A Note on Spectral Independence for the Hardcore Model

Zongchen Chen

September 5, 2022

These notes were based on the lectures of the author in the 2022 Summer School on New tools for optimal mixing of Markov chains: Spectral independence and entropy decay, which was held at the University of California, Santa Barbara (UCSB) from August 8, 2022 to August 12, 2022. More information on the summer school is available at: https://sites.cs.ucsb.edu/vigoda/School/

1 Preliminaries

In this note we will show general techniques for establishing spectral independence. As we saw in the previous lectures of Kuikui Liu, this will imply $O(n \log n)$ mixing time of the Glauber dynamics for constant-degree graphs. We will focus on two techniques. In the first part we will show that correlation decay approaches as used in Weitz’s algorithm [Wei06] imply spectral independence. In the second part we will show that stability of the partition function, so-called zero-freeness, also implies spectral independence; such conditions were used in the approximate counting algorithm introduced by Barvinok [Bar16].

1.1 Hardcore model

For a graph $G$, let $\mathcal{I}(G)$ denote the collection of all independent sets of $G$. The hardcore model on a graph $G = (V, E)$ describes a distribution $\mu_{G, \lambda}$ over $\mathcal{I}(G)$, called the Gibbs distribution, with the density of each independent set $I \in \mathcal{I}(G)$ given by

$$
\mu_{G, \lambda}(I) = \frac{\lambda^{|I|}}{Z_G(\lambda)},
$$

where $\lambda > 0$ is a parameter called the fugacity and $Z_G(\lambda)$ is the partition function (also called the independence polynomial) defined as

$$
Z_G(\lambda) = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}.
$$

We say a vertex $v \in V$ is occupied if $v \in I$, and unoccupied otherwise. The occupancy ratio $R_{G, \lambda}(v)$ at $v$ is defined as

$$
R_{G, \lambda}(v) = \frac{\mu_{G, \lambda}(v)}{\mu_{G, \lambda}(v)},
$$

where “$v$” represents the event “$v$ is occupied” and “$\bar{v}$” represents “$v$ is unoccupied”.

In some cases we use Greek characters $\sigma, \tau, \xi...$ to represent independent sets where we view $\sigma = 1_I \in \{0, 1\}^V$ as an indicator vector. For a subset $\Lambda \subseteq V$ of vertices, a partial configuration $\tau \in \{0, 1\}^\Lambda$ is feasible if it can be extended to an independent set of $G$ (i.e., $\tau$ is an independent set
of \(G[\Lambda]\)). We call \(\tau\) a *pinning* if it is a feasible partial configuration. We further define the hardcore model conditional on a pinning \(\tau \in \{0,1\}^\Lambda\) by considering only independent sets with \(\Lambda\) fixed to be \(\tau\); this allows us to define the conditional Gibbs distribution \(\mu_{G,\lambda}^\tau = \mu_{G,\lambda}(\cdot \mid X_\Lambda = \tau)\) and the corresponding partition function \(Z_{G,\lambda}(\lambda)\) and occupancy ratios \(R_{G,\lambda}^\tau(v)\). Observe that conditioning on \(\tau\) is equivalent to removing all unoccupied vertices in \(\Lambda\) and removing all occupied vertices in \(\Lambda\) together with their neighbors from \(G\). Finally, notice that the occupancy ratio can be written as

\[
R_{G,\lambda}(v) = \frac{\lambda Z_{G,\lambda}^v(\lambda)}{Z_{G,\lambda}(\lambda)},
\]

where “\(v\)” represents the pinning \(\tau(v) = 1\) on \(\Lambda = \{v\}\), and “\(\bar{v}\)” represents the pinning \(\tau(v) = 0\).

### 1.2 Tree-uniqueness threshold

Fix an integer \(d \geq 2\) and a real \(\lambda > 0\). Consider the hardcore model on a complete \(d\)-ary tree of height \(h\), denoted by \(T_h = T_{d,h}\). The *tree recursion* is a function \(F = F_{d,\lambda}\) that can be used to compute the occupancy ratio at the root, defined as

\[
F : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad F(R) = \frac{\lambda}{(1 + R)^d}. \tag{5}
\]

Denote by \(R_h = R_{d,\lambda,h}\) the root occupancy ratio for \(T_h\); e.g., \(R_0 = \lambda, R_1 = \lambda/(1+\lambda)^d\). Then one can easily show that \(R_h = F(R_{h-1})\). A natural and important question is whether the sequence \(\{R_h\}\) converges when \(h\) tends to infinity, which is closely related to the Gibbs measure on the infinite \(d\)-ary tree. The answer to this question is determined by whether the (unique) fixed point of \(F\) is attractive or repulsive. Denote the unique positive fixed point of \(F\) by \(R^*\), i.e., \(R^*(1 + R^*)^d = \lambda\). Define the critical fugacity by

\[
\lambda_c(\Delta) = \frac{(\Delta - 1)^{\Delta - 1}}{(\Delta - 2)^\Delta}, \tag{6}
\]

where \(\Delta = d + 1\) is the maximum degree of complete \(d\)-ary trees. It can be shown that if \(\lambda \leq \lambda_c(\Delta)\) then the fixed point \(R^*\) is attractive and \(R_h \to R^*\) as \(h \to \infty\), and if instead \(\lambda > \lambda_c(\Delta)\) then the fixed point \(R^*\) is repulsive and \(R_{2h-1} \to R^*\) as \(h \to \infty\) for some \(R'' < R^* < R'^*\).

Let \(\Delta \geq 3\) be an integer, and let \(\mathcal{G}_\Delta\) be the family of all graphs of maximum degree at most \(\Delta\). The critical fugacity \(\lambda_c(\Delta)\) captures phase transitions for the hardcore model in multiple aspects.

- When \(\lambda \leq \lambda_c(\Delta)\) there exists a unique Gibbs measure on the infinite \(d\)-array tree; meanwhile, when \(\lambda > \lambda_c(\Delta)\) there are multiple Gibbs measures. For this reason the critical value \(\lambda_c(\Delta)\) is called the tree-uniqueness threshold.

- When \(\lambda < \lambda_c(\Delta)\), for complete \(d\)-ary trees we have \(|R_h - R_{h-1}| = \exp(-\Theta(h))\), which can be viewed as the difference of root occupancy ratios on \(T_{h+1}\) between fixing all leaves to be unoccupied (corresponding to \(R_h\) on \(T_h\)) and fixing all leaves to be occupied (corresponding to \(R_{h-1}\) on \(T_{h-1}\)). This describes a spatial mixing/correlation decay property with exponential decay rate, which fails when \(\lambda > \lambda_c(\Delta)\).

More generally, when \(\lambda < \lambda_c(\Delta)\), for any graph \(G \in \mathcal{G}_\Delta\) of maximum degree at most \(\Delta\), for any vertex \(v \in V\) and any two pinnings \(\sigma, \tau\) on a subset of vertices \(\Lambda \subseteq V \setminus v\), it holds \(\left| R^\sigma(v) - R^\tau(v) \right| = \exp(-\Omega(\ell))\) where \(\ell\) is the distance from \(v\) to a closest vertex \(u \in \Lambda\) such that \(\sigma(u) \neq \tau(u)\). This is known as the strong spatial mixing property with exponential decay rate; see [Wei06].
• There exists an open set $\Gamma$ of complex numbers containing the interval $[0, \lambda_c(\Delta))$ such that, for all $G \in \mathcal{G}_\Delta$, one has $Z_G(\lambda) \neq 0$ whenever $\lambda \in \Gamma$. Meanwhile, the (complex) zeros of $Z_G(\lambda)$ can be arbitrarily close to $\lambda_c(\Delta)$ for $G \in \mathcal{G}_\Delta$. See [PR19].

• When $\lambda < \lambda_c(\Delta)$, there exists a fully polynomial-time approximation scheme (FPTAS) for the partition function $Z_G(\lambda)$ for all $G \in \mathcal{G}_\Delta$ [Wei06, Bar16, PR17], and the Glauber dynamics for sampling from $\mu_{G,\lambda}$ converges in $O(n \log n)$ steps (see Theorem 1 below). Meanwhile, when $\lambda > \lambda_c(\Delta)$ there is no FPTAS/FPRAS for estimating the partition function for $G \in \mathcal{G}_\Delta$ assuming $\text{RP} \neq \text{NP}$ [Sly10, SS14, GˇSV16], and the Glauber dynamics has exponential mixing time on random $\Delta$-regular bipartite graphs. The behavior at the critical point $\lambda = \lambda_c(\Delta)$ is still not fully understood yet.

1.3 Spectral independence

Consider the hardcore model on a graph $G = (V, E)$ with fugacity $\lambda > 0$. For two distinct vertices $u, v \in V$, the (pairwise) influence of $u$ on $v$ is defined by

$$\Psi_{G,\lambda}(u \to v) = \mu_{G,\lambda}(v | u) - \mu_{G,\lambda}(v | \bar{u}).$$

(7)

We also define $\Psi_{G,\lambda}(u \to u) = 0$.\footnote{In [CLV20] or Kuikui Liu’s lectures it was defined differently as $\Psi_{G,\lambda}(u \to u) = 1$.} For a pinning $\tau$, we also define the influence matrix $\Psi_{G,\lambda}^\tau$ for the conditional Gibbs distribution $\mu_{G,\lambda}^\tau$, where we let $\Psi_{G,\lambda}(u \to v) = 0$ if $\tau$ forces $u$ to be unoccupied (note that in this case $\Psi_{G,\lambda}(v \to u) = 0$ by definition). The Gibbs distribution $\mu_{G,\lambda}$ is said to be $\eta$-spectrally independent if for any pinning $\tau$, the maximum eigenvalue of the influence matrix $\Psi_{G,\lambda}^\tau$ is at most $\eta$. Note that all eigenvalues of $\Psi_{G,\lambda}^\tau$ are reals, see [ALO20].

The main purpose of this note is to establish the following spectral independence result for the hardcore model in the tree-uniqueness regime.

**Theorem 1.** Let $\Delta \geq 3$ be an integer and $\delta \in (0, 1)$ be a real. There exists a constant $\eta > 0$, such that for any graph $G \in \mathcal{G}_\Delta$, any vertex $u \in V(G)$, and any $\lambda \leq (1 - \delta)\lambda_c(\Delta)$, it holds

$$\sum_{v \in V} |\Psi_{G,\lambda}(u \to v)| \leq \eta.$$  

(8)

As a consequence, for any graph $G \in \mathcal{G}_\Delta$ the hardcore distribution $\mu_{G,\lambda}$ is $\eta$-spectrally independent, and the mixing time of the Glauber dynamics for $\mu_{G,\lambda}$ is $O(n \log n)$ where $n = |V(G)|$.

**Remark 2.** To see why Eq. (8) implies spectral independence, notice that for any $G \in \mathcal{G}_\Delta$ the maximum eigenvalue of $\Psi_{G,\lambda}$ is upper bounded by $\|\Psi_{G,\lambda}\|_\infty = \max_{u \in V} \sum_{v \in V} |\Psi_{G,\lambda}(u \to v)|$, which is at most $\eta$ by Eq. (8). Meanwhile, for any pinning $\tau \in \{0,1\}^\Lambda$, the conditional Gibbs distribution corresponds to the hardcore model on a smaller graph (removing $\Lambda$ and neighbors of occupied vertices in $\Lambda$) which is also in $\mathcal{G}_\Delta$. Since Eq. (8) applies to all graphs in $\mathcal{G}_\Delta$, the maximum eigenvalue of $\Psi_{G,\lambda}^\tau$ is at most $\eta$ also. Hence, $\eta$-spectral independence follows and optimal mixing of the Glauber dynamics follows from a sequence of recent works, see [CLV21a] for constant-degree graphs and more recently [CFYZ22, CE22] for unbounded maximum degree.

1.4 Relating influences and occupancy ratios

Here we give a lemma relating the influences of a vertex $u$ and the occupancy ratio at $u$. It is helpful to consider a more general setting where every vertex $v$ has a distinct fugacity $\lambda_v$. Let
\(\lambda = (\lambda_v)_{v \in V}\) be a vector of fugacity, and the hardcore distribution is then defined by

\[
\mu_{G, \lambda}(I) = \frac{\prod_{v \in I} \lambda_v}{Z_G(\lambda)},
\]  

(9)

where the multivariate partition function (independence polynomial) is defined as

\[
Z_G(\lambda) = \sum_{I \in \mathcal{I}(G)} \prod_{v \in I} \lambda_v.
\]  

(10)

Viewing the influences and the occupancy ratios as rational functions of \(\lambda\), we have the following relationship.

**Claim 3.** For two distinct vertices \(u, v \in V\), we have

\[
\Psi_G(u \rightarrow v; \lambda) = \frac{\partial \log R_G(u; \lambda)}{\partial \log \lambda_v} = \frac{\lambda_v}{R_G(u; \lambda)} \frac{\partial R_G(u; \lambda)}{\partial \lambda_v}.
\]

Proof. Similarly as Eq. (4), we have

\[
R_G(u; \lambda) = \frac{\lambda_u Z^u_G(\lambda)}{Z_G^u(\lambda)},
\]  

(11)

and hence

\[
\frac{\partial \log R_G(u; \lambda)}{\partial \log \lambda_v} = \frac{\partial}{\partial \log \lambda_v} \log \left( Z^u_G(\lambda) \right) - \frac{\partial}{\partial \log \lambda_v} \log \left( Z^\bar{u}_G(\lambda) \right).
\]

We compute that

\[
\frac{\partial}{\partial \lambda_v} \log \left( Z^u_G(\lambda) \right) = \frac{\lambda_v}{Z^u_G(\lambda)} \frac{\partial}{\partial \lambda_v} Z^u_G(\lambda) = \frac{\lambda_v}{Z^u_G(\lambda)} \sum_{I \in \mathcal{I}(G): u \in I} \prod_{w \in I \setminus \{v\}} \lambda_w = \frac{\lambda_v}{Z^u_G(\lambda)} \sum_{I \in \mathcal{I}(G): u, v \in I} \prod_{w \in I \setminus \{u, v\}} \lambda_w = \frac{\lambda_v Z^uv_G(\lambda)}{Z^u_G(\lambda)} = \mu_{G, \lambda}(v | u).
\]

Similarly, we have

\[
\frac{\partial}{\partial \log \lambda_v} \log \left( Z^\bar{u}_G(\lambda) \right) = \mu_{G, \lambda}(v | \bar{u}).
\]

Therefore, we conclude that

\[
\frac{\partial \log R_G(u; \lambda)}{\partial \log \lambda_v} = \mu_{G, \lambda}(v | u) - \mu_{G, \lambda}(v | \bar{u}) = \Psi_G(u \rightarrow v; \lambda),
\]

as claimed.

Since the occupancy ratios were intensively studied in previous works for establishing properties like correlation decay or zero-freeness, by Claim 3 we can transform these properties or their proof approaches into results for influences and thus establish spectral independence.
2 Spectral Independence via Correlation Decay

In this section, we show Theorem 1 for \( \eta = O(1/\delta) \) using an approach based on the strong spatial mixing (correlation decay) property, which appeared in [ALO20, CLV20] and was based on techniques in [Wei06, LLY13].

2.1 Proof approach

- We need to show that for any graph \( G \in G_\Delta \) and any vertex \( u \in V(G) \) it holds
  \[
  \sum_{v \in V} |\Psi_G(u \rightarrow v)| = O(1/\delta).
  \]

- For a graph \( G \in G_\Delta \) and a vertex \( u \in V(G) \), we associate them with a tree rooted at \( u \) called the self-avoiding walk tree \( T = T_{\text{SAW}}(G, u) \), which enumerates all self-avoiding walks starting from \( u \). The tree \( T \) is in general exponentially large and each vertex of \( G \) can appear multiple times in \( T \). The maximum degree of \( T \) is at most \( \Delta \) as well. We can define a hardcore model on \( T \), such that
  
  - The occupancy ratio at \( u \) is preserved:
    \[
    R_G(u) = R_T(u).
    \]
  
  - The influence from \( u \) to another vertex \( v \) is preserved:
    \[
    \Psi_G(u \rightarrow v) = \sum_{w \in \mathcal{C}_T(v)} \Psi_T(u \rightarrow w),
    \]
    where \( \mathcal{C}_T(v) \) denotes the set of all copies of \( v \) in \( T \).

- Then, it suffices to show that for any tree \( T \in G_\Delta \) and any vertex \( u \in V(T) \) it holds
  \[
  \sum_{v \in V} |\Psi_T(u \rightarrow v)| = O(1/\delta).
  \]
  This can be proved via the potential function method [RST+13, LLY13].

2.2 Self-avoiding walk tree

We now define the self-avoiding walk tree more formally. Suppose that there is a total order “<” of vertices of \( G \).

**Definition 4** (Self-avoiding walk tree). Let \( G = (V, E) \) be a connected graph and \( u \in V \) be a vertex. The self-avoiding walk (SAW) tree \( T = T_{\text{SAW}}(G, u) \) of \( G \) rooted at \( u \) is a tree consisting of all self-avoiding walks starting from \( u \), defined as follows.

- The root of \( T \) is \( u \);

- Every path from the root \( u \) to a leaf corresponds to a “maximal” self-avoiding walk in \( G \). More precisely, if \( u = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{k-1} \rightarrow v_k \) is a path from \( u \) to a leaf \( v \), then it corresponds to a walk in \( G \) (i.e., \( \{v_{i-1}, v_i\} \in E \) such that:
$G = (V, E)$

$T = T_{\text{SAW}}(G, u)$

$R_G(u) = R^T_\xi(u)$

$\Psi_G(u \to v) = \sum_{v' : \text{copy of } v} \Psi^T_\xi(u \to v')$

Figure 1: An example of the self-avoiding walk tree and the hardcore model on it. Solid red vertices are those fixed to be occupied in the pinning $\xi$, and hollow blue vertices are those fixed to be unoccupied.

Remark 5. (1) The maximum degree of $T$ is the same as that of $G$.

(2) Leaves in $T$ correspond to either “pendant vertices” (those of degree 1) in $G$ or those closing a cycle.

(3) Each vertex of $G$ possibly appear multiple times in $T$. We denote the set of all copies in $T$ of a vertex $v \in V(G)$ by $C_T(v)$.

(4) If $G$ itself is a tree, then $T = G$. However, in general $T$ can be exponentially larger than $G$.

Now for the hardcore model defined on $G$ with the fugacity vector $\lambda = (\lambda_v)_{v \in V}$, we define an associated hardcore model on the self-avoiding walk tree with a specific pinning on some leaves.

Definition 6 (Hardcore model on $T_{\text{SAW}}(G, u)$). Let $G = (V, E)$ be a connected graph and $u \in V$ be a vertex. Let $T = T_{\text{SAW}}(G, u)$ be the SAW tree of $G$ rooted at $u$. Define the hardcore model on $T$ with a pinning $\xi$ as follows.

- For each vertex $v \in V(G)$, every copy $w \in C_T(v)$ of $v$ has the same fugacity $\lambda_w = \lambda_v$.

- We define a pinning $\xi$ on a subset of leaves in the following way. Let $u = v_0 - v_1 - \cdots - v_{\ell-1} - v_\ell = v$ be a path from $u$ to a leaf $v$.
  
  - If $\deg_G(v) = 1$, then $v$ is not pinned;
  
  - Otherwise $v = v_\ell = v_i$ for some $i \leq \ell - 2$, and we fix $v_i$ to be occupied if $v_{i+1} < v_{\ell-1}$, and unoccupied if instead $v_{i+1} > v_{\ell-1}$.
Remark 7. For a path $u-v_1-\cdots-v_{i-1}-v_{i+1}\cdots-v_{\ell-1}-v$ from the root $u$ to a leaf $v$ such that $v = v_\ell = u$, there is also a path from $u$ to another copy of $v$ in the SAW tree given by $u-v_1-\cdots-v_{i-1}-v_{i+1}\cdots-v_{\ell-1}-v$, i.e., the order of the cycle is reversed. The pinnings at the two copies of $v$ (both are leaves) are opposite of each other.

Below we still use $\lambda$ to denote the fugacity vector for the hardcore model on $T$. The following lemma relates the hardcore models on $G$ and on the corresponding SAW tree $T$.

**Lemma 8** ([CLV20]). Let $T = T_{\text{saw}}(G,u)$ be the SAW tree of $G$ rooted at $u$. Consider the hardcore model on $T$ with the pinning $\xi$ as defined in Definition 6.

1. $Z_G(\lambda)$ divides $Z^\xi_T(\lambda)$. Moreover, there exists a polynomial $P(\lambda)$ independent of $\lambda_u$, such that
   \[ Z^\xi_T(\lambda) = Z_G(\lambda)P(\lambda). \]

2. [Wei06] The occupancy ratio at $u$ is preserved:
   \[ R_G(u; \lambda) = R^\xi_T(u; \lambda). \]

3. The influence of $u$ on another vertex $v$ is preserved:
   \[ \Psi_G(u \to v; \lambda) = \sum_{w \in C_T(v)} \Psi^\xi_T(u \to w; \lambda). \]

Proof of “(2) ⇒ (3)”.

We deduce from Claim 3 and the chain rule that
\[
\Psi_G(u \to v; \lambda) = \frac{\partial \log R_G(u; \lambda)}{\partial \log \lambda_v} \tag{Claim 3}
= \frac{\partial \log R^\xi_T(u; \lambda)}{\partial \log \lambda_v} \tag{Part (2)}
= \sum_{w \in C_T(v)} \frac{\partial \log R^\xi_T(u; \lambda)}{\partial \log \lambda_w} \frac{\partial \log \lambda_w}{\partial \log \lambda_v} \tag{Chain rule}
= \sum_{w \in C_T(v)} \Psi^\xi_T(u \to w; \lambda), \tag{Claim 3}
\]
which shows Part (3).

Remark 9. (1) If $w$ is fixed by $\xi$, then $\Psi^\xi_T(u \to w; \lambda) = 0$ by definition.

(2) One can show “(1) ⇒ (2)” using a similar strategy. More generally, it can be shown that the SAW tree $T = T_{\text{saw}}(G,u)$ preserves all cumulants involving $u$.

**Consequence of Lemma 8.** By the triangle inequality, we deduce that
\[
\sum_{v \in V(G)} |\Psi_G(u \to v)| \leq \sum_{w \in V(T)} |\Psi_T(u \to w)|.
\]

Hence, we reduce the problem to trees (albeit a possibly exponentially large tree).
2.3 Bounding influences on trees

In this subsection, we bound the absolute sum of influences of the root on bounded-degree trees. In particular, we show the following result.

**Lemma 10.** Let $T \in \mathcal{G}_\Delta$ be a tree rooted at $u$. For an integer $k \in \mathbb{N}^+$ and a vertex $v \in V(T)$, let $L_v(k)$ denote the set of all descendants at distance $k$ from $v$. Then for all $k \in \mathbb{N}^+$ we have

$$
\sum_{v \in L_u(k)} |\Psi_T(u \rightarrow v)| \leq a(1 - \delta)^ck
$$

where $a, c > 0$ are absolute constants.

The following claim is helpful to us.

**Claim 11.** Let $T$ be a tree and $u, v$ be two distinct vertices. If $w$ is a vertex on the unique path from $u$ to $v$, then

$$
\Psi_T(u \rightarrow v) = \Psi_T(u \rightarrow w) \Psi_T(w \rightarrow v).
$$

**Proof.** Using the Markov property of the hardcore model (i.e., conditional on the value of $w$, the two vertices $u$ and $v$ are independent), we deduce that

$$
\Psi_T(u \rightarrow v) = \mu_T(v | u) - \mu_T(v | \bar{u}) = \mu_T(w | u)\mu_T(v | w) + \mu_T(\bar{w} | u)\mu_T(v | \bar{w}) - \mu_T(w | \bar{u})\mu_T(v | w) - \mu_T(\bar{w} | \bar{u})\mu_T(v | \bar{w}) = \Psi_T(u \rightarrow w) \Psi_T(w \rightarrow v),
$$

as claimed.

**Sketch proof of Lemma 10.** By Claim 11 we have

$$
\sum_{v \in L_u(k)} |\Psi_T(u \rightarrow v)| = \sum_{w \in L_u(k-1)} \sum_{v \in L_w(1)} |\Psi_T(u \rightarrow v)| = \sum_{w \in L_u(k-1)} |\Psi_T(u \rightarrow w)| \sum_{v \in L_w(1)} |\Psi_T(w \rightarrow v)|.
$$

If we can show for all $w$,

$$
\sum_{v \in L_w(1)} |\Psi_T(w \rightarrow v)| \leq (1 - \delta)^c(12)
$$

for some constant $c > 0$, then we are done by induction. This is true in the case of the Ising model when $|\beta| < \beta_c(\Delta)$ in the tree-uniqueness regime. For the hardcore model, Eq. (12) is not true for all $\lambda \leq (1 - \delta)\lambda_c(\Delta)$, though it is easy to check that Eq. (12) holds when $\lambda \leq \frac{1 - \delta}{\Delta - 2}$ which is below the uniqueness threshold. To overcome this we use the potential function method.

The multivariate tree recursion is a function $F = F_{d,\lambda} : \mathbb{R}_{\geq 0}^d \rightarrow \mathbb{R}_{\geq 0}$ given by

$$
R = F(R_1, \ldots, R_d) := \frac{\lambda}{\prod_{i=1}^d (1 + R_i)},
$$

8
which means that for a tree rooted at \( w \) with \( d \) children \( v_1, \ldots, v_d \), the occupancy ratio \( R_T(w) \) at the root is given by \( R_T(w) = F(R_{T_1}(v_1), \ldots, R_{T_d}(v_d)) \) where \( T_i \) is the subtree rooted at \( v_i \) and \( R_{T_i}(v_i) \) is the root occupancy ratio of \( T_i \). It would be helpful to consider the logarithm of occupancy ratios in the spirit of Claim 3. Writing \( x = \log R \) and \( x_i = \log R_i \), we define a multivariate function \( H = H_{d, \lambda} : \mathbb{R}^d \to \mathbb{R} \) by

\[
x = H(x_1, \ldots, x_d) := \log \lambda - \sum_{i=1}^{d} \log(1 + e^{x_i}).
\]

One can check that \( \frac{\partial H}{\partial x_i} = \Psi_T(w \to v_i) \)
which is similar to Claim 3, and therefore

\[
\| \nabla H \|_1 = \sum_{i=1}^{d} |\Psi_T(w \to v_i)|.
\]

Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a suitable potential function that is monotone increasing, and we consider the tree recursion composed with \( \varphi \). That is, let \( y = \varphi(x) \) and \( y_i = \varphi(x_i) \), and then for \( H^\varphi = \varphi \circ H \circ \varphi^{-1} \) we have

\[
y = H^\varphi(y_1, \ldots, y_d).
\]

Moreover, it is easy to check that

\[
\| \nabla H^\varphi \|_1 = \sum_{i=1}^{d} \frac{\varphi'(x)}{\varphi'(x_i)} |\Psi_T(w \to v_i)|.
\]

Hence, if we choose \( \varphi \) nicely such that \( \| \nabla H^\varphi \|_1 \leq (1 - \delta)^c \) and \( \varphi' \) is bounded, then we can prove by induction that

\[
\sum_{v \in T_u(k)} \frac{\varphi'(x_u)}{\varphi'(x_v)} |\Psi_T(u \to v)| \leq (1 - \delta)^ck,
\]

and Lemma 10 follows immediately.

- **Base case:** For \( k = 1 \), we can find constant \( a > 0 \) such that

\[
\sum_{v \in T_u(1)} \frac{\varphi'(x_u)}{\varphi'(x_v)} |\Psi_T(u \to v)| \leq a(1 - \delta)^c.
\]

- **Inductive step:** Suppose Eq. (14) holds for \( k - 1 \). Then

\[
\sum_{v \in T_u(k)} \frac{\varphi'(x_u)}{\varphi'(x_v)} |\Psi_T(u \to v)| = \sum_{w \in T_L(k-1)} \frac{\varphi'(x_u)}{\varphi'(x_w)} |\Psi_T(u \to w)| \sum_{v \in T_w(1)} \frac{\varphi'(x_w)}{\varphi'(x_v)} |\Psi_T(w \to v)|
\leq \sum_{w \in T_L(k-1)} \frac{\varphi'(x_u)}{\varphi'(x_w)} |\Psi_T(u \to w)| \cdot (1 - \delta)^c \leq a(1 - \delta)^ck,
\]

by Claim 11, Eq. (13), and \( \| \nabla H^\varphi \|_1 \leq (1 - \delta)^c \).
Note that \( u \) can have \( \Delta \) children while any other vertex has at most \( \Delta - 1 \) children. It remains to choose a suitable potential function. We mention two choices for the hardcore model.

(1) The first one is from [RST+13]:

\[
\varphi(x) = \log \left( e^x + \frac{1}{\Delta} \right).
\]

For any integer \( d \leq \Delta - 1 \), and for any \( x_i \) and \( x = H(x_1, \ldots, x_d) \), it holds

\[
\|\nabla \log Z(x)\|_1 = \sum_{i=1}^{d} \frac{e^x}{e^x + 1} \frac{e^{x_i} + 1}{\Delta e^{x_i} + 1} \leq (1 - \delta)^c.
\]

(2) The second is from [LLY13]:

\[
\varphi(x) = \log \left( e^{x/2} + \sqrt{e^x + 1} \right).
\]

For any integer \( d \leq \Delta - 1 \), and for any \( x_i \) and \( x = H(x_1, \ldots, x_d) \), it holds

\[
\|\nabla \log Z(x)\|_1 = \sum_{i=1}^{d} \sqrt{\frac{e^x}{e^x + 1}} \sqrt{\frac{e^{x_i}}{e^{x_i} + 1}} \leq (1 - \delta)^c.
\]

In fact, a general construction of potential functions was given in [LLY13] which works for all two-spin systems, including the hardcore and Ising models, in the tree-uniqueness regime.

### 3 Spectral Independence via Zero-Freeness

In this section, we prove Theorem 1 using the zero-freeness of the partition function. The approach here is based on [AASV21] with appropriate modifications and generalizations; see also [CLV21b] for a more general setting.

#### 3.1 Some preliminaries

For a complex number \( \zeta \in \mathbb{C} \) and a real number \( r > 0 \), let

\[
\mathbb{D}(\zeta, r) = \{ z \in \mathbb{C} : |z - \zeta| < r \}
\]

be the open disk around \( \zeta \) of radius \( r \). Furthermore, for a subset \( A \subseteq \mathbb{C} \) of complex numbers define

\[
\mathbb{D}(A, r) = \bigcup_{\zeta \in A} \mathbb{D}(\zeta, r).
\]

Let \( \overline{\mathbb{D}}(\zeta, r) \) and \( \overline{\mathbb{D}}(A, r) \) denote their closure.

Consider the multivariate independence polynomial defined by Eq. (10). The following stability (zero-freeness) result is known and is the basis of our approach.

**Theorem 12** ([PR19, Theorem 4.2]). Let \( \Delta \geq 3 \) be an integer. For any \( \delta \in (0, 1) \), there exists \( \varepsilon > 0 \) such that for any graph \( G \in \mathcal{G}_\Delta \), we have \( Z(\lambda) \neq 0 \) whenever \( \lambda_\varepsilon \in \mathbb{D}([0, (1 - \delta)\lambda_c(\Delta)], \varepsilon) \) for each vertex \( v \).

We also need the following lemma from complex analysis.

**Lemma 13** (Schwarz–Pick lemma). Let \( f : \mathbb{D}(0, 1) \to \overline{\mathbb{D}}(0, 1) \) be a holomorphic function. Then

\[
|f'(0)| \leq 1 - |f(0)|^2 \leq 1.
\]
3.2 Proof approach

- We need to show that for any graph $G \in \mathcal{G}_\Delta$ and any vertex $u \in V$, it holds

$$\sum_{v \in V} |\Psi_{G,\lambda}(u \rightarrow v)| = O(1).$$

- Consider the multivariate case where every vertex has its own fugacity. For a complex number $\zeta \in \mathbb{C}$, define $\lambda(\zeta)$ to be some perturbation of the fugacity vector such that:
  - $\lambda(0) = \lambda 1$ is the uniform fugacity vector.
  - Consider the complex function
    $$f(\zeta) = \lambda \log \left( R_G (u; \lambda(\zeta)) \right).$$

Then by Theorem 12 $f$ is holomorphic around 0, and by Claim 3 we have

$$f'(0) = \sum_{v \in V} |\Psi_{G,\lambda}(u \rightarrow v)|$$

for a suitable choice of $\lambda(\zeta)$.

In the actual proof we define $f$ differently from above, so that it is easier to describe the image of $f$ as needed in the next step.

- Show that the function $f$ is holomorphic in $D(0, \varepsilon)$ and the image of $f$ is contained in $D(0, B)$ where $\varepsilon, B > 0$ are constants. So, Lemma 13, applied to the function $g(z) = \frac{1}{\varepsilon} f(\varepsilon z)$, implies that

$$\sum_{v \in V} |\Psi_{G,\lambda}(u \rightarrow v)| = f'(0) = \frac{B}{\varepsilon} g'(0) \leq \frac{B}{\varepsilon}.$$

3.3 Proofs

Fix the graph $G$ and the vertex $u$. For each $v \neq u$ define

$$s_v = \text{sgn}(\Psi_{G,\lambda}(u \rightarrow v)) := \begin{cases} 1, & \Psi_{G,\lambda}(u \rightarrow v) \geq 0; \\ -1, & \Psi_{G,\lambda}(u \rightarrow v) < 0. \end{cases}$$

Note that $|\Psi_{G,\lambda}(u \rightarrow v)| = s_v \Psi_{G,\lambda}(u \rightarrow v)$. We then define the perturbed fugacity vector $\lambda(\zeta)$ by $\lambda_v(\zeta) = \lambda + s_v \zeta$ for $v \neq u$ and $\lambda_u(\zeta) = \lambda$. Consider the complex function

$$f(\zeta) = \frac{\lambda}{R_{G,\lambda}(u)} R_G (u; \lambda(\zeta)).$$

Claim 14. The complex function $f$ is holomorphic in $D(0, \varepsilon)$.

Proof. As in Eqs. (4) and (11), we can write

$$R_G (u; \lambda(\zeta)) = \frac{\lambda Z_G^u(\lambda(\zeta))}{Z_G^u(\lambda(\zeta))}.$$

For any $\zeta \in D(0, \varepsilon)$, we observe that $\lambda_v(\zeta) = \lambda + s_v \zeta \in D(\lambda, \varepsilon) \subseteq D([0, (1 - \delta)\lambda_c(\Delta)], \varepsilon)$ for any $v \neq u$, and hence $Z_G^u(\lambda(\zeta)) \neq 0$ by Theorem 12 (note that $Z_G^u$ is the independence polynomial for the graph $G \setminus u \in \mathcal{G}_\Delta$). Thus, $R_G (u; \lambda(\zeta))$ is holomorphic in $D(0, \varepsilon)$ and so is $f$. \qed
Claim 15. We have

\[ f'(0) = \sum_{v \in V} |\Psi_{G,\lambda}(u \rightarrow v)|. \]

Proof. By the chain rule, we have

\[
\begin{align*}
\frac{\lambda}{R_{G,\lambda}(u)} \frac{d}{d\zeta} R_{G}(u; \lambda(\zeta)) \bigg|_{\zeta=0} &= \sum\limits_{v \in V \setminus u} \frac{\lambda}{R_{G,\lambda}(u)} \left( \frac{\partial}{\partial \lambda_v} R_{G}(u; \lambda(\zeta)) \right) \bigg|_{\zeta=0} \left( \frac{d\lambda_v}{d\zeta} \right) \bigg|_{\zeta=0} \\
&= \sum\limits_{v \in V \setminus u} |\Psi_{G,\lambda}(u \rightarrow v)| \text{ by Claim 3}
\end{align*}
\]

as claimed. \qed

Claim 16. The image of \( f \) is contained in \( \mathbb{D}(0, \lambda^2/(\varepsilon R_{G,\lambda}(u))) \).

Proof. Observe that

\[
\begin{align*}
R_G(u; \lambda(\zeta)) &= y \\
\Leftrightarrow \frac{\lambda Z^u_G(\lambda(\zeta))}{Z^u_G(\lambda(\zeta))} &= y \\
\Leftrightarrow \left( -\frac{\lambda}{y} \right) Z^u_G(\lambda(\zeta)) + Z^u_G(\lambda(\zeta)) &= 0 \\
\Leftrightarrow Z_G(\rho(\zeta)) &= 0,
\end{align*}
\]

where

\[
\rho_v(\zeta) = \begin{cases} 
\lambda_v(\zeta), & v \neq u; \\
-\frac{\lambda}{y}, & v = u.
\end{cases}
\]

Hence, we deduce from Theorem 12 that

\[-\frac{\lambda}{y} \notin \mathbb{D}([0, (1 - \delta)\lambda c(\Delta)], \varepsilon).\]

In particular,

\[-\frac{\lambda}{y} \notin \mathbb{D}(0, \varepsilon) \implies y \in \mathbb{D} \left( 0, \frac{\lambda}{\varepsilon} \right).\]

The claim then follows. \qed

With the arguments in Section 3.2, we conclude that

\[
\sum_{v \in V} |\Psi_{G,\lambda}(u \rightarrow v)| \leq \frac{\lambda^2}{\varepsilon^2 R_{G,\lambda}(u)} = O(\lambda/\varepsilon^2),
\]

since \( R_{G,\lambda}(u) \geq \lambda/(1 + \lambda)\Delta = \Omega(\lambda) \) when \( \lambda = O(1/\Delta) \) is in the tree-uniqueness regime.

Remark 17. Note that \( \varepsilon^2 \) in the final spectral independence bound comes from two places in Theorem 12. One of them is the zero-free radius around \( \lambda \), and the other is around 0.
References


