

Sampling for Colorings

k -coloring: Assignment of $\leq k$ labels to vertices so no neighbors have the same label. Straightforward when $k > \Delta$ due to extra color.

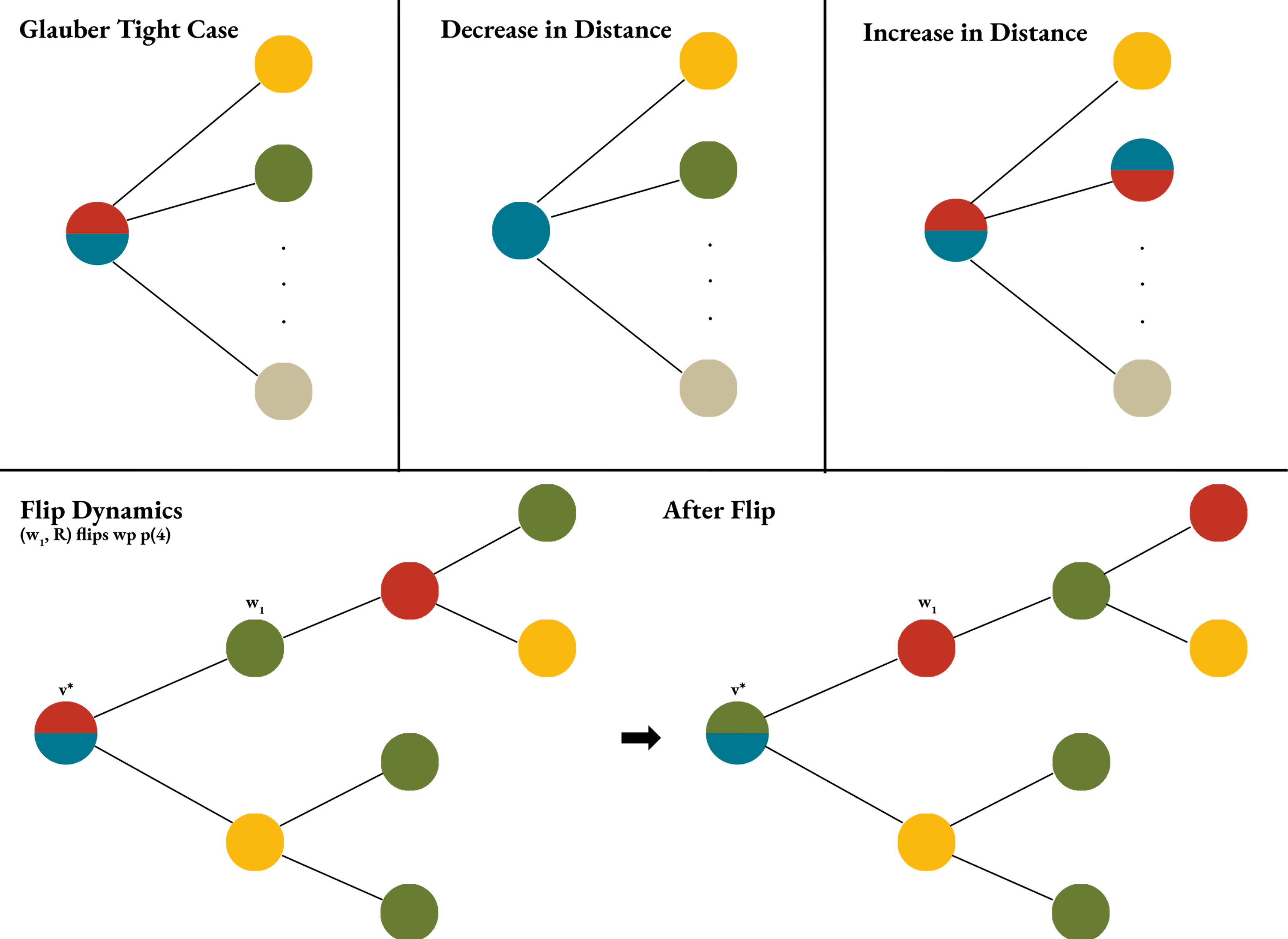
Goal: Generate a random coloring in polynomial time on general graphs with maximum degree Δ . Note, the number of colorings is $\Omega(\exp(|V|))$.

Problem: What is the minimum number of colors (k) needed to do so in $O(n \log n)$ time?

Flip Dynamics: A vertex, v , and color, c , are selected. Rooted at v , an alternating two-colored component with length L “flips” with probability $p(L)$. Typically, $p(1) = 1, p(2) = 1/3, \dots, p(\geq 7) = 0$.

Glauber Dynamics: A version of flip dynamics where $p(1) = 1$ and all other flip probabilities are 0. As shown by [Jerrum ‘95], with particular coupling techniques, k can be reduced to 2Δ . **Right:** tight Glauber case and increase + decrease in distance.

Dynamics

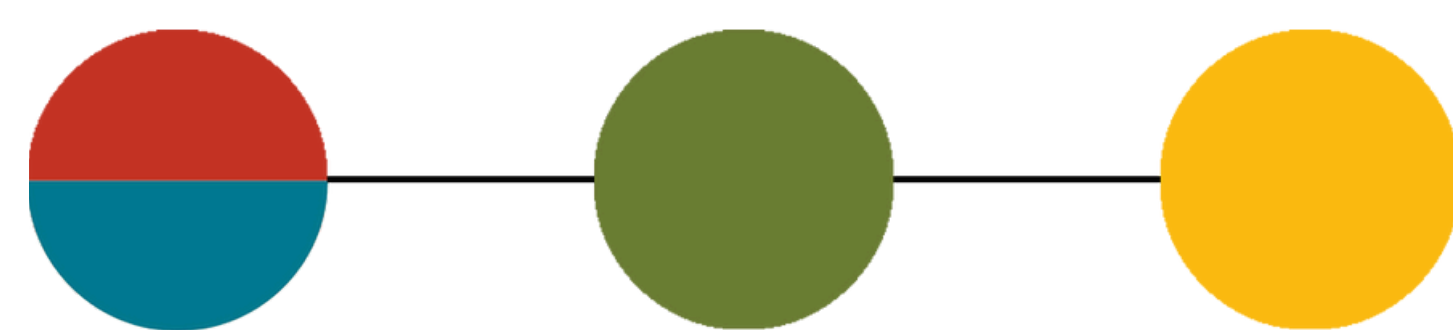


Relevant Results

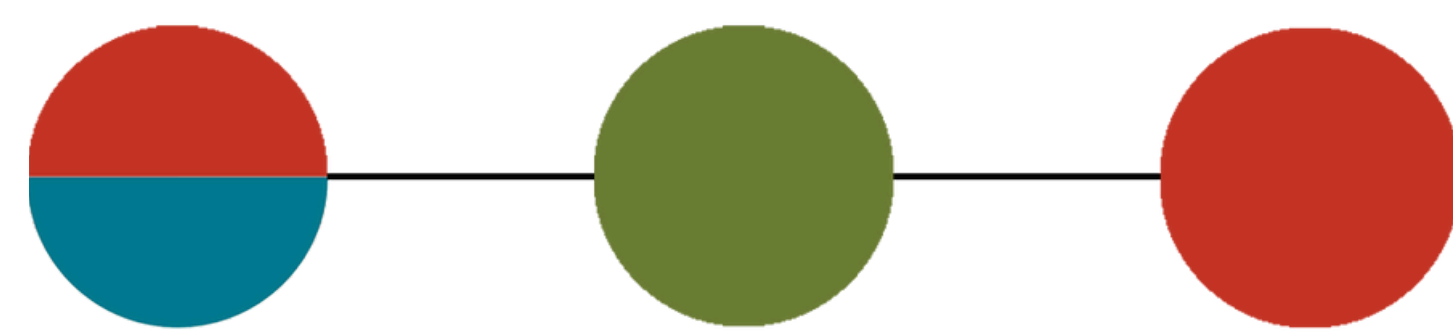
Paper	Result ($k >$)	Flip / Glauber	Analysis
Jerrum '95	2Δ	Glauber	Simple
Vigoda '99	$(11/6)\Delta$	Flip	Metric
Chen et al. '19	1.83332Δ	Flip	Variable Length
Carlson, Vigoda '25	1.809Δ	Flip	Metric
Our Work	1.832Δ	Flip	Variable Length

Blocked Neighbors

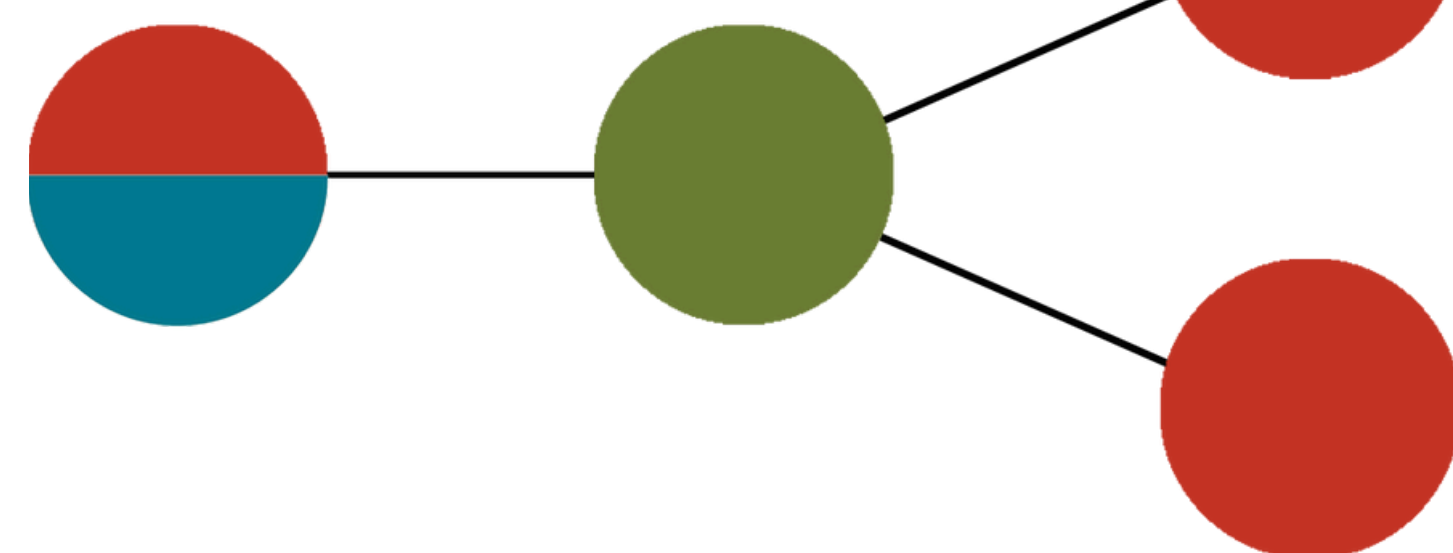
Unblocked



Singly-blocked



Multi-blocked



Our Variable Length Approach

Path Coupling: Enough to consider coupled pairs of states differing at one vertex and show the expected distance contracts.

Variable Length Coupling: A path coupling approach that utilizes multi-step analysis. In this case, consider until the Hamming distance changes.

Layered Approach: Previous approaches didn't allow for back-and-forth transitions between states. Layered approach unrolls transitions with no restrictions for L steps.

States:

- **Good:** ≥ 1 unblocked neighbor.
- **Bad:** 0 unblocked, ≥ 1 singly-blocked
- **Other:** 0 unblocked, 0 singly-blocked, ≥ 1 multi-blocked

Analog after Hamming distance has changed: END states.

Conclusion

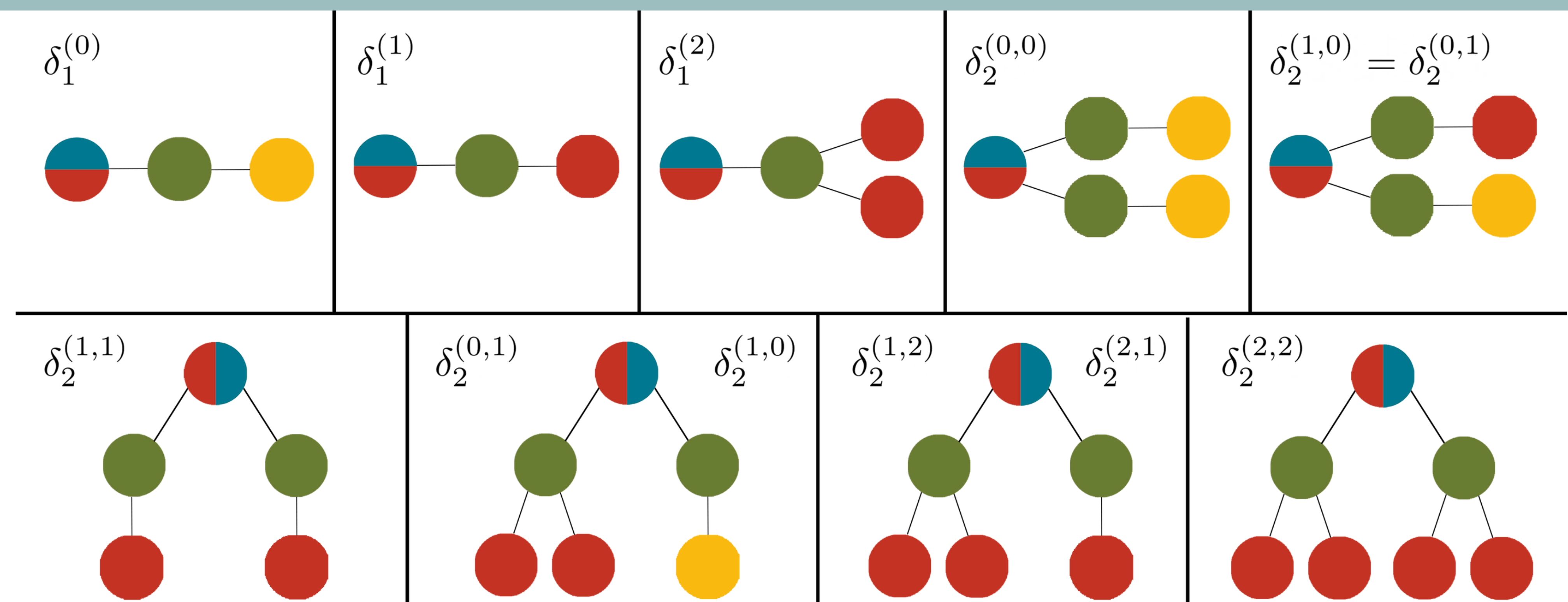
Main Result/Takeaway: $k > 1.832\Delta$. Layered transitions can improve state-based variable length methods through complex transitions.

Limitations: There seems to exist a bottleneck with variable length methods that prevents the capture of “tradeoffs,” something captured well by a metric analysis.

Acknowledgment

Infinite thanks to Dr. Eric Vigoda and Dr. Charlie Carlson for all of their guidance over the past year.

Latest Work on Improving 1.809 Δ



Current work indicates improved analysis with metric could lower k to 1.8066Δ through:

- Tightening the probability of the disagreement vertex changing colors.
- Considering cost of neighbor “types” flipping and the related extremal cases.
- Introducing a more complex version of the metric to encourage transitions between adjacent neighborhood types.