# A Note on Spectral Independence for the Hardcore Model

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## 1 Preliminaries

In this note we will show general techniques for establishing spectral independence. As we saw in the previous lectures of Kuikui Liu, this will imply $O(n \log n)$ mixing time of the Glauber dynamics for constant-degree graphs. We will focus on two techniques. In the first part we will show that correlation decay approaches as used in Weitz's algorithm [Wei06] imply spectral independence. In the second part we will show that stability of the partition function, so-called zero-freeness, also implies spectral independence; such conditions were used in the approximate counting algorithm introduced by Barvinok [Bar16].

### 1.1 Hardcore model

For a graph $G$, let $\mathcal{I}(G)$ denote the collection of all independent sets of $G$. The *hardcore model* on a graph $G = (V, E)$ describes a distribution $\mu_{G,\lambda}$ over $\mathcal{I}(G)$, called the *Gibbs distribution*, with the density of each independent set $I \in \mathcal{I}(G)$ given by

$$\mu_{G,\lambda}(I) = \frac{\lambda^{|I|}}{Z_G(\lambda)}, \tag{1}$$

where $\lambda > 0$ is a parameter called the *fugacity* and $Z_G(\lambda)$ is the *partition function* (also called the *independence polynomial*) defined as

$$Z_G(\lambda) = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}. \tag{2}$$

We say a vertex $v \in V$ is *occupied* if $v \in I$, and *unoccupied* otherwise. The *occupancy ratio* $R_{G,\lambda}(v)$ at $v$ is defined as

$$R_{G,\lambda}(v) = \frac{\mu_{G,\lambda}(v)}{\mu_{G,\lambda}(\bar{v})}, \tag{3}$$

where "$v$" represents the event "$v$ is occupied" and "$\bar{v}$" represents "$v$ is unoccupied".

In some cases we use Greek characters $\sigma, \tau, \xi$... to represent independent sets where we view $\sigma = \mathbf{1}_I \in \{0,1\}^V$ as an indicator vector. For a subset $\Lambda \subseteq V$ of vertices, a partial configuration $\tau \in \{0,1\}^\Lambda$ is feasible if it can be extended to an independent set of $G$ (i.e., $\tau$ is an independent set of $G[\Lambda]$). We call $\tau$ a *pinning* if it is a feasible partial configuration. We further define the hardcore model conditional on a pinning $\tau \in \{0,1\}^\Lambda$ by considering only independent sets with $\Lambda$ fixed to be $\tau$; this allows us to define the conditional Gibbs distribution $\mu^\tau_{G,\lambda} = \mu_{G,\lambda}(\cdot \mid X_\Lambda = \tau)$ and the corresponding partition function $Z^\tau_G(\lambda)$ and occupancy ratios $R^\tau_{G,\lambda}(v)$. Observe that conditioning on $\tau$ is equivalent to removing all unoccupied vertices in $\Lambda$ and removing all occupied vertices in $\Lambda$ together with their neighbors from $G$. Finally, notice that the occupancy ratio can be written as

$$R_{G,\lambda}(v) = \frac{\lambda Z^v_G(\lambda)}{Z^{\bar{v}}_G(\lambda)}, \tag{4}$$

where "$v$" represents the pinning $\tau(v) = 1$ on $\Lambda = \{v\}$, and "$\bar{v}$" represents the pinning $\tau(v) = 0$.

## 1.2 Tree-uniqueness threshold

Fix an integer $d \geq 2$ and a real $\lambda > 0$. Consider the hardcore model on a complete $d$-ary tree of height $h$, denoted by $T_h = T_{d,h}$. The *tree recursion* is a function $F = F_{d,\lambda}$ that can be used to compute the occupancy ratio at the root, defined as

$$F : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad F(R) = \frac{\lambda}{(1+R)^d}. \tag{5}$$

Denote by $R_h = R_{d,\lambda,h}$ the root occupancy ratio for $T_h$; e.g., $R_0 = \lambda$, $R_1 = \lambda/(1+\lambda)^d$. Then one can easily show that $R_h = F(R_{h-1})$. A natural and important question is whether the sequence $\{R_h\}$ converges when $h$ tends to infinity, which is closely related to the Gibbs measure on the *infinite $d$-ary tree*. The answer to this question is determined by whether the (unique) fixed point of $F$ is attractive or repulsive. Denote the unique positive fixed point of $F$ by $R^*$, i.e., $R^*(1+R^*)^d = \lambda$. Define the *critical fugacity* by

$$\lambda_c(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^\Delta}, \tag{6}$$

where $\Delta = d+1$ is the maximum degree of complete $d$-ary trees. It can be shown that if $\lambda \leq \lambda_c(\Delta)$ then the fixed point $R^*$ is attractive and $R_h \to R^*$ as $h \to \infty$, and if instead $\lambda > \lambda_c(\Delta)$ then the fixed point $R^*$ is repulsive and $R_{2h-1} \to R'$, $R_{2h} \to R''$ as $h \to \infty$ for some $R' < R^* < R''$.

Let $\Delta \geq 3$ be an integer, and let $\mathcal{G}_\Delta$ be the family of all graphs of maximum degree at most $\Delta$. The critical fugacity $\lambda_c(\Delta)$ captures phase transitions for the hardcore model in multiple aspects.

- When $\lambda \leq \lambda_c(\Delta)$ there exists a unique Gibbs measure on the infinite $d$-array tree; meanwhile, when $\lambda > \lambda_c(\Delta)$ there are multiple Gibbs measures. For this reason the critical value $\lambda_c(\Delta)$ is called the *tree-uniqueness threshold*.

- When $\lambda < \lambda_c(\Delta)$, for complete $d$-ary trees we have $|R_h - R_{h-1}| = \exp(-\Theta(h))$, which can be viewed as the difference of root occupancy ratios on $T_{h+1}$ between fixing all leaves to be unoccupied (corresponding to $R_h$ on $T_h$) and fixing all leaves to be occupied (corresponding to $R_{h-1}$ on $T_{h-1}$). This describes a spatial mixing/correlation decay property with exponential decay rate, which fails when $\lambda > \lambda_c(\Delta)$.

  More generally, when $\lambda < \lambda_c(\Delta)$, for any graph $G \in \mathcal{G}_\Delta$ of maximum degree at most $\Delta$, for any vertex $v \in V$ and any two pinnings $\sigma, \tau$ on a subset of vertices $\Lambda \subseteq V \setminus v$, it holds $|R^\sigma(v) - R^\tau(v)| = \exp(-\Omega(\ell))$ where $\ell$ is the distance from $v$ to a closest vertex $u \in \Lambda$ such that $\sigma(u) \neq \tau(u)$. This is known as the *strong spatial mixing* property with exponential decay rate; see [Wei06].

- There exists an open set $\Gamma$ of complex numbers containing the interval $[0, \lambda_c(\Delta))$ such that, for all $G \in \mathcal{G}_\Delta$, one has $Z_G(\lambda) \neq 0$ whenever $\lambda \in \Gamma$. Meanwhile, the (complex) zeros of $Z_G(\lambda)$ can be arbitrarily close to $\lambda_c(\Delta)$ for $G \in \mathcal{G}_\Delta$. See [PR19].

- When $\lambda < \lambda_c(\Delta)$, there exists a *fully polynomial-time approximation scheme (FPTAS)* for the partition function $Z_G(\lambda)$ for all $G \in \mathcal{G}_\Delta$ [Wei06, Bar16, PR17], and the Glauber dynamics for sampling from $\mu_{G,\lambda}$ converges in $O(n \log n)$ steps (see Theorem 1 below). Meanwhile, when $\lambda > \lambda_c(\Delta)$ there is no FPTAS/FPRAS for estimating the partition function for $G \in \mathcal{G}_\Delta$ assuming $\mathsf{RP} \neq \mathsf{NP}$ [Sly10, SS14, GŠV16], and the Glauber dynamics has exponential mixing time on random $\Delta$-regular bipartite graphs. The behavior at the critical point $\lambda = \lambda_c(\Delta)$ is still not fully understood yet.

### 1.3 Spectral independence

Consider the hardcore model on a graph $G = (V, E)$ with fugacity $\lambda > 0$. For two distinct vertices $u, v \in V$, the *(pairwise) influence* of $u$ on $v$ is defined by

$$\Psi_{G,\lambda}(u \to v) = \mu_{G,\lambda}(v \mid u) - \mu_{G,\lambda}(v \mid \bar{u}). \tag{7}$$

We also define $\Psi_{G,\lambda}(u \to u) = 0$.[1] For a pinning $\tau$, we also define the influence matrix $\Psi^\tau_{G,\lambda}$ for the conditional Gibbs distribution $\mu^\tau_{G,\lambda}$, where we let $\Psi^\tau_{G,\lambda}(u \to v) = 0$ if $\tau$ forces $u$ to be unoccupied (note that in this case $\Psi^\tau_{G,\lambda}(v \to u) = 0$ by definition). The Gibbs distribution $\mu_{G,\lambda}$ is said to be $\eta$-spectrally independent if for any pinning $\tau$, the maximum eigenvalue of the influence matrix $\Psi^\tau_{G,\lambda}$ is at most $\eta$. Note that all eigenvalues of $\Psi^\tau_{G,\lambda}$ are reals, see [ALO20].

The main purpose of this note is to establish the following spectral independence result for the hardcore model in the tree-uniqueness regime.

**Theorem 1.** *Let $\Delta \geq 3$ be an integer and $\delta \in (0, 1)$ be a real. There exists a constant $\eta > 0$, such that for any graph $G \in \mathcal{G}_\Delta$, any vertex $u \in V(G)$, and any $\lambda \leq (1 - \delta)\lambda_c(\Delta)$, it holds*

$$\sum_{v \in V} |\Psi_{G,\lambda}(u \to v)| \leq \eta. \tag{8}$$

*As a consequence, for any graph $G \in \mathcal{G}_\Delta$ the hardcore distribution $\mu_{G,\lambda}$ is $\eta$-spectrally independent, and the mixing time of the Glauber dynamics for $\mu_{G,\lambda}$ is $O(n \log n)$ where $n = |V(G)|$.*

*Remark* 2. To see why Eq. (8) implies spectral independence, notice that for any $G \in \mathcal{G}_\Delta$ the maximum eigenvalue of $\Psi_{G,\lambda}$ is upper bounded by $\|\Psi_{G,\lambda}\|_\infty = \max_{u \in V} \sum_{v \in V} |\Psi_{G,\lambda}(u \to v)|$, which is at most $\eta$ by Eq. (8). Meanwhile, for any pinning $\tau \in \{0, 1\}^\Lambda$, the conditional Gibbs distribution corresponds to the hardcore model on a smaller graph (removing $\Lambda$ and neighbors of occupied vertices in $\Lambda$) which is also in $\mathcal{G}_\Delta$. Since Eq. (8) applies to all graphs in $\mathcal{G}_\Delta$, the maximum eigenvalue of $\Psi^\tau_{G,\lambda}$ is at most $\eta$ also. Hence, $\eta$-spectral independence follows and optimal mixing of the Glauber dynamics follows from a sequence of recent works, see [CLV21a] for constant-degree graphs and more recently [CFYZ22, CE22] for unbounded maximum degree.

### 1.4 Relating influences and occupancy ratios

Here we give a lemma relating the influences of a vertex $u$ and the occupancy ratio at $u$. It is helpful to consider a more general setting where every vertex $v$ has a distinct fugacity $\lambda_v$. Let

[1] In [CLV20] or Kuikui Liu's lectures it was defined differently as $\Psi_{G,\lambda}(u \to u) = 1$.

$\boldsymbol{\lambda} = (\lambda_v)_{v\in V}$ be a vector of fugacity, and the hardcore distribution is then defined by

$$\mu_{G,\boldsymbol{\lambda}}(I) = \frac{\prod_{v\in I}\lambda_v}{Z_G(\boldsymbol{\lambda})}, \tag{9}$$

where the multivariate partition function (independence polynomial) is defined as

$$Z_G(\boldsymbol{\lambda}) = \sum_{I\in\mathcal{I}(G)}\prod_{v\in I}\lambda_v. \tag{10}$$

Viewing the influences and the occupancy ratios as rational functions of $\boldsymbol{\lambda}$, we have the following relationship.

**Claim 3.** *For two distinct vertices $u, v \in V$, we have*

$$\Psi_G(u\to v;\boldsymbol{\lambda}) = \frac{\partial \log R_G(u;\boldsymbol{\lambda})}{\partial \log \lambda_v} = \frac{\lambda_v}{R_G(u;\boldsymbol{\lambda})}\frac{\partial R_G(u;\boldsymbol{\lambda})}{\partial \lambda_v}.$$

*Proof.* Similarly as Eq. (4), we have

$$R_G(u;\boldsymbol{\lambda}) = \frac{\lambda_u Z_G^u(\boldsymbol{\lambda})}{Z_G^{\bar{u}}(\boldsymbol{\lambda})}, \tag{11}$$

and hence

$$\frac{\partial \log R_G(u;\boldsymbol{\lambda})}{\partial \log \lambda_v} = \frac{\partial}{\partial \log \lambda_v}\log\left(Z_G^u(\boldsymbol{\lambda})\right) - \frac{\partial}{\partial \log \lambda_v}\log\left(Z_G^{\bar{u}}(\boldsymbol{\lambda})\right).$$

We compute that

$$\begin{aligned}
\frac{\partial}{\partial \log \lambda_v}\log\left(Z_G^u(\boldsymbol{\lambda})\right) &= \frac{\lambda_v}{Z_G^u(\boldsymbol{\lambda})}\frac{\partial}{\partial \lambda_v}Z_G^u(\boldsymbol{\lambda}) \\
&= \frac{\lambda_v}{Z_G^u(\boldsymbol{\lambda})}\sum_{I\in\mathcal{I}(G):\, u\in I}\frac{\partial}{\partial \lambda_v}\prod_{w\in I\setminus\{u\}}\lambda_w \\
&= \frac{\lambda_v}{Z_G^u(\boldsymbol{\lambda})}\sum_{I\in\mathcal{I}(G):\, u,v\in I}\prod_{w\in I\setminus\{u,v\}}\lambda_w \\
&= \frac{\lambda_v Z_G^{uv}(\boldsymbol{\lambda})}{Z_G^u(\boldsymbol{\lambda})} = \mu_{G,\boldsymbol{\lambda}}(v \mid u).
\end{aligned}$$

Similarly, we have

$$\frac{\partial}{\partial \log \lambda_v}\log\left(Z_G^{\bar{u}}(\boldsymbol{\lambda})\right) = \mu_{G,\boldsymbol{\lambda}}(v \mid \bar{u}).$$

Therefore, we conclude that

$$\frac{\partial \log R_G(u;\boldsymbol{\lambda})}{\partial \log \lambda_v} = \mu_{G,\boldsymbol{\lambda}}(v \mid u) - \mu_{G,\boldsymbol{\lambda}}(v \mid \bar{u}) = \Psi_G(u\to v;\boldsymbol{\lambda}),$$

as claimed. □

Since the occupancy ratios were intensively studied in previous works for establishing properties like correlation decay or zero-freeness, by Claim 3 we can transform these properties or their proof approaches into results for influences and thus establish spectral independence.

## 2 Spectral Independence via Correlation Decay

In this section, we show Theorem 1 for $\eta = O(1/\delta)$ using an approach based on the strong spatial mixing (correlation decay) property, which appeared in [ALO20, CLV20] and was based on techniques in [Wei06, LLY13].

### 2.1 Proof approach

- We need to show that for any graph $G \in \mathcal{G}_\Delta$ and any vertex $u \in V(G)$ it holds

$$\sum_{v \in V} |\Psi_G(u \to v)| = O(1/\delta).$$

- For a graph $G \in \mathcal{G}_\Delta$ and a vertex $u \in V(G)$, we associate them with a tree rooted at $u$ called the *self-avoiding walk tree* $T = T_{\text{SAW}}(G, u)$, which enumerates all self-avoiding walks starting from $u$. The tree $T$ is in general exponentially large and each vertex of $G$ can appear multiple times in $T$. The maximum degree of $T$ is at most $\Delta$ as well. We can define a hardcore model on $T$, such that
  - The occupancy ratio at $u$ is preserved:

$$R_G(u) = R_T(u).$$

  - The influence from $u$ to another vertex $v$ is preserved:

$$\Psi_G(u \to v) = \sum_{w \in \mathcal{C}_T(v)} \Psi_T(u \to w),$$

    where $\mathcal{C}_T(v)$ denotes the set of all copies of $v$ in $T$.

- Then, it suffices to show that for any *tree* $T \in \mathcal{G}_\Delta$ and any vertex $u \in V(T)$ it holds

$$\sum_{v \in V} |\Psi_T(u \to v)| = O(1/\delta).$$

  This can be proved via the *potential function method* [RST+13, LLY13].

### 2.2 Self-avoiding walk tree

We now define the self-avoiding walk tree more formally. Suppose that there is a total order "<" of vertices of $G$.

**Definition 4** (Self-avoiding walk tree)**.** Let $G = (V, E)$ be a connected graph and $u \in V$ be a vertex. The *self-avoiding walk (SAW) tree* $T = T_{\text{SAW}}(G, u)$ of $G$ rooted at $u$ is a tree consisting of all self-avoiding walks starting from $u$, defined as follows.

- The root of $T$ is $u$;
- Every path from the root $u$ to a leaf corresponds to a "maximal" self-avoiding walk in $G$. More precisely, if $u = v_0\text{-}v_1\text{-}\cdots\text{-}v_{\ell-1}\text{-}v_\ell = v$ is a path from $u$ to a leaf $v$, then it corresponds to a walk in $G$ (i.e., $\{v_{i-1}, v_i\} \in E$) such that:

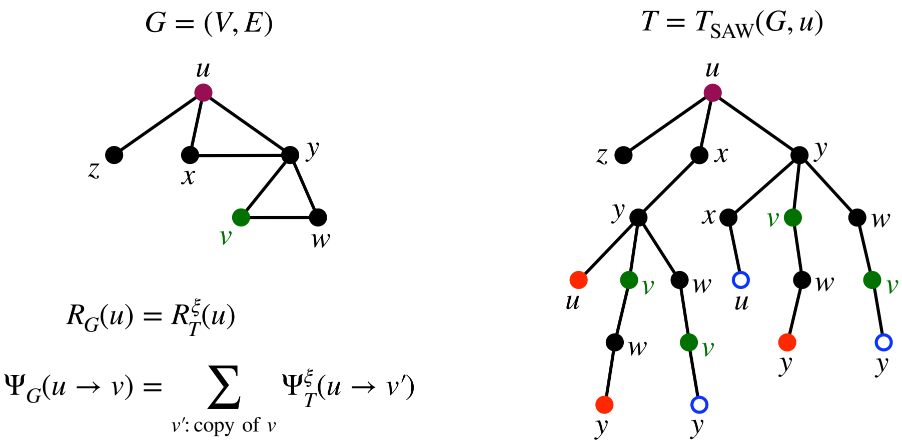


Figure 1: An example of the self-avoiding walk tree and the hardcore model on it. Solid red vertices are those fixed to be occupied in the pinning $\xi$, and hollow blue vertices are those fixed to be unoccupied.

- either $u = v_0, v_1, \dots, v_{\ell-1}, v_\ell = v$ are all distinct vertices (so they form a self-avoiding walk), and $\deg_G(v) = 1$ (so the self-avoiding walk is maximal);
- or $u = v_0, v_1, \dots, v_{\ell-1}$ are all distinct vertices (so they form a self-avoiding walk), and $v = v_\ell = v_i$ for some $i \le \ell - 2$ (so the last vertex $v$ makes a cycle, and the self-avoiding walk is "maximal" in some sense).

*Remark* 5. (1) The maximum degree of $T$ is the same as that of $G$.

(2) Leaves in $T$ correspond to either "pendant vertices" (those of degree 1) in $G$ or those closing a cycle.

(3) Each vertex of $G$ possibly appear multiple times in $T$. We denote the set of all copies in $T$ of a vertex $v \in V(G)$ by $\mathcal{C}_T(v)$.

(4) If $G$ itself is a tree, then $T = G$. However, in general $T$ can be exponentially larger than $G$.

Now for the hardcore model defined on $G$ with the fugacity vector $\boldsymbol{\lambda} = (\lambda_v)_{v \in V}$, we define an associated hardcore model on the self-avoiding walk tree with a specific pinning on some leaves.

**Definition 6** (Hardcore model on $T_{\text{SAW}}(G,u)$)**.** Let $G = (V, E)$ be a connected graph and $u \in V$ be a vertex. Let $T = T_{\text{SAW}}(G, u)$ be the SAW tree of $G$ rooted at $u$. Define the hardcore model on $T$ with a pinning $\xi$ as follows.

- For each vertex $v \in V(G)$, every copy $w \in \mathcal{C}_T(v)$ of $v$ has the same fugacity $\lambda_w = \lambda_v$.
- We define a pinning $\xi$ on a subset of leaves in the following way. Let $u = v_0\text{-}v_1\text{-}\cdots\text{-}v_{\ell-1}\text{-}v_\ell = v$ be a path from $u$ to a leaf $v$.
  - If $\deg_G(v) = 1$, then $v$ is not pinned;
  - Otherwise $v = v_\ell = v_i$ for some $i \le \ell - 2$, and we fix $v_i$ to be occupied if $v_{i+1} < v_{\ell-1}$, and unoccupied if instead $v_{i+1} > v_{\ell-1}$.

*Remark* 7. For a path $u$-$v_1$-$\cdots$-$v_{i-1}$-$v$-$v_{i+1}$-$\cdots$-$v_{\ell-1}$-$v$ from the root $u$ to a leaf $v$ such that $v = v_\ell = v_i$, there is also a path from $u$ to another copy of $v$ in the SAW tree given by $u$-$v_1$-$\cdots$-$v_{i-1}$-$v$-$v_{\ell-1}$-$\cdots$-$v_{i+1}$-$v$, i.e., the order of the cycle is reversed. The pinnings at the two copies of $v$ (both are leaves) are opposite of each other.

Below we still use $\boldsymbol{\lambda}$ to denote the fugacity vector for the hardcore model on $T$. The following lemma relates the hardcore models on $G$ and on the corresponding SAW tree $T$.

**Lemma 8** ([CLV20])**.** *Let $T = T_{\text{SAW}}(G, u)$ be the SAW tree of $G$ rooted at $u$. Consider the hardcore model on $T$ with the pinning $\xi$ as defined in Definition 6.*

*(1) $Z_G(\boldsymbol{\lambda})$ divides $Z_T^\xi(\boldsymbol{\lambda})$. Moreover, there exists a polynomial $P(\boldsymbol{\lambda})$* independent of $\lambda_u$*, such that*

$$Z_T^\xi(\boldsymbol{\lambda}) = Z_G(\boldsymbol{\lambda})P(\boldsymbol{\lambda}).$$

*(2) [Wei06] The occupancy ratio at $u$ is preserved:*

$$R_G(u; \boldsymbol{\lambda}) = R_T^\xi(u; \boldsymbol{\lambda}).$$

*(3) The influence of $u$ on another vertex $v$ is preserved:*

$$\Psi_G(u \to v; \boldsymbol{\lambda}) = \sum_{w \in \mathcal{C}_T(v)} \Psi_T^\xi(u \to w; \boldsymbol{\lambda}).$$

*Proof of "(2) ⇒ (3)".* We deduce from Claim 3 and the chain rule that

$$\begin{aligned}
\Psi_G(u \to v; \boldsymbol{\lambda}) &= \frac{\partial \log R_G(u; \boldsymbol{\lambda})}{\partial \log \lambda_v} && \text{(Claim 3)} \\
&= \frac{\partial \log R_T^\xi(u; \boldsymbol{\lambda})}{\partial \log \lambda_v} && \text{(Part (2))} \\
&= \sum_{w \in \mathcal{C}_T(v)} \frac{\partial \log R_T^\xi(u; \boldsymbol{\lambda})}{\partial \log \lambda_w} \frac{\partial \log \lambda_w}{\partial \log \lambda_v} && \text{(Chain rule)} \\
&= \sum_{w \in \mathcal{C}_T(v)} \Psi_T^\xi(u \to w; \boldsymbol{\lambda}), && \text{(Claim 3)}
\end{aligned}$$

which shows Part (3). □

*Remark* 9. (1) If $w$ is fixed by $\xi$, then $\Psi_T^\xi(u \to w; \boldsymbol{\lambda}) = 0$ by definition.

(2) One can show "(1) ⇒ (2)" using a similar strategy. More generally, it can be shown that the SAW tree $T = T_{\text{SAW}}(G, u)$ preserves all cumulants involving $u$.

**Consequence of Lemma 8.** By the triangle inequality, we deduce that

$$\sum_{v \in V(G)} |\Psi_G(u \to v)| \le \sum_{w \in V(T)} |\Psi_T(u \to w)|.$$

Hence, we reduce the problem to trees (albeit a possibly exponentially large tree).

### 2.3 Bounding influences on trees

In this subsection, we bound the absolute sum of influences of the root on bounded-degree trees. In particular, we show the following result.

**Lemma 10.** *Let $T \in \mathcal{G}_\Delta$ be a tree rooted at $u$. For an integer $k \in \mathbb{N}^+$ and a vertex $v \in V(T)$, let $L_v(k)$ denote the set of all descendants at distance $k$ from $v$. Then for all $k \in \mathbb{N}^+$ we have*

$$\sum_{v \in L_u(k)} |\Psi_T(u \to v)| \le a(1-\delta)^{ck}$$

*where $a, c > 0$ are absolute constants.*

The following claim is helpful to us.

**Claim 11.** *Let $T$ be a tree and $u, v$ be two distinct vertices. If $w$ is a vertex on the unique path from $u$ to $v$, then*

$$\Psi_T(u \to v) = \Psi_T(u \to w)\, \Psi_T(w \to v).$$

*Proof.* Using the Markov property of the hardcore model (i.e., conditional on the value of $w$, the two vertices $u$ and $v$ are independent), we deduce that

$$\begin{aligned}
\Psi_T(u \to v) &= \mu_T(v \mid u) - \mu_T(v \mid \bar{u}) \\
&= \mu_T(w \mid u)\mu_T(v \mid w) + \mu_T(\bar{w} \mid u)\mu_T(v \mid \bar{w}) \\
&\quad - \mu_T(w \mid \bar{u})\mu_T(v \mid w) - \mu_T(\bar{w} \mid \bar{u})\mu_T(v \mid \bar{w}) \\
&= (\mu_T(w \mid u) - \mu_T(w \mid \bar{u}))\, \mu_T(v \mid w) - (\mu_T(\bar{w} \mid \bar{u}) - \mu_T(\bar{w} \mid u))\, \mu_T(v \mid \bar{w}) \\
&= (\mu_T(w \mid u) - \mu_T(w \mid \bar{u}))\, (\mu_T(v \mid w) - \mu_T(v \mid \bar{w})) \\
&= \Psi_T(u \to w)\, \Psi_T(w \to v),
\end{aligned}$$

as claimed. ☐

*Sketch proof of Lemma 10.* By Claim 11 we have

$$\begin{aligned}
\sum_{v \in L_u(k)} |\Psi_T(u \to v)| &= \sum_{w \in L_u(k-1)} \sum_{v \in L_w(1)} |\Psi_T(u \to v)| \\
&= \sum_{w \in L_u(k-1)} |\Psi_T(u \to w)| \sum_{v \in L_w(1)} |\Psi_T(w \to v)|\,.
\end{aligned}$$

If we can show for all $w$,

$$\sum_{v \in L_w(1)} |\Psi_T(w \to v)| \le (1-\delta)^c \tag{12}$$

for some constant $c > 0$, then we are done by induction. This is true in the case of the Ising model when $|\beta| < \beta_c(\Delta)$ in the tree-uniqueness regime. For the hardcore model, Eq. (12) is not true for all $\lambda \le (1-\delta)\lambda_c(\Delta)$, though it is easy to check that Eq. (12) holds when $\lambda \le \frac{1-\delta}{\Delta-2}$ which is below the uniqueness threshold. To overcome this we use the *potential function method*.

The multivariate tree recursion is a function $F = F_{d,\lambda} : \mathbb{R}^d_{\ge 0} \to \mathbb{R}_{\ge 0}$ given by

$$R = F(R_1, \dots, R_d) := \frac{\lambda}{\prod_{i=1}^d (1 + R_i)},$$

which means that for a tree rooted at $w$ with $d$ children $v_1, \ldots, v_d$, the occupancy ratio $R_T(w)$ at the root is given by $R_T(w) = F(R_{T_1}(v_1), \ldots, R_{T_d}(v_d))$ where $T_i$ is the subtree rooted at $v_i$ and $R_{T_i}(v_i)$ is the root occupancy ratio of $T_i$. It would be helpful to consider the logarithm of occupancy ratios in the spirit of Claim 3. Writing $x = \log R$ and $x_i = \log R_i$, we define a multivariate function $H = H_{d,\lambda} : \mathbb{R}^d \to \mathbb{R}$ by

$$x = H(x_1, \ldots, x_d) := \log \lambda - \sum_{i=1}^{d} \log(1 + e^{x_i}).$$

One can check that

$$\frac{\partial H}{\partial x_i} = \Psi_T(w \to v_i)$$

which is similar to Claim 3, and therefore

$$\|\nabla H\|_1 = \sum_{i=1}^{d} |\Psi_T(w \to v_i)| \,.$$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a suitable potential function that is monotone increasing, and we consider the tree recursion composed with $\varphi$. That is, let $y = \varphi(x)$ and $y_i = \varphi(x_i)$, and then for $H^\varphi = \varphi \circ H \circ \varphi^{-1}$ we have

$$y = H^\varphi(y_1, \ldots, y_d).$$

Moreover, it is easy to check that

$$\|\nabla H^\varphi\|_1 = \sum_{i=1}^{d} \frac{\varphi'(x)}{\varphi'(x_i)} |\Psi_T(w \to v_i)| \,. \tag{13}$$

Hence, if we choose $\varphi$ nicely such that $\|\nabla H^\varphi\|_1 \le (1-\delta)^c$ and $\varphi'$ is bounded, then we can prove by induction that

$$\sum_{v \in L_u(k)} \frac{\varphi'(x_u)}{\varphi'(x_v)} |\Psi_T(u \to v)| \le (1-\delta)^{ck}, \tag{14}$$

and Lemma 10 follows immediately.

- Base case: For $k = 1$, we can find constant $a > 0$ such that

$$\sum_{v \in L_u(1)} \frac{\varphi'(x_u)}{\varphi'(x_v)} |\Psi_T(u \to v)| \le a(1-\delta)^c.$$

- Inductive step: Suppose Eq. (14) holds for $k-1$. Then

$$\begin{aligned}\sum_{v \in L_u(k)} \frac{\varphi'(x_u)}{\varphi'(x_v)} |\Psi_T(u \to v)| &= \sum_{w \in L_u(k-1)} \frac{\varphi'(x_u)}{\varphi'(x_w)} |\Psi_T(u \to w)| \sum_{v \in L_w(1)} \frac{\varphi'(x_w)}{\varphi'(x_v)} |\Psi_T(w \to v)| \\ &\le \sum_{w \in L_u(k-1)} \frac{\varphi'(x_u)}{\varphi'(x_w)} |\Psi_T(u \to w)| \cdot (1-\delta)^c \le a(1-\delta)^{ck},\end{aligned}$$

  by Claim 11, Eq. (13), and $\|\nabla H^\varphi\|_1 \le (1-\delta)^c$.

Note that $u$ can have $\Delta$ children while any other vertex has at most $\Delta - 1$ children. It remains to choose a suitable potential function. We mention two choices for the hardcore model.

(1) The first one is from [RST+13]:

$$\varphi(x) = \log\left(e^x + \frac{1}{\Delta}\right).$$

For any integer $d \leq \Delta - 1$, and for any $x_i$ and $x = H(x_1, \ldots, x_d)$, it holds

$$\|\nabla H^\varphi\|_1 = \sum_{i=1}^{d} \frac{e^x}{e^x + \frac{1}{\Delta}} \frac{e^{x_i} + \frac{1}{\Delta}}{e^{x_i} + 1} \leq (1-\delta)^c.$$

(2) The second is from [LLY13]:

$$\varphi(x) = \log\left(e^{x/2} + \sqrt{e^x + 1}\right).$$

For any integer $d \leq \Delta - 1$, and for any $x_i$ and $x = H(x_1, \ldots, x_d)$, it holds

$$\|\nabla H^\varphi\|_1 = \sum_{i=1}^{d} \sqrt{\frac{e^x}{e^x + 1}} \sqrt{\frac{e^{x_i}}{e^{x_i} + 1}} \leq (1-\delta)^c.$$

In fact, a general construction of potential functions was given in [LLY13] which works for all two-spin systems, including the hardcore and Ising models, in the tree-uniqueness regime. □

# 3 Spectral Independence via Zero-Freeness

In this section, we prove Theorem 1 using the zero-freeness of the partition function. The approach here is based on [AASV21] with appropriate modifications and generalizations; see also [CLV21b] for a more general setting.

## 3.1 Some preliminaries

For a complex number $\zeta \in \mathbb{C}$ and a real number $r > 0$, let

$$\mathbb{D}(\zeta, r) = \{z \in \mathbb{C} : |z - \zeta| < r\}$$

be the open disk around $\zeta$ of radius $r$. Furthermore, for a subset $A \subseteq \mathbb{C}$ of complex numbers define

$$\mathbb{D}(A, r) = \bigcup_{\zeta \in A} \mathbb{D}(\zeta, r).$$

Let $\overline{\mathbb{D}}(\zeta, r)$ and $\overline{\mathbb{D}}(A, r)$ denote their closure.

Consider the multivariate independence polynomial defined by Eq. (10). The following stability (zero-freeness) result is known and is the basis of our approach.

**Theorem 12** ([PR19, Theorem 4.2])**.** *Let $\Delta \geq 3$ be an integer. For any $\delta \in (0,1)$, there exists $\varepsilon > 0$ such that for any graph $G \in \mathcal{G}_\Delta$, we have $Z(\boldsymbol{\lambda}) \neq 0$ whenever $\lambda_v \in \mathbb{D}([0, (1-\delta)\lambda_c(\Delta)], \varepsilon)$ for each vertex $v$.*

We also need the following lemma from complex analysis.

**Lemma 13** (Schwarz–Pick lemma)**.** *Let $f : \mathbb{D}(0,1) \to \overline{\mathbb{D}}(0,1)$ be a holomorphic function. Then*

$$|f'(0)| \leq 1 - |f(0)|^2 \leq 1.$$

### 3.2 Proof approach

- We need to show that for any graph $G \in \mathcal{G}_\Delta$ and any vertex $u \in V$, it holds

$$\sum_{v \in V} |\Psi_{G,\lambda}(u \to v)| = O(1).$$

- Consider the multivariate case where every vertex has its own fugacity. For a complex number $\zeta \in \mathbb{C}$, define $\boldsymbol{\lambda}(\zeta)$ to be some perturbation of the fugacity vector such that:
  - $\boldsymbol{\lambda}(0) = \lambda \mathbf{1}$ is the uniform fugacity vector.
  - Consider the complex function

$$f(\zeta) = \lambda \log \big( R_G \left(u; \boldsymbol{\lambda}(\zeta)\right) \big).$$

    Then by Theorem 12 $f$ is holomorphic around 0, and by Claim 3 we have

$$f'(0) = \sum_{v \in V} |\Psi_{G,\lambda}(u \to v)|$$

    for a suitable choice of $\boldsymbol{\lambda}(\zeta)$.

  In the actual proof we define $f$ differently from above, so that it is easier to describe the image of $f$ as needed in the next step.

- Show that the function $f$ is holomorphic in $\mathbb{D}(0, \varepsilon)$ and the image of $f$ is contained in $\overline{\mathbb{D}}(0, B)$ where $\varepsilon, B > 0$ are constants. So, Lemma 13, applied to the function $g(z) = \frac{1}{B} f(\varepsilon z)$, implies that

$$\sum_{v \in V} |\Psi_{G,\lambda}(u \to v)| = f'(0) = \frac{B}{\varepsilon} g'(0) \leq \frac{B}{\varepsilon}.$$

### 3.3 Proofs

Fix the graph $G$ and the vertex $u$. For each $v \neq u$ define

$$s_v = \operatorname{sgn}\left(\Psi_{G,\lambda}(u \to v)\right) := \begin{cases} 1, & \Psi_{G,\lambda}(u \to v) \geq 0; \\ -1, & \Psi_{G,\lambda}(u \to v) < 0. \end{cases}$$

Note that $|\Psi_{G,\lambda}(u \to v)| = s_v \Psi_{G,\lambda}(u \to v)$. We then define the perturbed fugacity vector $\boldsymbol{\lambda}(\zeta)$ by $\lambda_v(\zeta) = \lambda + s_v \zeta$ for $v \neq u$ and $\lambda_u(\zeta) = \lambda$. Consider the complex function

$$f(\zeta) = \frac{\lambda}{R_{G,\lambda}(u)} R_G\big(u; \boldsymbol{\lambda}(\zeta)\big).$$

**Claim 14.** *The complex function $f$ is holomorphic in $\mathbb{D}(0, \varepsilon)$.*

*Proof.* As in Eqs. (4) and (11), we can write

$$R_G\big(u; \boldsymbol{\lambda}(\zeta)\big) = \frac{\lambda Z_G^u\big(\boldsymbol{\lambda}(\zeta)\big)}{Z_G^{\bar{u}}\big(\boldsymbol{\lambda}(\zeta)\big)}.$$

For any $\zeta \in \mathbb{D}(0, \varepsilon)$, we observe that $\lambda_v(\zeta) = \lambda + s_v \zeta \in \mathbb{D}(\lambda, \varepsilon) \subseteq \mathbb{D}([0, (1-\delta)\lambda_c(\Delta)], \varepsilon)$ for any $v \neq u$, and hence $Z_G^{\bar{u}}\big(\boldsymbol{\lambda}(\zeta)\big) \neq 0$ by Theorem 12 (note that $Z_G^{\bar{u}}$ is the independence polynomial for the graph $G \setminus u \in \mathcal{G}_\Delta$). Thus, $R_G\big(u; \boldsymbol{\lambda}(\zeta)\big)$ is holomorphic in $\mathbb{D}(0, \varepsilon)$ and so is $f$. □

**Claim 15.** *We have*

$$f'(0) = \sum_{v \in V} \left|\Psi_{G,\lambda}(u \to v)\right|.$$

*Proof.* By the chain rule, we have

$$\begin{aligned} f'(0) &= \frac{\lambda}{R_{G,\lambda}(u)} \frac{\mathrm{d}}{\mathrm{d}\zeta} R_G\big(u; \boldsymbol{\lambda}(\zeta)\big)\bigg|_{\zeta=0} \\ &= \sum_{v \in V \setminus u} \underbrace{\frac{\lambda}{R_{G,\lambda}(u)} \left(\frac{\partial}{\partial \lambda_v} R_G\big(u; \boldsymbol{\lambda}(\zeta)\big)\right)\bigg|_{\zeta=0}}_{=\Psi_{G,\lambda}(u \to v) \text{ by Claim 3}} \underbrace{\left(\frac{\mathrm{d}\lambda_v}{\mathrm{d}\zeta}\right)\bigg|_{\zeta=0}}_{=s_v} \\ &= \sum_{v \in V \setminus u} \left|\Psi_{G,\lambda}(u \to v)\right|, \end{aligned}$$

as claimed. ☐

**Claim 16.** *The image of $f$ is contained in $\overline{\mathbb{D}}\big(0, \lambda^2/(\varepsilon R_{G,\lambda}(u))\big)$.*

*Proof.* Observe that

$$\begin{aligned} & R_G\big(u; \boldsymbol{\lambda}(\zeta)\big) = y \\ \iff \quad & \frac{\lambda Z_G^u\big(\boldsymbol{\lambda}(\zeta)\big)}{Z_G^{\bar{u}}\big(\boldsymbol{\lambda}(\zeta)\big)} = y \\ \iff \quad & \left(-\frac{\lambda}{y}\right) Z_G^u\big(\boldsymbol{\lambda}(\zeta)\big) + Z_G^{\bar{u}}\big(\boldsymbol{\lambda}(\zeta)\big) = 0 \\ \iff \quad & Z_G\big(\boldsymbol{\rho}(\zeta)\big) = 0, \end{aligned}$$

where

$$\rho_v(\zeta) = \begin{cases} \lambda_v(\zeta), & v \neq u; \\ -\frac{\lambda}{y}, & v = u. \end{cases}$$

Hence, we deduce from Theorem 12 that

$$-\frac{\lambda}{y} \notin \mathbb{D}([0, (1-\delta)\lambda_c(\Delta)], \varepsilon).$$

In particular,

$$-\frac{\lambda}{y} \notin \mathbb{D}(0, \varepsilon) \quad \implies \quad y \in \overline{\mathbb{D}}\left(0, \frac{\lambda}{\varepsilon}\right).$$

The claim then follows. ☐

With the arguments in Section 3.2, we conclude that

$$\sum_{v \in V} \left|\Psi_{G,\lambda}(u \to v)\right| \leq \frac{\lambda^2}{\varepsilon^2 R_{G,\lambda}(u)} = O(\lambda/\varepsilon^2),$$

since $R_{G,\lambda}(u) \geq \lambda/(1+\lambda)^{\Delta} = \Omega(\lambda)$ when $\lambda = O(1/\Delta)$ is in the tree-uniqueness regime.

*Remark* 17. Note that $\varepsilon^2$ in the final spectral independence bound comes from two places in Theorem 12. One of them is the zero-free radius around $\lambda$, and the other is around 0.

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