Lecture 2 (4)

Today our goal is to prove the following theorem of [KO, AL] about convergence of RW in local spectral expanders.

- Thm [KO, AL] [Convergence of RW in local spectral expanders]

If X is a dim d s.c $(\gamma_{-1}, \gamma_0 \dots \gamma_{k-2})$-local spectral expander

$\gamma_i := \max \{ \lambda_\tau := \lambda_2 (X_\tau(0), X_\tau(1)) \}$ - [second e.v of the 1-skel of the link of τ]

$-1 \le i \le k-2$ [In γ-local spec exp $\gamma_i \le \gamma$ for all i].

$\lambda_2 (M^+_{k-1}) = \lambda_2 (M^-_k) = 1 - \frac{1}{k+1} \prod_{i=-1}^{k-2} (1-\gamma_i)$

M^+_{k-1}, M^-_k have same non zero eigenvalues, including multiplicities, alway true for AA^*, A^*A operators.

$M^+_{k-1} = A^*A$, $d^*_{k-1} d_{k-1}$

$M^-_k = d_{k-1} d^*_{k-1}$

$\langle d_{k-1} F, G \rangle = \langle F, d^*_{k-1} G \rangle$ $\quad$ $\langle d_{k-1} F, G \rangle = \langle F, d^*_{k-1} G \rangle$

$(d_{k-1})^* = d^*_{k-1}$ $\quad$ $(d_{k-1})^* = d^*_{k-1}$

- Today: main steps based on new analysis based on work with (Roy Gotlib)

- Recall

$d_k: C^k \to C^{k+1}$ $\quad d_k F(\sigma) = \mathbb{E}[F(\tau) \mid \tau \le \sigma] = \sum_{\tau \in \binom{\sigma}{k+1}} \frac{w(\sigma)}{k+2} \frac{1}{w(\sigma)} F(\tau) = \sum_{\tau \in \binom{\sigma}{k+1}} \frac{1}{k+2} F(\tau)$

$d^*_k : C^{k+1} \to C^k$ $\quad d^*_k F(\tau) \quad \mathbb{E}[F(\sigma) \mid \tau \le \sigma] = \sum_{\substack{\sigma \in X(k+1) \\ \tau \le \sigma}} w_\tau(\sigma) F(\sigma) = \mathbb{E}_{\sigma \in X_\tau(0)} [F_\tau(\sigma)]$

$M^+_{k-1} = d^*_{k-1} d_{k-1} \quad M^-_k = d_{k-1} d^*_{k-1}$

(1) Lemma (4) $(M')^+_k$ non lazy version of M^+_k: $\frac{k+2}{k+1} M^+_k - \frac{1}{k+1} I = (M')^+_k$ no lazy version of M^+_k

$M^+_k = \frac{k+1 (M')^+_k + I}{k+2} = M^+_k = \frac{k+1}{k+2} (M')^+_k + \frac{1}{k+2} I$

- Localization

Def-localization: Let $\sigma \in X(i)$ $\quad F \in C^k(X, R)$ $\quad F_\sigma(\tau) := F(\sigma \cup \tau)$ $\quad F_\sigma \in C^{k-|\sigma|}(X_\sigma, R)$

example for $F \in C^1(X, R)$ $\quad F_v \in C^0(X_v, R)$ $\quad F_v(\tau) = F(\tau \cup v)$

localization lemma

$G, F \in C^k(X, R)$

(2) $\langle F, G \rangle = \mathbb{E}_{\sigma \in X(i)} \langle F_\sigma, G_\sigma \rangle$

(3) $\langle (M')^+_k F, F \rangle = \mathbb{E}_{v \in X(0)} \langle (M')^+_{k-1,v} F_v, F_v \rangle$ $\quad (M')^+_{k-1,v}$ non lazy $k-1$ up down walk in the link of v.

- i-level co-chains

$F \in C^k(X, R)$ is an *i level cochain* if localization

$F \in \operatorname{Ker}(d^*_{i-1}, \dots, d^*_{k-1}) \Leftrightarrow \forall \sigma \in X(i-1) \ \langle F_\sigma, 1 \rangle_\sigma = 0$

0-level-cochain orthogonal to constants

1-level orthogonal to constant at every link

note if F is an i-level cochain of vertex it is also $i-1$ level-cochain.

$F \in C^k(X, R)$ is a proper-i-level-cochain if

$F \in \operatorname{Ker}(d^*_{i-1} \dots d^*_{k-1}) \cap \operatorname{Im}(d_{k-1} \dots d_i)$

for every $F \in C^k(X, R)$ there exist a decomposition $F = \sum_{i=0}^k F_i$ $\quad F_i$ are proper level co-chains orthogonal to constants.

$\forall i j \ \langle F_i, F_j \rangle = 0$.

Let $C^k_i(X, R)$ denote the space of i-level-cochains. (30)

Decomposition Thm: Let $F \in C^k(X, R)$ $\quad F = \sum_{i=0}^k F^i$ s.t $F^i \in C^k_i(X, R)$

$\gamma_i = \gamma_{\emptyset, i} := \max_{\sigma \in X(i)} \{ \{\lambda_2(X_\sigma)\} \lambda_\sigma \}$ $\quad -1 \le i \le k-1$ then

$\langle (M')^+_k F, F \rangle \le \sum_{j=0}^k \lambda_{\sigma, i} \| F^i \|^2 + \sum_{i \ne j} c_{ij} \langle F_i, F_j \rangle$ $\quad = \frac{k+1 \left(1 - \frac{1}{k+1} \prod_{j=-1}^{k-1} (1-\gamma_j) \right) + 1}{k+2}$

(3) where $\lambda_{\emptyset, i, k} = 1 - \frac{1}{k+1-i} \prod_{j=i-1}^{k-1} (1-\gamma_j)$ $\quad \lambda_2(M^+_k) = 1 - \frac{1}{k+2} \prod_{j=-1}^{k-1} (1-\gamma_j)$

Claim: Decomposition Thm $\Rightarrow$ RW Thm.

choose F_i's to be proper level co-chain of level i $\quad F_i \in C^k_i \cap (C^k_{i-1})^\perp$ then $\langle F_i, F_j \rangle = 0$

For $i \ne j$ this implies $\lambda_2((M')^+_k) \le 1 - \frac{1}{k+1} \prod_{j=-1}^{k-1} (1-\gamma_j)$ by Lemma (4) $\lambda_2(M^+_{k-1}) = \lambda_2(M^-_k) = 1 - \frac{1}{k+1} \prod_{j=-1}^{k-2} (1-\gamma_j)$

Thus, in order to prove random walk Thm, we need to prove the decomposition theorem (5)

The proof will be based on induction.

For $k=0$ this is reduced to usual expansion (trivial)

$F \in C^0(X,\mathbb{R}) \quad \langle (M')^+_0 F, F\rangle \leq \gamma_{-1}\|F\|^2$.

$\lambda_{\emptyset,0} = \gamma_{-1}$ expansion of the link of the empty set underlying expansion in the complex graph.

$$\lambda_{v,i,k-1} = 1 - \frac{1}{(k-1)-(i)+1}\prod_{j=i-1}^{k-2}(1-\gamma_{v,j})$$

towards proof by induction let perform following Defs:

$\gamma_{v,j} = \max_{\sigma \in X_v(i)} \lambda_\sigma = \lambda_2(X_{v\cup\sigma}) = \lambda_2(X_{v\cup\sigma}) \quad \lambda_2(X_v)_\sigma$

$\gamma_{v_j} \quad \max\{\lambda_2((X_v)_\sigma) \mid \sigma \in X_v(j)\}$

$\gamma_j = \lambda_2(X) \max\{\lambda_2(X_\sigma) \mid \sigma \in X(j)\}$

- $\lambda_{v,i-1,k-1} = 1 - \frac{1}{k}\prod_{j=(i-1)-1}^{(k-1)-1}(1-\gamma_{v_j}) = 1 - \frac{1}{k-i+1}\prod_{j=i-2}^{k-2}(1-\gamma_{v,j})$

$$\lambda_{v,i-1,k-1} = 1 - \frac{1}{k-i+1}\prod_{j=i-2}^{k-2}(1-\gamma_{v,j})$$

- $\lambda_{\emptyset,i,k} = 1 - \frac{1}{k+1-i}\prod_{j=i-1}^{k-1}(1-\gamma_j) = 1 - \frac{1}{k-i+1}\prod_{j=i-1}^{k-1}(1-\gamma_j)$

$$\lambda_{\emptyset,i} = 1 - \frac{1}{k+1-i}\prod_{j=i-1}^{k-1}(1-\gamma_j)$$

- $\boxed{\lambda_{\emptyset,i,k} = \max_v\{\lambda_{v,i-1,k-1}\}} \leq 1 - \frac{1}{k-i+1}\prod_{j=i-1}^{k-1}(1-\gamma_j) \leq \lambda_{\emptyset,i,k}$ so

$\boxed{\begin{array}{l}\lambda_{v,i-1} \leq \lambda_{\emptyset,i} \quad (4)\\ \lambda_{\emptyset,i,k} \leq \lambda_{\emptyset,i}\end{array}}$ $\quad \gamma_j \geq \gamma_{v,j-1}$

$\max_v\{\lambda_{v,i-1,k-1}\} \leq \lambda_{\emptyset,i}$

$\boxed{\lambda_{\emptyset,1,k} \leq \lambda_{\emptyset,0,k}}$

- $\langle (M')^+_k F, F\rangle = E_{v\in X(0)} \langle (M')^+_{k-1,v} F_v, F_v\rangle$
  $= E_v \langle (M')^+_{k-1,v} F_v^\perp, F_v^\perp\rangle + E_v\langle (M')^+_{k-1} F_v^{\|}, F_v^{\|}\rangle$ (2)
  by orthogonality between $F_v^{\|}, F_v^\perp$

$F_v^\perp$ proj of F_v to part orthogonal to constants

$F_v^{\|}$ proj of F_v to constants part

$\|F_v^\perp\|^2 + \|F_v^{\|}\|^2 = \|F_v\|^2$

We want to estimate (k) & (2)

We want to study (k) by applying induction hypothesis.

We need to define for $F_v^\perp$ level functions s.t. $F_v^\perp = \sum_{i=0}^{k-1} F_{v,i}$ $\quad F_{v,i}$ is an i level function.

$F_v = \left(\sum_{i=0}^{k} F^i\right)_v \qquad F_{v,i} \in C^{k-1}_{i,v}(X_v,\mathbb{R})$

$F_v^\perp = \sum_{i=1}^{k} F_v^i + F_v^{0\perp}$ the contribution for F^i's $i\geq 1$ is clearly orthogonal to constants.

$F_v^{\|} = F_v^{0\|}$

Define $F_{v,i} := F_v^{i+1}$ if $i \geq 1 \qquad F_{v,i} = F_v^1 + F_v^{0\perp} \quad i=0$.

(k) $\leq$ by induction hypothesis $E_{v\in X(0)}\left[\sum_{i=0}^{k-1}\lambda_{v,i}\|F_{v,i}\|^2 + \text{Mixed term contributions}_{MT}\right]$ $\quad F_v^0 - F_v^{0\|}$

Note even if F^i's are orthogonal $F_{v,i}$'s are not such!

$\underset{(4)}{\leq} \sum_{i=1}^{k-1}\lambda_{\emptyset,i+1}\|F^{i+1}\|^2 + E_v\left[\lambda_{v,0}\|F_v^1 + F_v^{0\perp}\|^2 + MT\right] \leq \sum_{i=1}^{k}\lambda_{\emptyset,i}\|F^i\|^2 + E_v\left[\lambda_{v,0}\|F_v^{0\perp}\|^2 + MT\right]$

localization thm $E_v\langle F_v^{i+1}, F_v^{i+1}\rangle = \langle F^{i+1}, F^{i+1}\rangle = \|F^{i+1}\|^2 \quad = \sum_{i=1}^{k}\lambda_{\emptyset,i}\|F^i\|^2 + E_v\left[\lambda_{v,0}\left[\|F_v^0\|^2 - \|F_v^{0\|}\|^2\right] + MT\right]$

(*) $= \sum_{i=1}^{k}\lambda_{\emptyset,i}\|F^i\|^2 + \lambda_{\emptyset,1,k}\|F^0\|^2 - \lambda_{\emptyset,1,k}E_v\left[\|F_v^{0\|}\|^2 + MT\right]$

$\lambda_{v,0} \leq \lambda_{\emptyset,1}$

(2) $F_v^{0\|} = d^*_{-1} \quad d^*_{k-2}F^0_v(\emptyset) = \left(d_0^* \ldots d^*_{k-1}F^0\right)_v(\emptyset) \quad d_0^* \qquad d^*_{k-1}F^0(v)$

(2) $= E_v\langle (M')^+_{k-1,v} F_v^{0\|}, F_v^{0\|}\rangle = E_v\langle F_v^{0\|}, F_v^{0\|}\rangle = E_v\langle d_0^* \ldots d^*_{k-1}F^0(v), d_0^* \ldots d^*_{k-1}F(v)\rangle$

(**) $= \|d_0^* \ldots d^*_{k-1}F^0\|^2 = E_v\|F_v^{0\|}\|^2$

(*) + (**) implies $\langle (M')^+_k F, F\rangle \leq \sum_{i=1}^{k}\lambda_{\emptyset,i}\|F^i\|^2 + \lambda_{\emptyset,1,k}\|F^0\|^2 + (1-\lambda_{\emptyset,1,k})E_v\left[\|F_v^{0\|}\|^2 + MT\right] \leq \sum_{i=0}^{k}\lambda_{\emptyset,i}\|F^i\|^2 + MT$.

(5) $\boxed{\lambda_{\emptyset,1,k}\|F^0\|^2 + (1-\lambda_{\emptyset,1,k})E_v\|F_v^{0\|}\|^2 \leq \lambda_{\emptyset,0}\|F^0\|^2}$

$\lambda_{\emptyset,1,k} \leq \lambda_{\emptyset,0}$

(20)

So we have reduced the understanding of the RW Thm (and of the decomposition theorem) [illegible]
to understand wheather the equation ⑤ over F^0 holds.
recursive relation

So we have reduced it to find enough expansion on F^0.

advantage lemma
⑥ Lemma $\quad E_v \|F_v^{0\|}\|^2 \leq \left(1 - \frac{k}{k+1}(1-\gamma^{-1})\right)\|F^0\|^2$ (the advantage) - lemma

[illegible]

Lets first see why given lemma ⑥ equation. ⑤ follows.

need to show

$$\lambda_{\emptyset,1k} + (1-\lambda_{\emptyset,1k})\left(1 - \tfrac{k}{k+1}(1-\gamma^{-1})\right) \leq \lambda_{\emptyset,0}$$

$$\cancel{\lambda_{\emptyset,1k}} + 1 - \tfrac{k}{k+1}(1-\gamma^{-1}) - \cancel{\lambda_{\emptyset,1k}} + \lambda_{\emptyset,1k}\tfrac{k}{k+1}(1-\gamma^{-1})$$

$$= 1 - \frac{k}{k+1}(1-\gamma^{-1}) + \frac{k}{k+1}(1-\gamma^{-1})\left(1 - \frac{1}{\cancel{k}}\prod_{j=0}^{k-1}(1-\gamma_j)\right) = 1 - \frac{1}{k+1}\prod_{j=-1}^{k-1}(1-\gamma_j) = \lambda_{\emptyset,0}.$$ as required.

[illegible]
exp decay of eigen value
Lemma ⑥ is proved using the following Lemma.

⑦ Lemma: For $F^0 \in C^k(X,\mathbb{R})$ $F^0 \in \mathrm{Im}(d_{k-1} \ldots d_0) \cap \mathrm{Ker}(d_{-1}^* \quad d_{k-1}^*)$
i.e F^0 is a proper 0-level co-chain for such function there exist

$F^{=0} \in C^0(X,\mathbb{R})$ s.t
a $\quad d_{-1}F^{=0} = 0$
b $\quad \|F^0\|^2 = \|F^{=0}\|^2$
c $\quad \|d_0^* \quad d_{k-1}^* F^0\|^2 = \|d_{k-1}\ldots d_0 F^{=0}\|^2$

$$E_v\|F_v^{0\|}\|^2 = \|d_0^* \ldots d_{k-1}^* F^0\|^2 = \|d_{k-1} \ldots d_0 F^{=0}\|^2 = \langle M_0^{+k}F^{=0}, F^{=0}\rangle$$

$$M_0^{+k} = d_0^* \ldots d_{k-1}^* d_{k-1} \ldots d_0$$

lemma ⑧ $\boxed{(M')_0^+ = \frac{k+1}{k}M_0^{+k} - \frac{1}{k}I}$ $\qquad M_0^{+k} = \frac{k}{k+1}(M')_0^+ + \frac{1}{k}I$

$$= \left\langle\left(\frac{k}{k+1}(M')_0^+ + \frac{1}{k+1}I\right)F^{=0}, F^{=0}\right\rangle \leq \left(\gamma^{-1}\frac{k}{k+1} + \frac{1}{k+1}\right)\|F^{=0}\|^2 = \left(1 - \frac{\cancel{k}}{k+1} + \gamma^{-1}\frac{k}{k+1}\right)\|F^{=0}\|^2$$

since $d_{-1}^*F^{=0} = 0$ and $\lambda^2((M')_0^+) = \gamma^{-1}$ $\qquad \left(1 - \frac{k}{k+1}(1-\gamma^{-1})\right)\|F^{=0}\|^2$

$$= \left(1 - \frac{k}{k+1}(1-\gamma^{-1})\right)\|F^0\|^2$$ so lemma ⑧ holds ⇒ ⑥ ⇒ ⑤ ⇒ decomposition Thm ⇒ RW - Thm.

$$d_{-1}^* \ldots d_{k-2}^* F_v^0(\emptyset) = d_0^* \ldots d_{k-1}^* F^0(v)$$

$$F_v^{0\|} = d_{-1}^* \ldots d_{k-2}^* F_v^0$$

$$E_v\|F_v^{0\|}\|^2 = \|d_0^* \ldots d_{k-1}^* F^0\|^2.$$