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Intro to High Dimensional Expanders and Local to Global theorems

These notes were prepared by Tushant Mittal based on the pair of lectures by Tali Kaufman. This is part of the 2022 Summer School on *New tools for optimal mixing of Markov chains: Spectral independence and entropy decay*, which was held at the University of California, Santa Barbara (UCSB) from August 8, 2022 to August 12, 2022. More information on the summer school is available at: https://sites.cs.ucsb.edu/~vigoda/School/

2.1 Introduction

In these two lectures, we will introduce *simplicial complexes* and prove two (local-to-global) results about them. Namely,

1. Trickling Down Theorem - Bound the spectral gap of the local walks on any link using the local walks on $(d-2)$-links (here d is the dimension of the complex).

2. Random Walk Theorem - Using the spectral gap of local walks for all links to bound the spectral gap of global walks, such as the Glauber dynamics.

2.2 Basic Definitions

A *simplicial complex* is a downward closed set system. It is called *pure* if every set is contained in a subset of maximal size. The *dimension* of the complex is d if the largest set in the system has size $d+1$. We denote $X(k)$ to be the sets of size exactly $k+1$ and define $X(-1) = \{\phi\}$. The *link* of a complex is related to the notion of conditioning. Formally,

Definition 2.1 (Link)**.** *Let $\tau \in X(k)$, then the link at τ is a $(d-k-1)$-dimensional simplicial complex, $X_\tau = \{\sigma \setminus \tau \mid \tau \subset \sigma\}$.*

Weights: Let $w : X(d) \to \mathbb{R}_{\geq 0}$ be a probability measure on the top faces, i.e., $\sum_{\sigma \in X(d)} w(\sigma) = 1$. We can define a measure on the lower levels $X(k)$ by trickling the weights down,

$$\forall \tau \in X(k), \;\; w(\tau) = \frac{1}{k+2} \sum_{\substack{\sigma \in X(k+1), \\ \tau \subset \sigma}} w(\sigma).$$

One can also define the measure on the links by taking marginals as follows,

$$\forall \sigma \in X_\tau, \;\; w_\tau(\sigma) = \frac{w(\tau \cup \sigma)}{\binom{|\tau|+|\sigma|}{|\tau|} w(\tau)}.$$

Cochains: The space of *k-cochains* denoted as $C^k(X,\mathbb{R}) := \{f : X(k) \to \mathbb{R}\}$ is a vector space with an inner product defined as $\langle f, g\rangle_w = \sum_{\sigma\in X(k)} w(\sigma)f(\sigma)g(\sigma)$. We will denote it simply as C^k when there is no ambiguity in the underlying complex.

Definition 2.2 (Up-Down Operators)**.** *For each $k < d$, we define the following pair of adjoint operators between C^k and C^{k+1}.*

(Up-Operator) $\mathrm{P}_k^{\uparrow} : C^k \to C^{k+1}$. Let f be a function that is defined on $(k+1)$-face, i.e, $f \in C^k$. The up operator lifts it to a function on $(k+2)$-faces by averaging over all subsets of size $k+1$. More formally,

$$\mathrm{P}_k^{\uparrow} f(\sigma) = \mathop{\mathbb{E}}_{\substack{\tau\sim X(k)\\ \tau\subset\sigma}}[f(\tau)] = \frac{1}{k+2}\sum_{\substack{\tau\sim X(k)\\ \tau\subset\sigma}} f(\tau)$$

(Down-Operator) $\mathrm{P}_k^{\downarrow} : C^{k+1} \to C^k$. Analogously, one can transform a function on $k+2$ faces to one on $k+1$ faces by averaging over all $(k+2)$-sized faces containing a given $(k+1)$-face.

$$\mathrm{P}_k^{\downarrow} f(\tau) = \mathop{\mathbb{E}}_{\substack{\sigma\sim X(k+1)\\ \tau\subset\sigma}}[f(\sigma)] = \sum_{\substack{\tau\in X(k+1)\\ \tau\subset\sigma}} w_\tau(\sigma)f(\sigma)$$

By composing these two operators appropriately, we can define two kinds of walks at level k.

- (Up-Down Chain) This walk corresponds to the following random process. Start from $\tau \in X(k)$. In the up-step, we sample a random j such that $\tau\cup\{j\} \in X(k+1)$; the probability of picking j is proportional to $w(\tau\cup\{j\})$. Then in the down step, we drop a uniform element of $\tau\cup\{j\}$.
$$\mathrm{P}_k^{\wedge} : C^k \to C^k, \quad \mathrm{P}_k^{\wedge} = \mathrm{P}_{k+1}^{\downarrow}\mathrm{P}_{k+1}^{\uparrow}.$$
This walk has a lazy component as there is a $\frac{1}{k+2}$ chance of returning to τ. To avoid this, we define the non-lazy version $\widetilde{\mathrm{P}}_k^{\wedge}$ by subtracting the identity component,
$$\widetilde{\mathrm{P}}_k^{\wedge} : C^k \to C^k, \quad \mathrm{P}_k^{\wedge} = \frac{k+1}{k+2}\widetilde{\mathrm{P}}_k^{\wedge} + \frac{1}{k+2}I.$$
- (Down-Up Chain) Similarly, we can first remove a uniformly random element i from τ first and then add an element j with probability proportional to the weight of the resulting set $w(\tau\cup\{j\}\setminus\{i\})$,
$$\mathrm{P}_k^{\vee} : C^{k+1} \to C^{k+1}, \quad \mathrm{P}_k^{\vee} = \mathrm{P}_{k+1}^{\uparrow}\mathrm{P}_{k+1}^{\downarrow}.$$

Remark 2.3 (Glauber dynamics). For a spin system, the down-up chain $\mathrm{P}_{n-1}^{\vee}$ for the appropriate simplicial complex is equivalent to the Glauber dynamics. A spin system is defined on a graph $G = (V, E)$ and the Gibbs distribution μ has support $\Omega \subset \{1,\ldots,q\}^V$ for integer $q \geq 2$. The elements of the corresponding simplicial complex are $(vertex, spin)$ pairs $(v, \sigma(v))$ where $v \in V$ and $\sigma(v) \in \{1,\ldots,q\}$.

Let us illustrate the above for the special case of independent sets (this is the hard-core model with $\lambda = 1$); here the Gibbs distribution is uniformly distributed over all independent sets (of any size) of G. The corresponding simplicial complex has dimension $n = |V|$; for every independent set I we have $\{(v,1) \,|\, v \in I\} \cup \{(v,0) \,|\, v \notin I\}$ in $X(n-1)$. The Glauber dynamics which updates the spin at a randomly chosen vertex is equivalent to $\mathrm{P}_{n-1}^{\vee}$.

We are now ready to define the notion of spectral expansion for a simplicial complex. Let $\tau \in X(k)$ for $k \leq d-2$. The *1-skeleton* of X_τ is the weighted graph $G_\tau = (X_\tau(0), X_\tau(1))$. The weight of vertices (or edges) in G_τ are obtained by taking the weights in X and dividing by $w(\tau)$. Note that the sum of the weights of all the vertices (or edges) in G_τ is 1. The eigenvalues of G_τ are solutions of $Ax = \lambda Dx$ where A is the weighted adjacency matrix and D is a diagonal matrix with vertex weights; the weight of a vertex is the sum of the weights of the adjacent edges, this follows from the above definition of vertex/edge weights.

Definition 2.4 (λ-local spectral expanders)**.** *A pure d-dimensional simplicial complex X is a λ-local spectral expander if for every $\tau \in X(k)$ such that $k \leq d-2$, the 1-skeleton of X_τ, which is the weighted graph $G_\tau = (X_\tau(0), X_\tau(1))$, satisfies $\lambda_2(G_\tau) \leq \lambda$.*

Local walk: We refer to the random walk $\widetilde{\mathrm{P}}_0^{\wedge}$ on the 1-skeleton of X_τ as the *local walk*. Note that the spectrum of the local walk on X_τ is the same as the spectrum of G_τ.

2.3 The Trickle-down theorem

Theorem 2.5 (Oppenheim [Opp18])**.** *If X is a pure simplicial complex such that*

(i) Its 1-skeleton is connected,

(ii) $\forall v \in X(0)$, $\lambda_2(X_v(0), X_v(1)) \leq \lambda$,

then, X is a $\frac{\lambda}{1-\lambda}$-local spectral expander.

Corollary 2.6 (Trickle with loss)**.** *If X is d-dimensional and all $(d-2)$-links are λ-expanders, then X is a $\frac{\lambda}{1-(d-2)\lambda}$-local spectral expander.*

Corollary 2.7 (Trickle without loss)**.** *If X is d-dimensional and all $(d-2)$-links are 0-local expanders, then X is a 0-local spectral expander.*

2.4 Proof of Trickle-Down

The goal is to show that if $\lambda_2(X_v) \leq \lambda$ for every $v \in X(0)$, then, we have $\lambda_2(X) \leq \frac{\lambda}{1-\lambda}$. To do so, we will need a way to study functions locally and the key notion we will use here will be that of restriction.

Restriction: Let $\tau \in X(i)$ and let $f \in C^k$. We define the restriction of f to X_τ, $f^\tau \in C^k(X_\tau, \mathbb{R})$, to be $f^\tau(\sigma) = f(\sigma)$. Note that here, $\sigma \in X_\tau(k)$ which means that $\sigma \cup \tau \in X(i+k+1)$ and by the downward closed property, $\sigma \in X(k)$ and therefore f^τ is well-defined.

We wish to bound the second largest eigenvalue of the (weighted) adjacency operator on the (global) graph $G = (X(0), X(1))$. This operator, which we denote as $\widetilde{\mathrm{P}}_0^{\wedge}$, can be seen as the non-lazy part of the up-down operator from vertices to edges and back to vertices. More formally, the up-down operator is $\mathrm{P}_0^{\wedge} = \mathrm{P}_1^{\downarrow}\mathrm{P}_1^{\uparrow} : C^0 \to C^0$ which decomposes as $\mathrm{P}_0^{\wedge} = \frac{1}{2}\widetilde{\mathrm{P}}_0^{\wedge} + \frac{1}{2}I$.

Lemma 2.8 (Restriction Lemma)**.** *For a d-dimensional simplicial complex X, cochains $f, g \in C^k(X, \mathbb{R})$ and $0 \leq i \leq d-k-2$, we have,*

1. $\langle f, g \rangle = \mathbb{E}_{\tau \in X(i)}[\langle f^\tau, g^\tau \rangle]$,

2. $\left\langle \widetilde{\mathrm{P}}_0^{\wedge} f, g \right\rangle = \mathbb{E}_{v\in X(0)}[\left\langle \widetilde{\mathrm{P}}_{0,v}^{\wedge} f^v, g^v \right\rangle]$.

Proof. Exercise. □

Now, we will prove the trickle-down theorem. Let $f \in C^0$ be an eigenfunction perpendicular to the constant function, that is, $(M_0')^+ f = \mu f$ and $\|f\| = 1$. We will bound μ in terms of the local spectral expansion λ.

Proof of Trickle-down.

$$\begin{aligned}
\mu &= \left\langle \widetilde{\mathrm{P}}_0^{\wedge} f, f \right\rangle \\
&= \mathop{\mathbb{E}}_{v\in X(0)} \left[\left\langle \widetilde{\mathrm{P}}_{0,v}^{\wedge} f^v, f^v \right\rangle \right] && \text{Restriction Lemma} \\
&= \mathop{\mathbb{E}}_{v\in X(0)} \left[\left[\left\langle \widetilde{\mathrm{P}}_{0,v}^{\wedge} f^{v\perp}, f^{v\perp} \right\rangle + \left\langle \widetilde{\mathrm{P}}_{0,v}^{\wedge} f^{v\|}, f^{v\|} \right\rangle \right] \right] \\
&\le \mathop{\mathbb{E}}_{v\in X(0)} \left[\left[\lambda \left\| f^{v\perp} \right\|^2 + \left\| f^{v\|} \right\|^2 \right] \right] && \text{Using local expansion} \\
&\le \lambda \mathop{\mathbb{E}}_{v\in X(0)} \left[\|f^v\|^2 \right] + (1-\lambda) \mathop{\mathbb{E}}_{v\in X(0)} \left[\left\| f^{v\|} \right\|^2 \right]. && \left\| f^{v\perp} \right\|^2 = \|f^v\|^2 - \left\| f^{v\|} \right\|^2
\end{aligned}$$

We can bound the second term using the following observation,

$$\left\| f^{v\|} \right\| = |\langle f^v, \mathbf{1}_v \rangle| = \left| \mathop{\mathbb{E}}_{(u,v)\in X(1)} [f^v(u)] \right| = |\widetilde{\mathrm{P}}_0^{\wedge} f(v)|. \tag{2.1}$$

Here, $\mathbf{1}_v$ denotes the constant all-ones function in $C^0(X_v, \mathbb{R})$.

$$\begin{aligned}
\mu &\le \lambda \mathop{\mathbb{E}}_{v\in X(0)} [\langle f^v, f^v \rangle] + (1-\lambda) \mathop{\mathbb{E}}_{v\in X(0)} \left[|\widetilde{\mathrm{P}}_0^{\wedge} f(v)|^2 \right] && \text{Using Eq. (2.1)} \\
&\le \lambda \langle f, f \rangle + (1-\lambda) \left\| \widetilde{\mathrm{P}}_0^{\wedge} f \right\|^2 && \text{Using Restriction lemma} \\
&= \lambda \|f\|^2 + (1-\lambda)\mu^2 \|f\|^2 .
\end{aligned}$$

Solving this finishes the proof,

$$\begin{aligned}
\mu &\le \lambda + (1-\lambda)\mu^2 \\
\mu - \mu^2 &\le \lambda(1-\mu^2) \\
\mu &\le \lambda(1+\mu) \\
\mu &\le \frac{\lambda}{1-\lambda}.
\end{aligned}$$

□

2.5 Random Walk Theorem

Today we will give a proof of the spectral gap of the random walks. Recall the following definitions from the first lecture. Recall the up-down operators from Definition 2.2. We wish to prove rapid mixing of the walks at level k.

Theorem 2.9 (Convergence of RW in local-spectral expander [KM17, DK17, KO20])**.** *If X is a pure d-dimensional simplicial complex which is a γ-local spectral expander, then for any $1 \leq k \leq d$,*

$$\lambda_2(\mathrm{P}_k^{\vee}) \leq 1 - \frac{1}{k+1} + O(\gamma k).$$

Theorem 2.10 ([AL20])**.** *If X is a $(\lambda_{-1}, \cdots, \lambda_{d-2})$-local spectral expander where $\gamma_j = 1 - \max_{\tau \in X(j)} \lambda_2(X_\tau)$, then for any $1 \leq k \leq d$,*

$$\lambda_2(\mathrm{P}_k^{\vee}) = \lambda_2(\mathrm{P}_k^{\wedge}) \leq 1 - \frac{1}{k+1} \prod_{i=-1}^{k-2} \gamma_i.$$

Localization Given a global function, we wish to define a local function on the links. Let $f \in C^k, \sigma \in X(i)$. We define $f_\sigma : X_\sigma(k-i) \to \mathbb{R}$ as $f_\sigma(\tau) = f(\sigma \cup \tau)$.

Example 2.11. Let $f \in C^1$, i.e., a function on edges. For a vertex v, we have $f_v(u) = f((u \cup v))$ where $u \in X_v(0)$. Therefore, $f_v \in C^0(X_v, \mathbb{R})$

This differs from the restriction we saw yesterday as the restriction of f gives a local function $f^v \in C(X_\tau^k)$ whereas the localization is $f_v \in C(X_\tau^{k-i})$. While restriction worked well with trickle down, localization is better suited for *Garland's method.*

Lemma 2.12 (Localization)**.** *Let $f, g \in C^k$. We have that,*

1. $\langle f, g \rangle = \mathbb{E}_{\sigma \in X(i)}[\langle f_\sigma, g_\sigma \rangle]$,
2. $\left\langle \widetilde{\mathrm{P}}_k^{\wedge} f, f \right\rangle = \mathbb{E}_{v \in X(0)}[\left\langle \widetilde{\mathrm{P}}_{k-1,v}^{\wedge} f_v, f_v \right\rangle]$.

Proof. Proof not provided in the lecture. □

Properness: A cochain $f \in C^k$ is an *i-level* co-chain if $f \in \ker(\mathrm{P}_{i-1}^{\downarrow} \cdots \mathrm{P}_{k-1}^{\downarrow})$. We denote this subspace as C_i^k. We say that $f \in C_i^k$ is a *proper i-level co-chain* if $f \in \mathrm{Im}(\mathrm{P}_{k-1}^{\uparrow} \cdots \mathrm{P}_i^{\uparrow})$. Essentially, every function at level i is also in $i-1$. Whereas a proper level i function is one not in level $i+1$.

Lemma 2.13 (Orthogonal Decomposition)**.** *Every $f \in C^k$ can be represented as $f = \sum_i f^i$ such that f^i is proper i-level cochain and $\langle f^i, f^j \rangle = 0$ whenever $i \neq j$.*

Proof. Proof not given in the lecture. □

Before we prove the main theorem, we make the following observation about the localization of a cochain f.

Claim 2.14 (Localization Properties)**.** *Let $f \in C^k$ with the decomposition $f = \sum_i f^i$, $f^i \in C_i^k$ and $v \in X(0)$. Then,*

- $f_v^{\perp} = \left(\sum_{i=1}^k f^i\right)_v + (f_v^0)^{\perp}$ *and* $(f_v)^{\parallel} = (f_v^0)^{\parallel}$.
- *Let* $f_{v,i} := (f_v)^i$*. Then, for* $i > 0$, $f_{v,i} = f_v^{i+1}$ *and* $f_{v,0} = f_v^1 + f_v^{0\perp}$.

Proof. We wish to show that for any $i > 0$, $(f^i)_v^\perp = (f^i)_v$ which is equivalent to saying that $\left\langle (f^i)_v, \mathbf{1}_v \right\rangle = 0$ where $\mathbf{1}_v$ is the constant function on the link X_v. Let $I_v \in C^0$ be the function that is $I_v(x) = 1$ iff $x = v$. Now, it is easy to see that $\mathrm{P}^\uparrow_{k-1} \cdots \mathrm{P}^\uparrow_0 I_v = \mathbf{1}_v$. Now,

$$\begin{aligned} \left\langle f_v^i, \mathbf{1}_v \right\rangle &= \left\langle f_v^i, \mathrm{P}^\uparrow_{k-1} \cdots \mathrm{P}^\uparrow_0 I_v \right\rangle \\ &= \left\langle \mathrm{P}^\downarrow_0 \cdots \mathrm{P}^\downarrow_{k-1} f_v^i, I_v \right\rangle \\ &= \langle 0, I_v \rangle . \end{aligned}$$

The last equality uses the definition of an i-level cochain. □

Theorem 2.15 (Decomposition Theorem)**.** *Let $f \in C^k$ and decompose $f = \sum_i f^i$ where $f^i \in C_i^k$. Then,*

$$\begin{aligned} \left\langle \mathrm{P}^\vee_k f, f \right\rangle &\leq \sum_{i=0}^{k} \lambda_{\phi,i,k} \left\| f^i \right\|^2 + \underbrace{\sum_{i \neq j} c_{ij} \left\langle f^i, f^j \right\rangle}_{\textit{Mixed Terms} \text{ (MT)}} \\ &\leq \lambda_{\phi,0,k} \left\| f \right\|^2 + \mathrm{MT}. \end{aligned}$$

where $\lambda_{\phi,i,k} = 1 - \frac{1}{k+1-i} \prod_{j=i-1}^{k-1} (1 - \lambda_j)$ *and* $\lambda_j = \max_{\tau \in X(j)} \lambda_2(X_\tau)$.

Proof sketch: The proof is quite similar to that of the Trickle Down Theorem from the previous lecture wherein we decompose the inner product as a sum over the perpendicular and parallel components of the local cochains f_v. The perpendicular part is easily bounded (as earlier) using the inductive hypothesis on the links X_v. We make the following notation to make the induction easier.

$$\lambda_{v,i,k} := 1 - \frac{1}{k+1-i} \prod_{j=i-1}^{k-1} (1 - \lambda_{v,j}) \text{ and } \lambda_{v,j} = \max_{\tau \in X_v(j)} \lambda_2(X_\tau)$$

The parallel part is handled using the "advantage lemma" Lemma 2.16 which can be seen as the technical core of the result.

Proof. We proceed by induction on k.

Base Case: When $k = 0$, $\left\langle \widetilde{\mathrm{P}}^\wedge_0 f, f \right\rangle \leq \lambda_{-1} \left\| f \right\|^2 = \lambda_{\phi,0,0} \left\| f \right\|^2$.

Inductive Case:

$$\begin{aligned} \left\langle \widetilde{\mathrm{P}}^\wedge_k f, f \right\rangle &= \mathop{\mathbb{E}}_{v \in X(0)} \left[\left\langle \widetilde{\mathrm{P}}^\wedge_{k-1,v} f_v, f_v \right\rangle \right] && \text{Localization Lemma} \\ &= \mathop{\mathbb{E}}_{v \in X(0)} \left[\left\langle \widetilde{\mathrm{P}}^\wedge_{k-1,v} f_v^\perp, f_v^\perp \right\rangle \right] + \mathop{\mathbb{E}}_{v \in X(0)} \left[\left\langle \widetilde{\mathrm{P}}^\wedge_{k-1,v} f_v^\parallel, f_v^\parallel \right\rangle \right] \\ &= \underbrace{\mathop{\mathbb{E}}_{v \in X(0)} \left[\left\langle \widetilde{\mathrm{P}}^\wedge_{k-1,v} f_v^\perp, f_v^\perp \right\rangle \right]}_{A} + \underbrace{\mathop{\mathbb{E}}_{v \in X(0)} \left[\left\langle f_v^{0\parallel}, f_v^{0\parallel} \right\rangle \right]}_{B} . && \text{Using Claim 2.14} \end{aligned}$$

Since $f_v^\perp$ is a cochain in $C^{k-1}(X_v, \mathbb{R})$, we can use the inductive hypotheses to obtain,

$$\left\langle \widetilde{\mathrm{P}}^{\wedge}_{k-1,v} f_v^{\perp}, f_v^{\perp} \right\rangle \leq \sum_{j=0}^{k-1} \lambda_{v,j,k} \left\| (f_v^{\perp})^j \right\|^2 + \mathrm{MT}$$

From Claim 2.14, we know that for $j > 0$, the j-level cochain of $f_v^{\perp}$ are the same the localization of the $j+1$-level cochain of f_v, i.e., is $(f_v^{\perp})^j = (f^{j+1})_v$. Plugging this in,

$$\sum_{j=0}^{k-1} \lambda_{v,j,k} \left\| (f_v^{\perp})^j \right\|^2 = \lambda_{v,0,k} \left\| (f_v^0)^{\perp} + f_v^1 \right\|^2 + \sum_{j=1}^{k-1} \lambda_{v,j,k} \left\| f_v^{j+1} \right\|^2 \quad (2.2)$$

$$\leq \lambda_{v,0,k} \left\| f_v^0 \right\|^2 + \sum_{j=0}^{k-1} \lambda_{v,j,k} \left\| f_v^{j+1} \right\|^2 . \quad (2.3)$$

Now we will use the observation that $\lambda_{\phi,i,k} \geq \lambda_{v,i-1,j}$ for any $v \in X(0)$ and $j \leq k$. Moreover, by localization lemma (Lemma 2.12), we get $\mathbb{E}_v[\left\| f_v^j \right\|^2] = \left\| f^j \right\|^2$. The term (A) then can be bounded as

$$(A) \leq \lambda_{\phi,1,k} \left\| f^0 \right\|^2 + \sum_{j=1}^{k-1} \lambda_{\phi,j,k} \left\| f^j \right\|^2 .$$

□

Lemma 2.16 (Advantage lemma)**.**

$$\mathop{\mathbb{E}}_{v \in X(0)} \left[\left\| f_v^{0\|} \right\|^2 \right] \leq \left(1 - \frac{k}{k+1}(1 - \lambda_{-1}) \right) \left\| f^0 \right\|^2 .$$

Proof. Proof not provided in the lecture. See [GK22, Lemma 7.8]. □

References

[AL20] Vedat Levi Alev and Lap Chi Lau. Improved analysis of higher order random walks and applications. In *Proceedings of the 52nd Annual ACM Symposium on Theory of Computing (STOC)*, pages 1198–1211, 2020.

[DK17] Irit Dinur and Tali Kaufman. High Dimensional Expanders Imply Agreement Expanders. In *58th IEEE Annual Symposium on Foundations of Computer Science (FOCS)*, pages 974–985, 2017.

[GK22] Roy Gotlib and Tali Kaufman. Beyond the Worst Case: Structured Convergence of High Dimensional Random Walks. *arXiv preprint arXiv:2208.03241*, 2022.

[KM17] Tali Kaufman and David Mass. High Dimensional Random Walks and Colorful Expansion. In *8th Innovations in Theoretical Computer Science Conference (ITCS 2017)*, volume 67 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 4:1–4:27, 2017.

[KO20] Tali Kaufman and Izhar Oppenheim. High Order Random Walks: Beyond Spectral Gap. *Combinatorica*, 40(2):245–281, 2020.

[Opp18] Izhar Oppenheim. Local Spectral Expansion Approach to High Dimensional Expanders Part I: Descent of Spectral Gaps. *Discrete & Computational Geometry*, 59(2):293–330, 2018.