) := F(\sigma) \qquad F^{\tau} \in C^k(X_\tau, R)$$

Restriction Lemma: For $F, G \in C^k(X,R)$, X of dim d, $0 \le i \le d-k-1$ $\boxed{\text{exercise}}$

① $\langle F, G\rangle = \mathbb{E}_{\tau \in X(i)} \langle F^{\tau}, G^{\tau}\rangle$ inner product can be expressed by averaging over links

non lazy version

② $\langle M_{0,k}^{+} F, G\rangle = \mathbb{E}_{\tau \in X(i)} \langle M_{\tau,0,k}^{+} F^{\tau}, G^{\tau}\rangle$ RW "

$(M'_\tau)^+_0$ weighted adj matrix of the 1-skeleton of X_τ

no-lazy RW from vertices to vertices via edges

M_0^+ is the ~~lazy~~ lazy RW from vertices to edges to vertices.

$(M')^+_0$ non lazy version of M_0^+ — weighted adj matrix of the 1-skeleton of X $\quad \boxed{\frac{1}{2}(M')^+_0 + \frac{1}{2} I = M_0^+}$

$\boxed{(M')^+_0 = 2M_0^+ - I}$ $\qquad \boxed{A_G = 2M_0^+ - I}$

Our goal now is to understand second e.v of $(M'_X)^+_0$ by eigenvalues in the links of the complex. $(M'_{X_v})^+_0$

Second e.v of the weighted adj matrix of the 1-skeleton of X given knowledge on the second e.v in links of vertices

$$\langle (M')^+_0 F, F\rangle \underset{②}{=} \mathbb{E}_{v \in X(0)} \langle (M')^+_{0,v} F^v, F^v\rangle$$

$F \in C^0(X,R)$ Let F be an eigenfunction of $(M')^+_0$ orthogonal to constants with e.v μ.

$\boxed{F \perp \text{constants}}$ $\quad \boxed{F^v \text{ may be no longer orthogonal to constant at links}}$ $\qquad F^{v\perp}$

$$= \mathbb{E}_{v \in X(0)} \left[\langle (M')^+_{0,v} F^{v\perp}, F^{v\perp}\rangle + \langle (M')^+_{0,v} F^{v\parallel}, F^{v\parallel}\rangle\right]$$

$$\le \mathbb{E}_v \left[\lambda \|F^{v\perp}\|^2 + \|F^{v\parallel}\|^2\right] =$$

$F^{v\parallel}$

$$\mathbb{E}_v \left[\lambda \|F^v\|^2 + (1-\lambda)\|F^{v\parallel}\|^2\right] \qquad \|F^{v\perp}\|^2 = \|F^v\|^2 - \|F^{v\parallel}\|^2$$

F is an eigenfunction of $(M')^+_0$ orthogonal to constant

Assume μ is an eigen~~function~~ val of $(M')^+_0$ (not the one correspond to constants)

$$\mu\|F\|^2 = \langle \mu F, F\rangle = \langle (M')^+_0 F, F\rangle \le \mathbb{E}_v \left[\lambda \|F^v\|^2 + (1-\lambda)\|F^{v\parallel}\|^2\right] \underset{①}{=} \lambda\|F\|^2 + (1-\lambda)\mathbb{E}_v \|F^{v\parallel}\|^2$$

$\sum_{u\sim v} w_v(u) F(u) = ((M')^+_0 F)(v)$

$$\mathbb{E}_v \|F^{v\parallel}\|^2 = ? \qquad F^{v\parallel} = \langle F^v, 1\rangle = (M')^+_0 F(v)$$

$$\mathbb{E}_v \|F^{v\parallel}\|^2 = \|(M')^+_0 F\|^2 = \langle (M')^+_0 F, (M')^+_0 F\rangle = \mu^2\|F\|^2$$

$w_v(u)$

when

$$\mu\|F\|^2 \le \lambda\|F\|^2 + (1-\lambda)\mathbb{E}_v \|F^{v\parallel}\|^2$$

when

so we got

$$\mu\|F\|^2 \le \lambda\|F\|^2 + (1-\lambda)\mu^2\|F\|^2 \qquad \mu \le \lambda + (1-\lambda)\mu^2$$

$$\mu - \mu^2 \le \lambda(1-\mu)(1+\mu) \quad / 1-\mu \qquad \mu \ne 1$$

$1 > \mu$

$$\mu \le \lambda(1+\mu)$$

$$\mu(1-\lambda) \le \lambda \qquad \mu \le \frac{\lambda}{1-\lambda}$$

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