Overview

- Data structures and associated operations

Data structures	*Associated operations*
Linked list	insert, delete, makenull
Stacks	push, pop
Queues	remove from head, insert from tail
Trees	insert, delete, traverse
Graphs	traverse, shortest path, strong components, etc.

- Data structures and associated operations are "tools" for building programs

Overview (cont.)

- Algorithm design
  - A sequence of operations which are
    - clearly defined (no ambiguity as to what to do next)
    - effective (component operations done in finite time)
    - terminate
  - E.g. *Sorting*
    - Data structures: an array of length *n*
    - Algorithms: comparing & swapping elements (bubble sort, insertion sort, selection sort, quick sort, merge sort, etc.)
  - Programs = Algorithms + Data structures

Overview (cont.)

- General principles
  - Divide-and-conquer
  - Greedy
  - Dynamic programming
  - Backtracking
  - Branch-and-bound
  - Randomized algorithms

Caveats

- There are a lot more principles for algorithm designs that we do not cover
  - Numerical algorithms
  - Graph algorithms
  - Geometrical algorithms (e.g., vision, graphics)
  - Probabilistical algorithms
- Multi-stage, discrete, countably many, unique

Divide-and-Conquer

Divide-and-Conquer

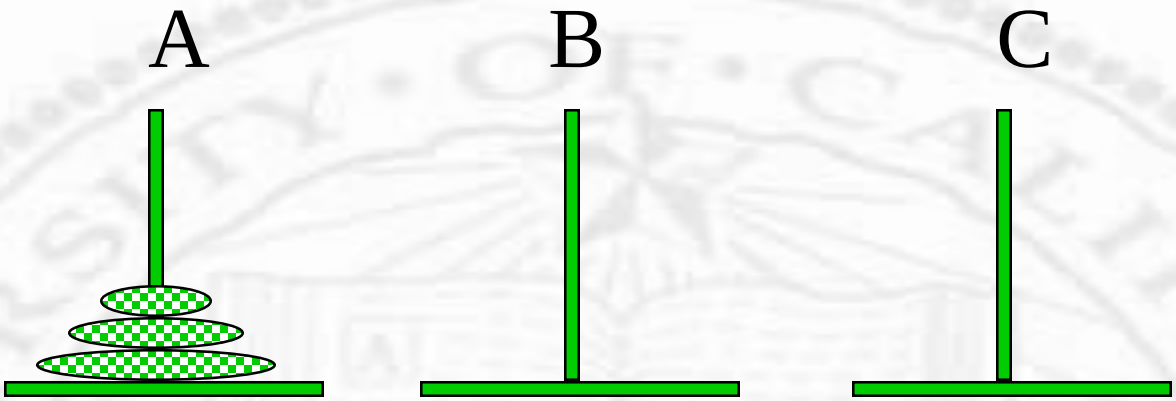
- Input $A(1{:}n)$: n elements stored in an array

```
Procdure DandC(p,q)
    if Small(p,q) then
        return (G(p,q))
    else m ← Divide(p,q)
        return Combine(DandC(p,m),DandC(m+1,q))
    end if
end DandC
```

Divide-and-Conquer (cont.)

- *Divide*: split a larger problem into sub-problems of smaller size
- *Combine*: merge the solutions of sub-problems into that of a larger problem
- *Small*: is the problem small enough?
- *G(p,q)*: easy solutions to small problems

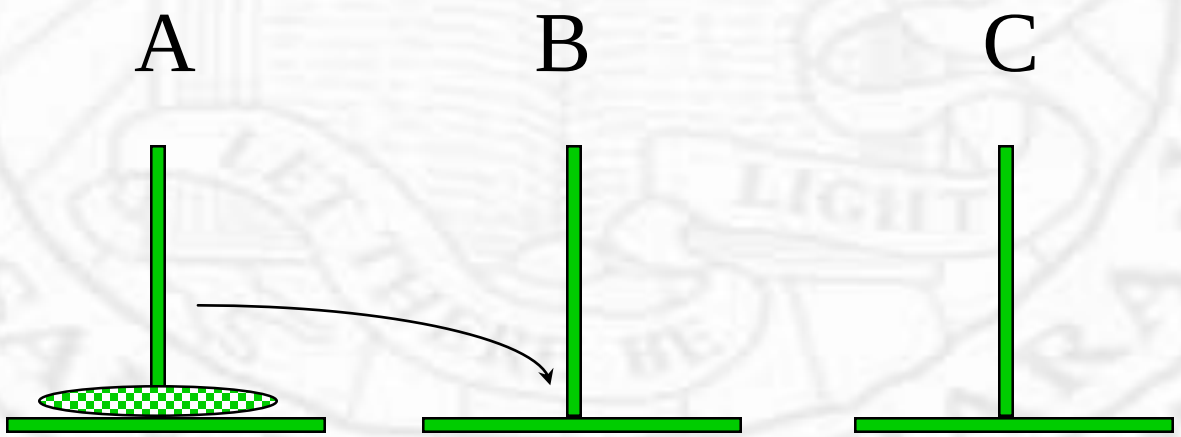
Hanoi Towers



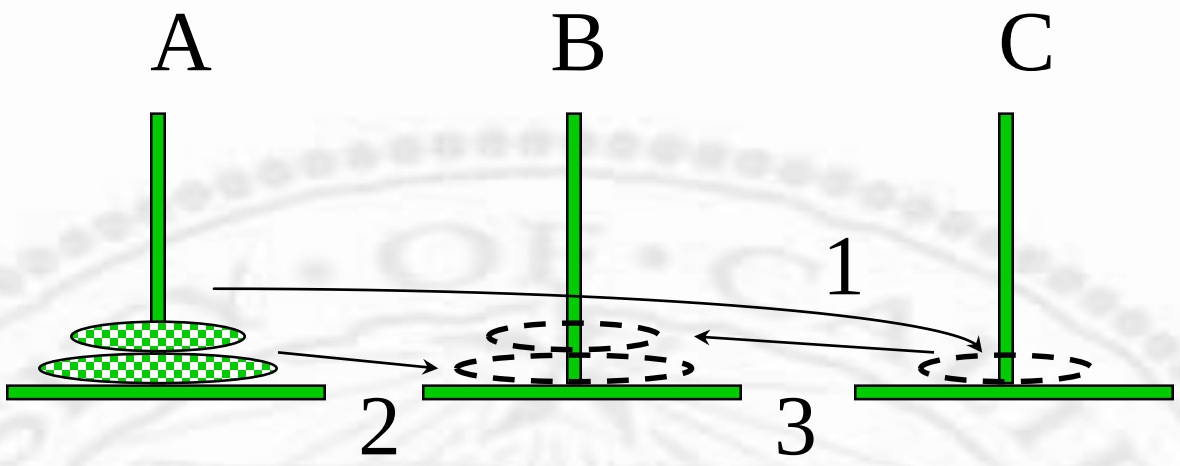
- Three pegs, A has *n* disks of different sizes stacked with smaller ones on top of bigger ones
- Move disks one at a time
- Never place a larger disk on top of a smaller one
- Move all disks onto B
- Trivial problem, the rule of the game dictates divide-and-conquer

- Hanoi(n, A, B, C)
  - n: number of disks
  - A: starting peg
  - B: end peg
  - C: temporary peg

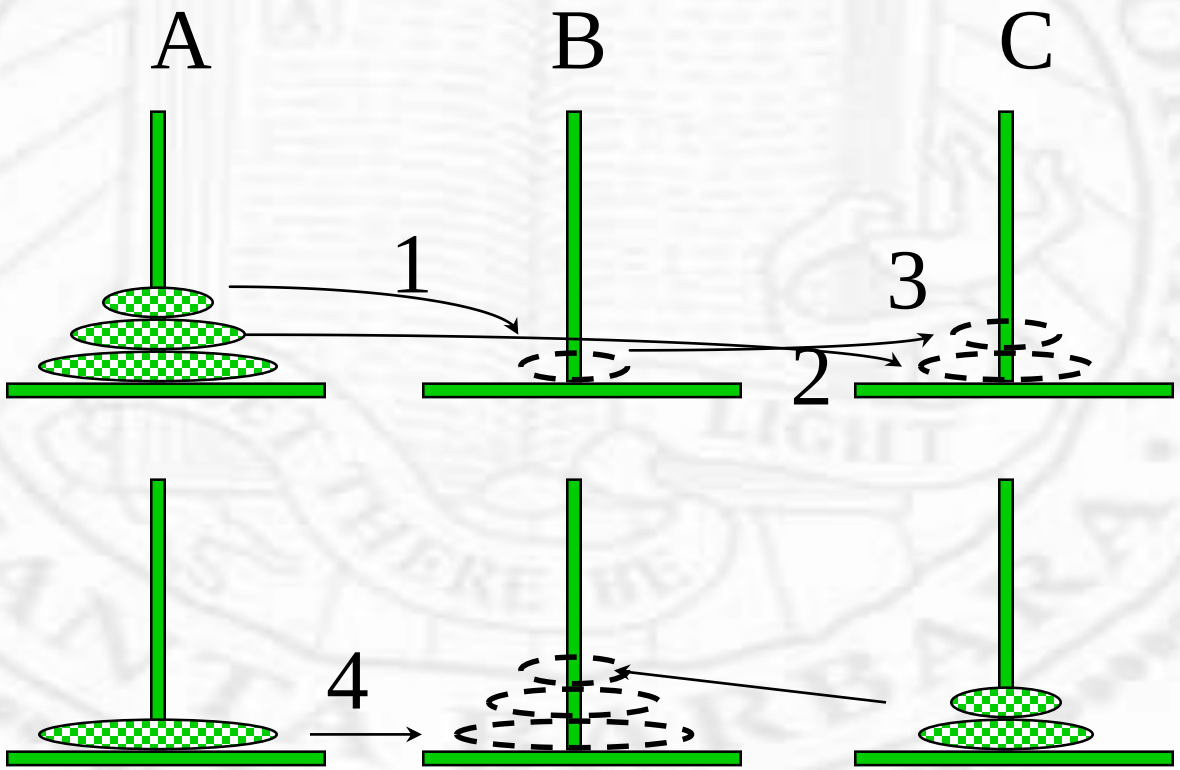
n=1



n=2



n=3



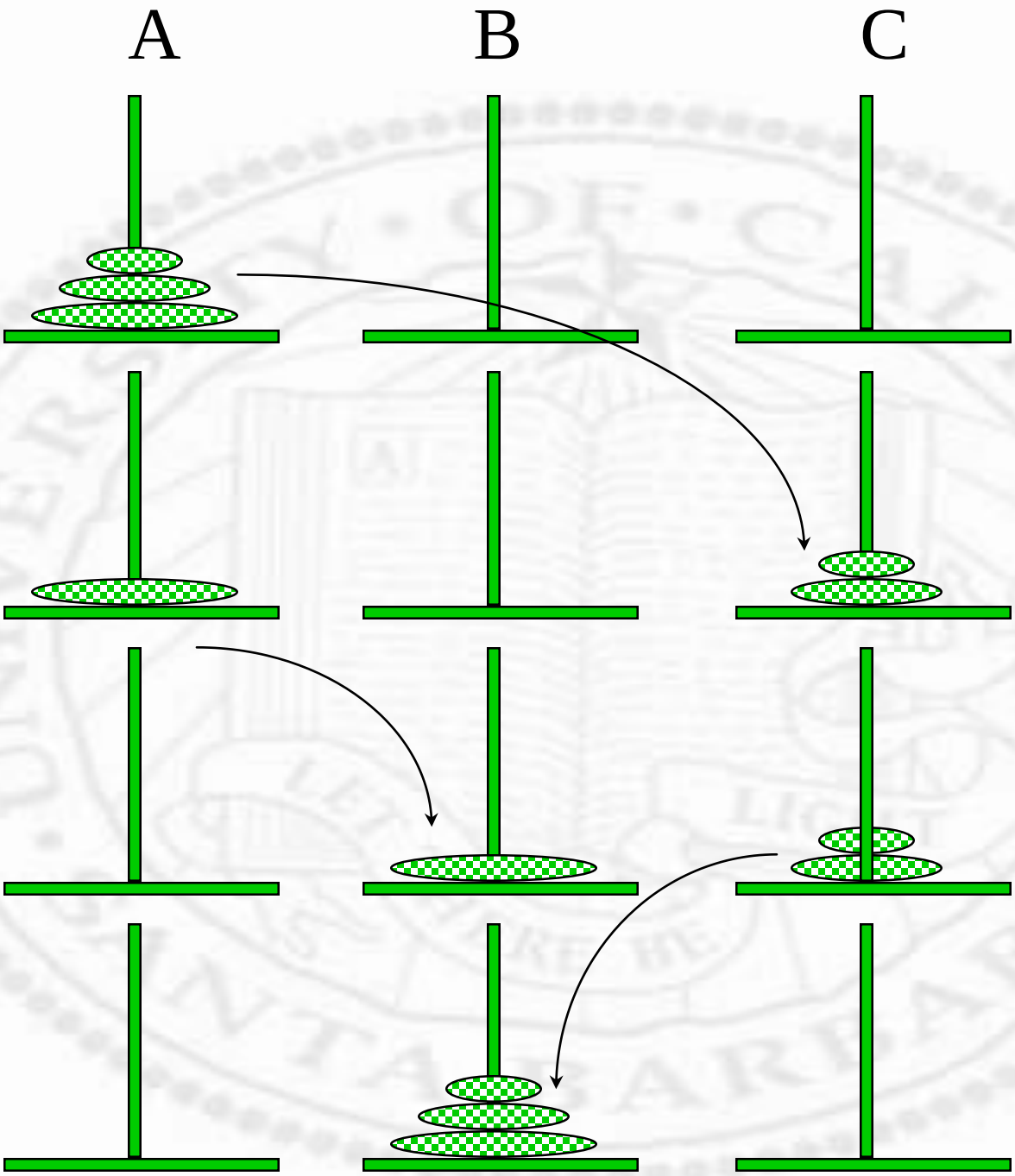
- Movement steps of *m* disks will be used in moving *n* disks (*n*>*m*)
- Problem is decomposable

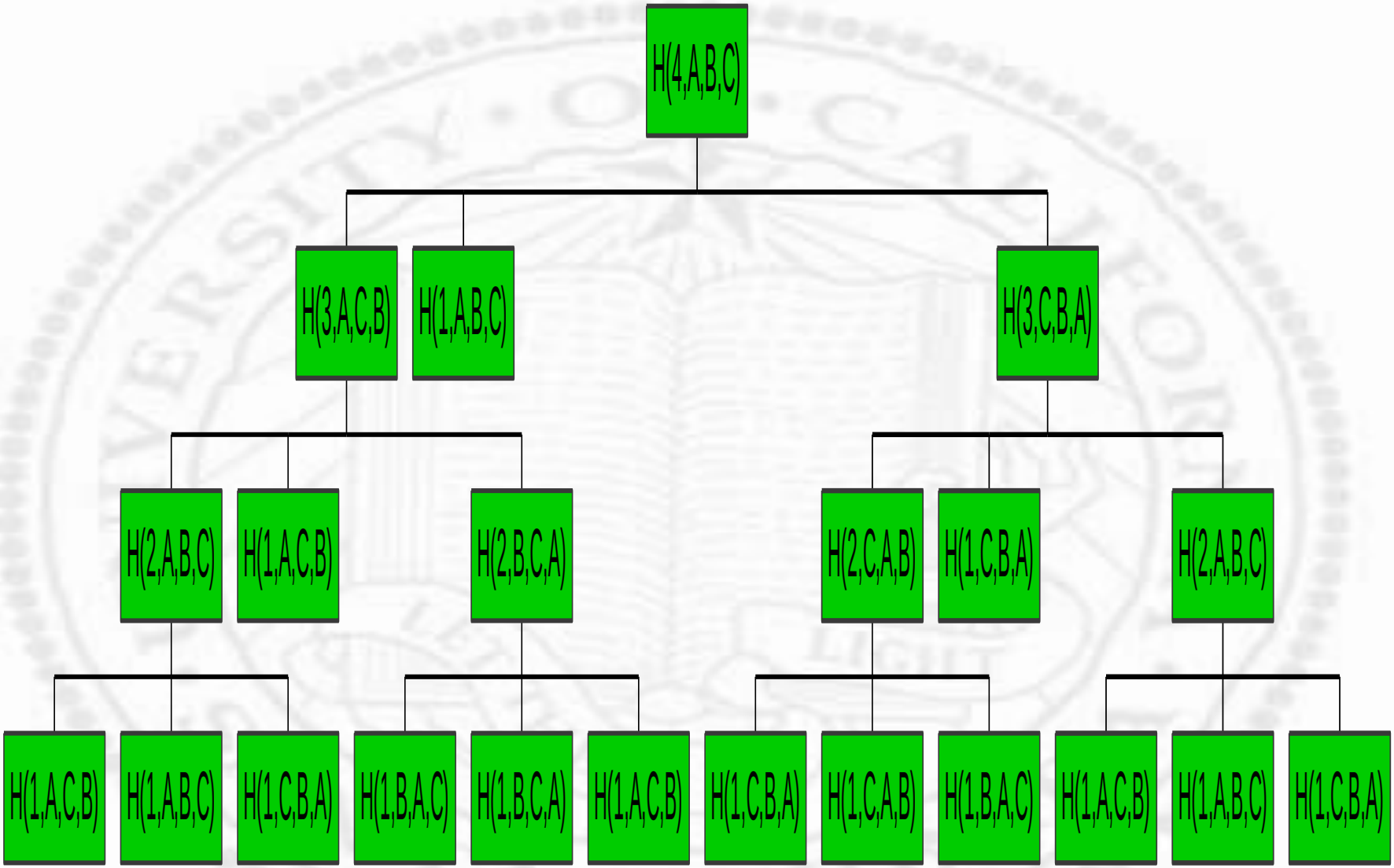
Hanoi(n,A,B,C) =

Hanoi(n-1,A,C,B)

+ *Hanoi(1,A,B,C)*

+ *Hanoi(n-1,C,B,A)*


A
B
C



- *Divide*: two sub-problems of size *n-1* and one sub-problem of size *1*
- *Small(p,q)*: when the problem size is *1*
- *G(p,q)*: move a disk from peg to peg
- *Combine*: sequential concatenation of moves

Time complexity

$$
\begin{aligned}
T(n) &= T(n-1)+c+T(n-1) && = 2T(n-1)+c \\
&= 2\{2T(n-2)+c\}+c && = 2^2T(n-2)+(1+2)c \\
&= 2^2\{2T(n-3)+c\}+(1+2)c && = 2^3T(n-3)+(1+2+2^2)c \\
&\ldots && \ldots \\
&= 2^{n-1}T(1)+(1+2+\ldots+2^{n-2})c \\
&= (1+2+\ldots+2^{n-2}+2^{n-1})c \\
&= O(2^n)
\end{aligned}
$$

Binary Search

- Input:
  - a list of elements sorted in *nondecreasing* order
  - an element *x*
- Output
  - determine whether *x* is present
  - if so, the position index *j*

- *Divide:*

$$BS(n, a_1, a_2, \ldots, a_n, x) =$$

$$BS(\left\lfloor \frac{n+1}{2} \right\rfloor - 1, a_1, a_2, \ldots, a_{\left\lfloor \frac{n+1}{2} \right\rfloor - 1}, x) +$$

$$BS(1, a_{\left\lfloor \frac{n+1}{2} \right\rfloor}, x) +$$

$$BS(n - \left\lfloor \frac{n+1}{2} \right\rfloor, a_{\left\lfloor \frac{n+1}{2} \right\rfloor + 1}, \ldots, a_n, x)$$

  - two problems of size approximately *n/2*, and one problem of size *1*

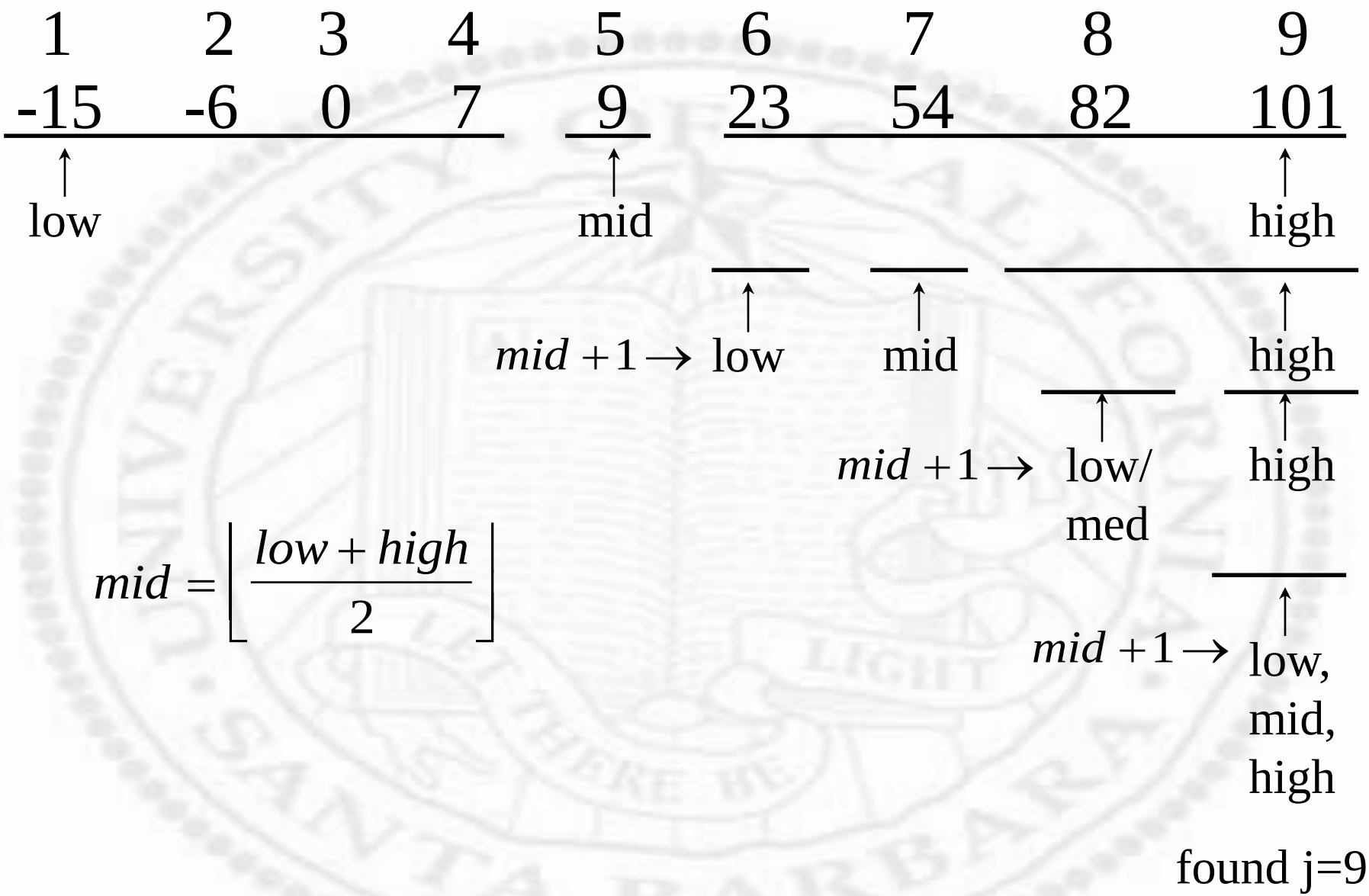
- *Small(p,q): when* the size of problem is 1
- *G(p,q):* compare the single element in the list with the search element
- *Combine:*

$$x = a_{\left\lfloor \frac{n+1}{2} \right\rfloor} \qquad j = \left\lfloor \frac{n+1}{2} \right\rfloor$$

$$x < a_{\left\lfloor \frac{n+1}{2} \right\rfloor} \quad \text{solve the first sub-problem}$$

$$x > a_{\left\lfloor \frac{n+1}{2} \right\rfloor} \quad \text{solve the third sub-problem}$$

x=101



$$mid = \left\lfloor \frac{low + high}{2} \right\rfloor$$

found j=9

x=30

1	2	3	4	5	6	7	8	9
-15	-6	0	7	9	23	54	82	101

low, mid, high

$mid + 1 \rightarrow$ low, mid, high

low, mid, high

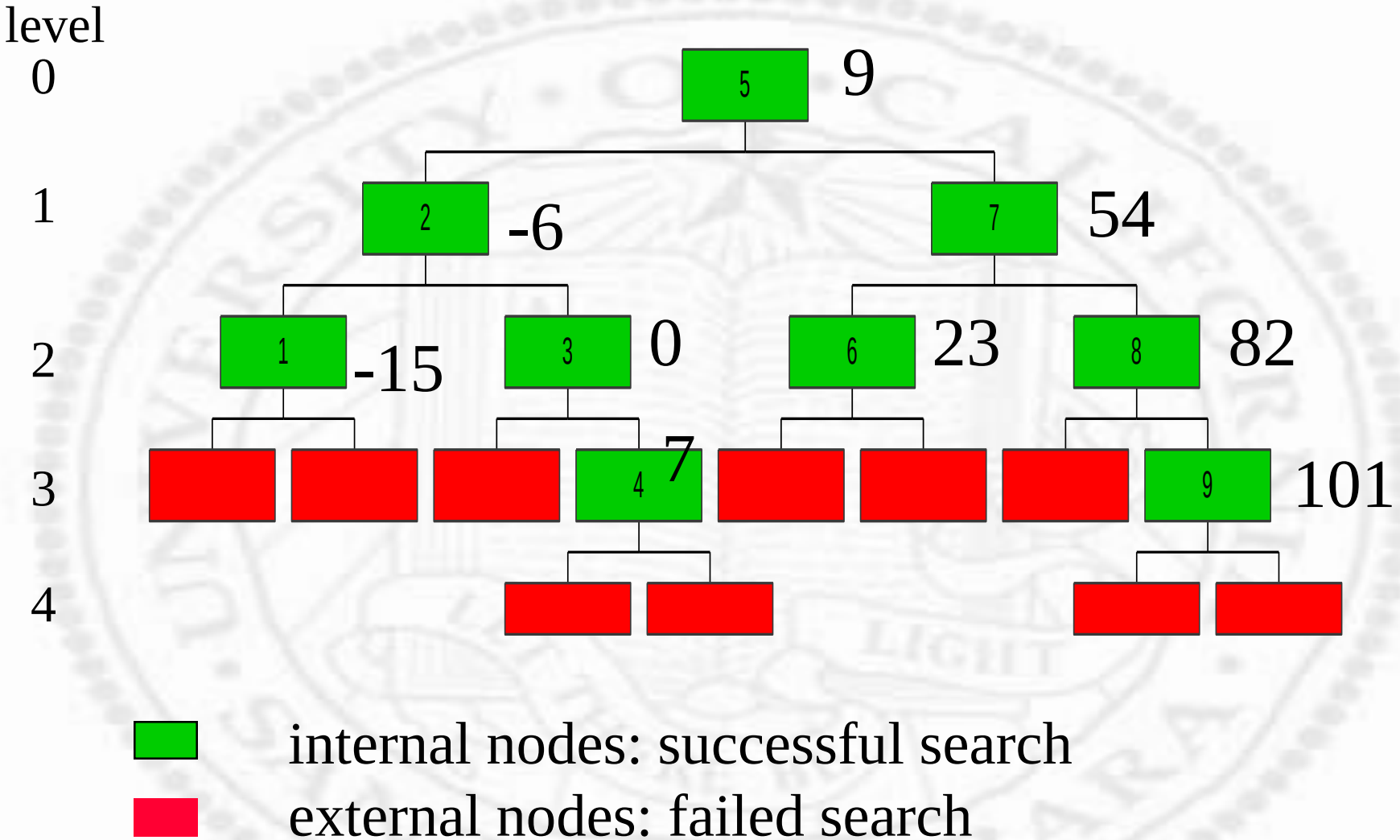
$mid - 1 \leftarrow$

high

$mid + 1 \rightarrow$ low>high, not found

$$mid = \left\lfloor \frac{low + high}{2} \right\rfloor$$

Time complexity



- Properties of binary search trees
  - balanced (root corresponds to the middle element, two subtrees are of approximately equal size)
  - with n elements $2^{k-1} \le n < 2^k$

Tree depth (k)	Min capacity (2^{k-1})	Max capacity (2^k)
1	1	2
2	2	4
3	4	8
4	8	16

    - internal nodes at levels of 0 to *k-1*

(successful searches make at most *k* comparisons)

    - external nodes at levels *k-1* and *k*

(failed searches make at least *k-1* ad at most *k* comparisons)

  - worst case is *O(k)* or *O(logn)* for both successful and failed searches
  - best case is *O(1)* for successful and *O(logn)* for failed searched

Average case - slightly more complicated

Average performance =

prob(S) × average performance of successful searches +

⇓

average # of comparisons in successful searches

⇓

average internal path length + 1

⇓

$$\frac{\text{total internal path length (I)}}{\text{\# of internal nodes (i)}} + 1$$

prob(F) × average performance of failed searches

⇓

average # of comparisons in failed searches

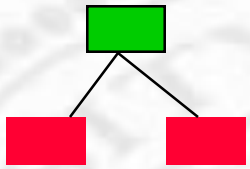
⇓

average external path length

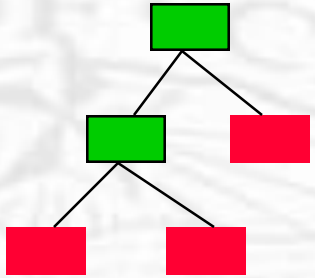
⇓

$$\frac{\text{total external path length (E)}}{\text{\# of external nodes (e)}}$$

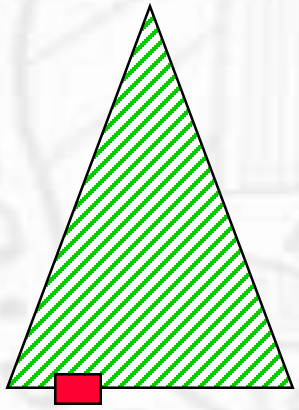
- # of internal (i) and external (e) nodes



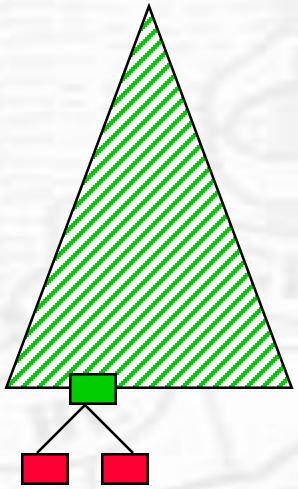
(i=1, e=2)



(i=2, e=3)

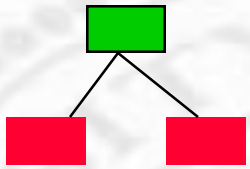


(i=n, e=n+1)

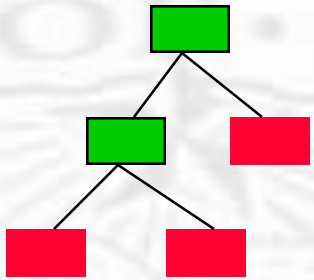


(i=n+1, e=n+2)

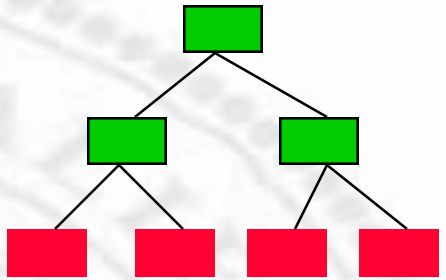
- total internal (I) and external (E) path length



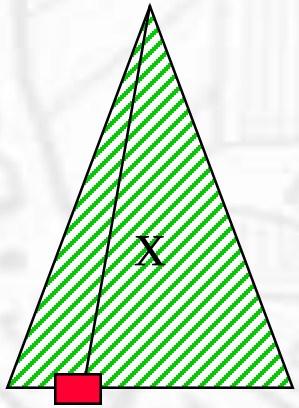
(I=0, E=2)



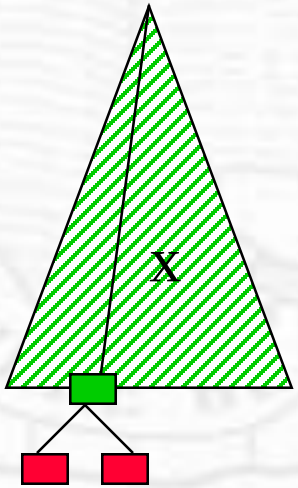
(I=1, E=5)



(I=2, E=8)



$E = I + 2i$



$$
\begin{aligned}
I' &= I + x \\
E' &= E + 2(x+1) - x \\
&= (I + 2i) + 2(x+1) - x \\
&= (I + x) + 2(i+1) \\
&= I' + 2i'
\end{aligned}
$$

$$If\ \ i\ is\ n$$

$$\Downarrow$$

$$I = E - 2i \approx (n+1)\log n - 2n$$

$$\Downarrow$$

$$\frac{I}{n} + 1 \approx \frac{(n+1)\log n - 2n}{n} + 1$$

$$\approx \log n - 1 = O(\log n)$$

$$\Rightarrow$$

$$Then\ e\ is\ n+1$$

$$\Downarrow$$

$$E \approx (n+1)\log n = O(n\log n)$$

$$\Downarrow$$

$$\Downarrow$$

$$\frac{E}{n+1} \approx \frac{(n+1)\log n}{n+1} = O(\log n)$$

average performanc e is O(logn) regardless

More Examples: Sorting

- brute force methods
  - bubble sort
  - selection sort
  - insertion sort

    $O(n^2)$

- "smart" methods
  - quick sort
  - merge sort

    $O(n \log n)$
  - based on Divide-and-Conquer

Bubble sort

```
for i=1 to n-1 do
    for  j=n downto i+1 do
        if a[j]<a[j-1] then
            swap(a[j-1], a[j])
```

$O(1)$

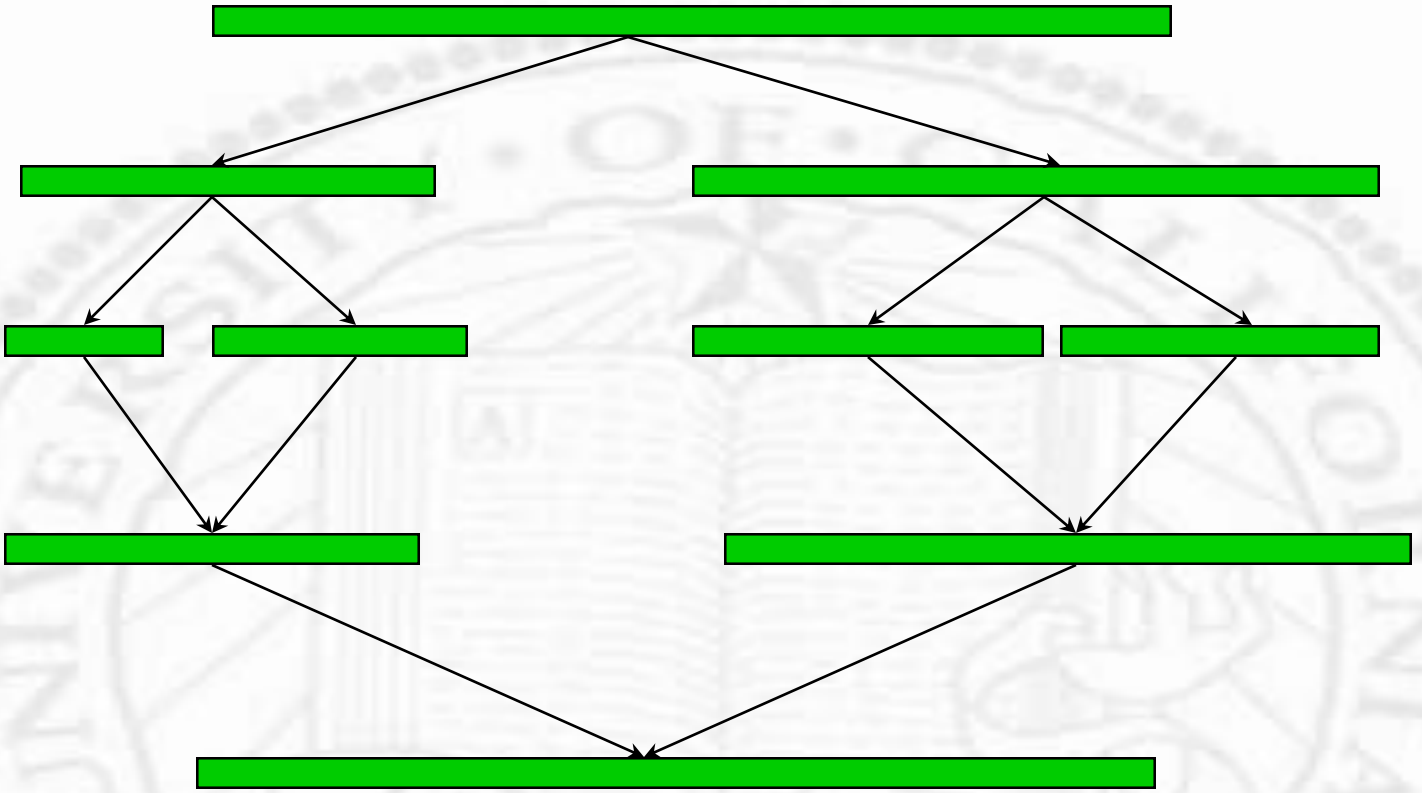
$O(n-i)$

$O(\sum_{i=1}^{n-1} n-i) = O(n^2)$

$i =$	1	2	3	4	5	6	7	8	9
1	65	85	70	75	80	60	55	50	45
2	45	65	85	70	75	80	60	55	50
3	45	50	65	85	70	75	80	60	55
4	45	50	55	65	85	70	75	80	60
5	45	50	55	60	65	85	70	75	80
6	45	50	55	60	65	70	85	75	80
7	45	50	55	60	65	70	75	85	80
8	45	50	55	60	65	70	75	80	85
9	45	50	55	60	65	70	75	80	85

- Each iteration places one element correctly
- Many elements are involved in many iterations
- Size of subproblems decrease very slowly through iterations

Sorting based on Divide-and-Conquer



- Quick sort
  - uneven division
  - simple concatenation
- Merge sort
  - even division
  - elaborate merge

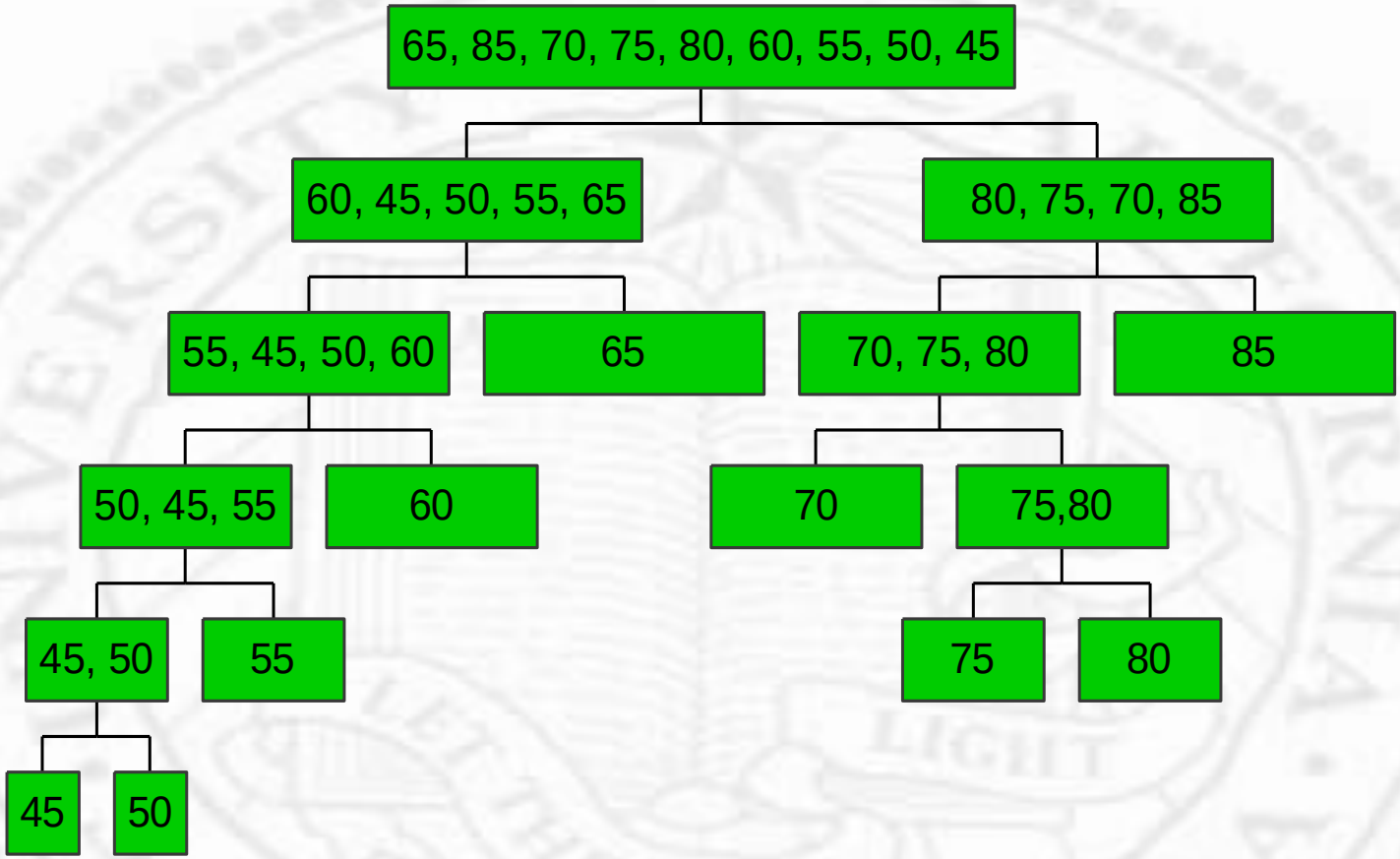
Quick Sort

- Input: a list of *n* elements
- Output: a list of the same elements sorted in nondecreasing order

- *Divide*

$$QS(n, a_1, a_2, \ldots, a_n) =$$
$$partition(1, n) +$$
$$QS(i, a_1, a_2, \ldots, a_i) +$$
$$QS(n - i, a_{i+1}, \ldots, a_n)$$

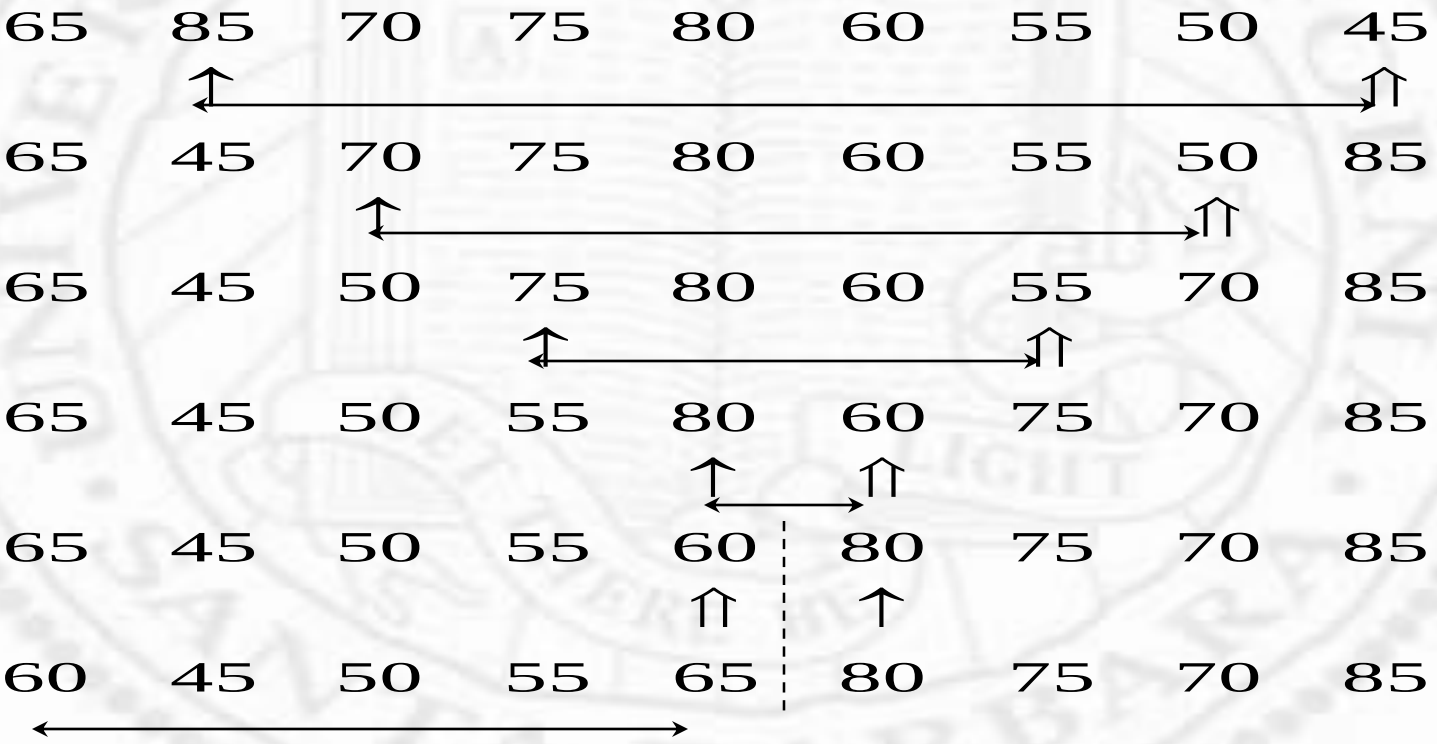
- *Small(p,q):* when the problem size becomes *1*
- *G(p,q):* nothing
- *Combine:* simple concatenation of solutions of two sorted lists

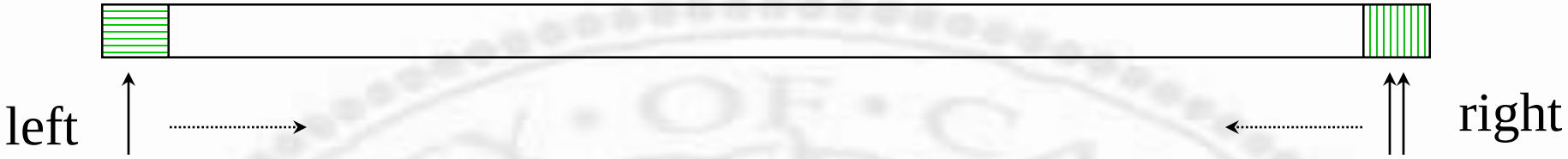


$$partition\,(p,q) \rightarrow i, such\ that\ a_i\ is\ the\ pivot$$

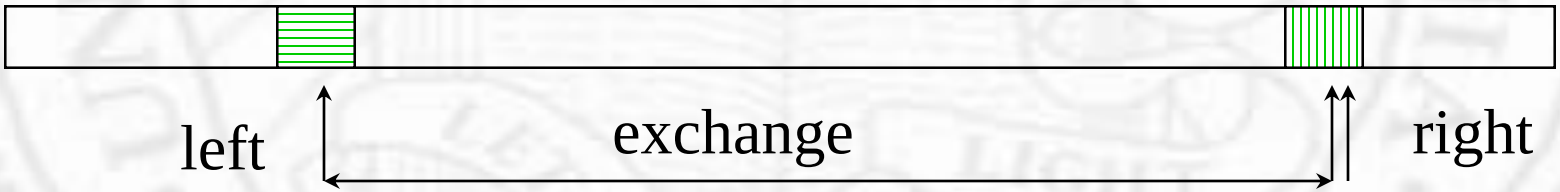
$$a_p, a_{p+1}, \ldots, a_i \leq a_i$$

$$a_{i+1}, a_{i+2}, \ldots, a_q > a_i$$

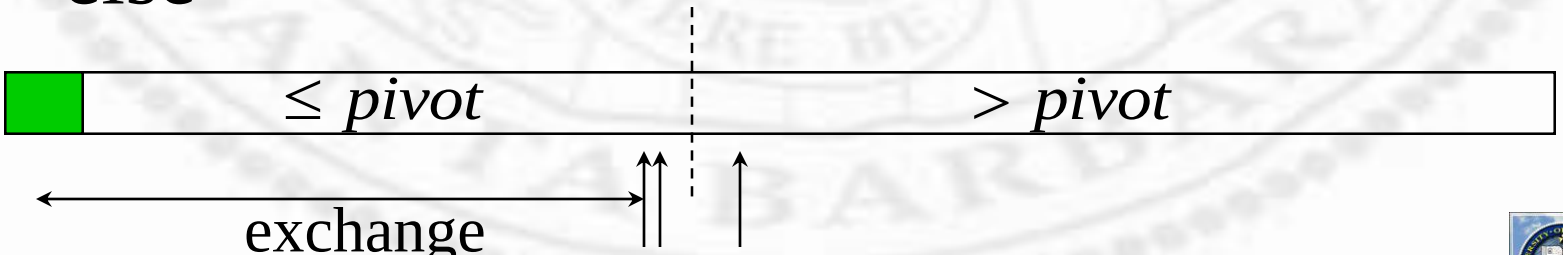




- left pointer moves right, until $*(left) > pivot$
- right point moves left, until $*(right) \leq pivot$
- if *left*<*right*, swap **(left)* and **(right)*

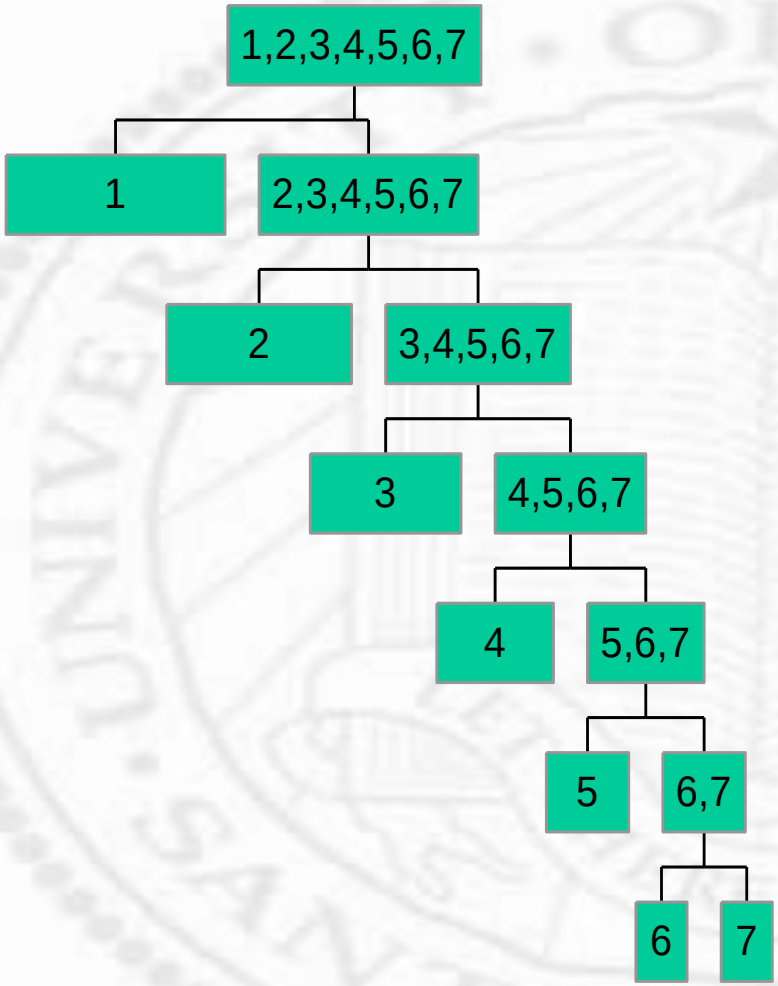


- else



- Array is scanned only once, at a particular location
  - no action is taken (advance pointer), or
  - swap elements and advance pointer
  - partition is *O(array length)*

Time complexity - worst case



$$\sum_{1}^{n-1}(\#\text{ of elements in the array})$$

$$= n + (n-1) + (n-2) + ... + 2$$

$$= O(n^2)$$

- Time complexity - average case
  - Assumptions:
    - the n elements are distinct
    - the pivot element can be equally likely the *ith* element in the sorted array

$$T(n) = \frac{1}{n}\sum_{i=1}^{n}\{T(i) + T(n-i)\} + cn$$

$$= O(n \log n)$$

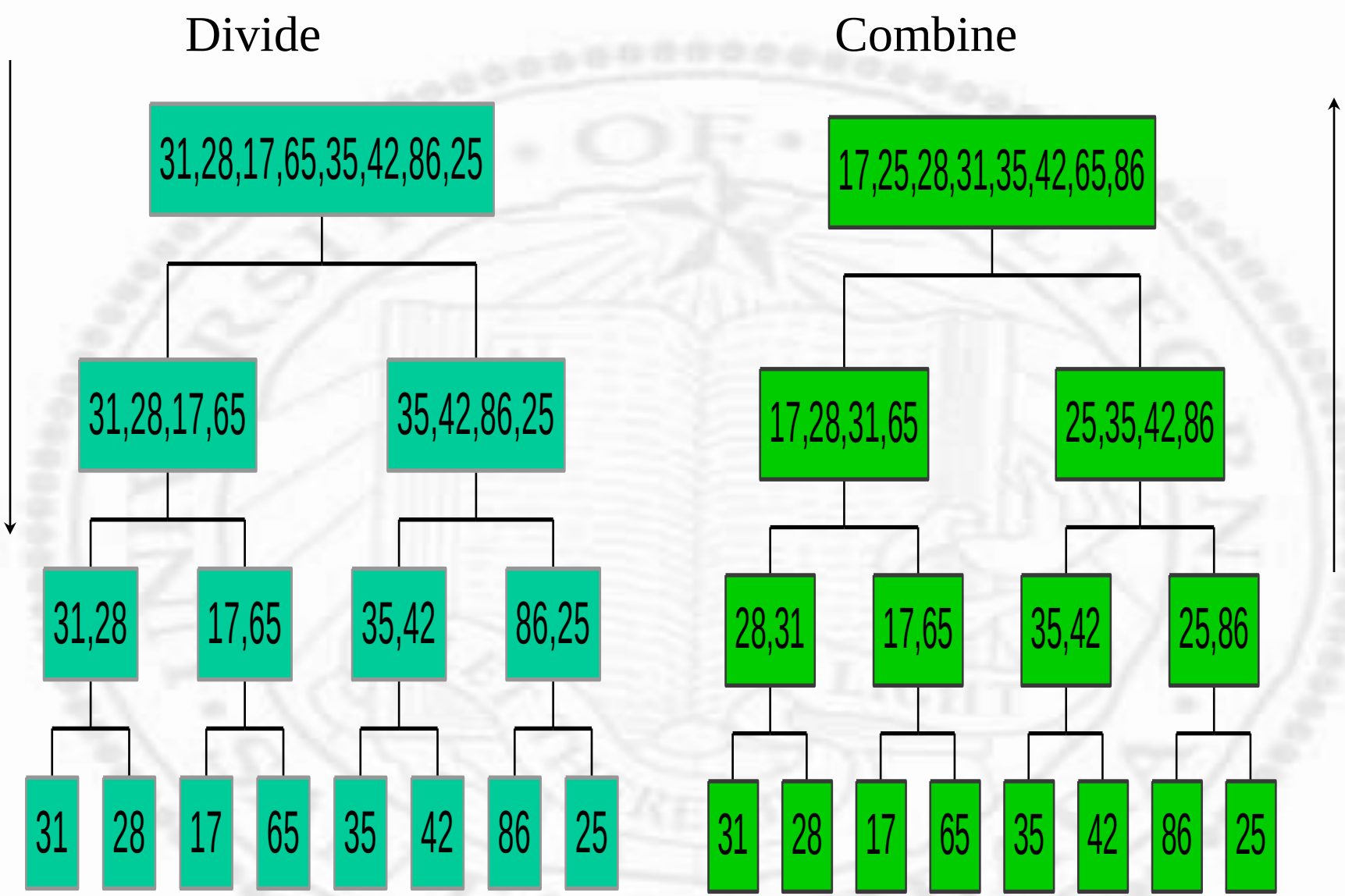
Merge Sort

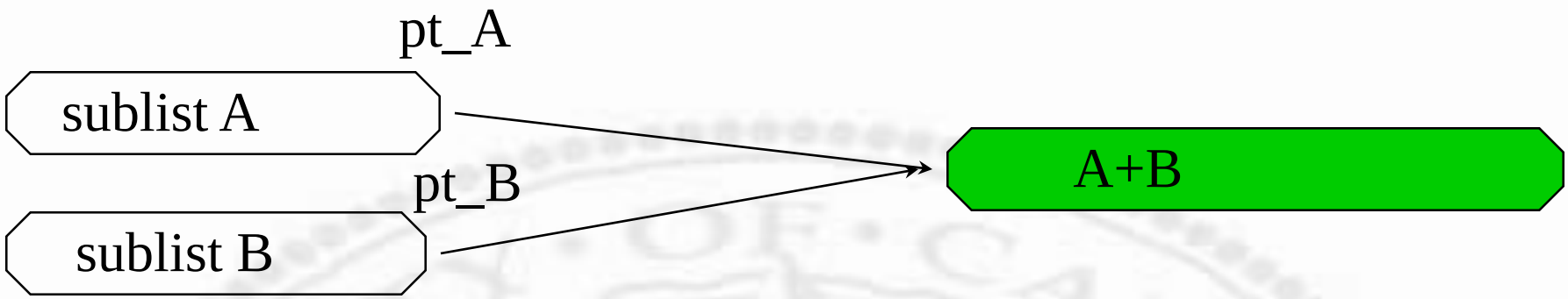
- Input: a list of *n* elements
- Output: a list of the same elements sorted in nondecreasing order

- *Divide*

$$MS(n, a_1, a_2, \ldots, a_n) =$$
$$MS(\lfloor \frac{n+1}{2} \rfloor, a_1, a_2, \ldots, a_{\lfloor \frac{n+1}{2} \rfloor}) +$$
$$MS(n - \lfloor \frac{n+1}{2} \rfloor, a_{\lfloor \frac{n+1}{2} \rfloor + 1}, \ldots, a_n) +$$
$$\text{merge the two sublists properly}$$

- *Small(p,q):* when the problem size becomes *1*
- *G(p,q):* nothing
- *Combine:* trace down the two sublists and merge them properly





if **(pt_A)* is *NULL*, append *B* to *A+B*

else if **(pt_B)* is *NULL*, append *A* to *A+B*

else if **(pt_A) < *(pt_B)*,

    append **(pt_A)* to *A+B*, increment *pt_A*

else

    append **(pt_B)* to *A+B*, increment *pt_B*

end if

$O(|A|+|B|)$ operations

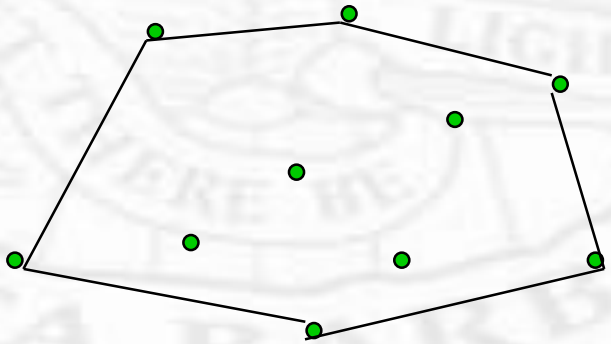
Time complexity

$$MS(n, a_1, a_2, \ldots, a_n)$$
$$= MS(\left\lfloor \frac{n+1}{2} \right\rfloor, a_1, a_2, \ldots, a_{\left\lfloor \frac{n+1}{2} \right\rfloor})$$
$$+ MS(n - \left\lfloor \frac{n+1}{2} \right\rfloor, a_{\left\lfloor \frac{n+1}{2} \right\rfloor + 1}, \ldots, a_n)$$
$$+ \text{merge}$$

$$\begin{aligned} T(n) &= T(\frac{n}{2}) + T(\frac{n}{2}) + cn = 2T(\frac{n}{2}) + cn \\ &= 2(2T(\frac{n}{4}) + c\frac{n}{2}) + cn = 2^2 T(\frac{n}{4}) + 2cn \\ &= 2^2(2T(\frac{n}{8}) + c\frac{n}{4}) + 2cn = 2^3 T(\frac{n}{8}) + 3cn \\ &\ldots \\ &= 2^k T(1) + kcn \qquad n = 2^k \\ &= an + cn\log n \\ &= O(n\log n) \end{aligned}$$

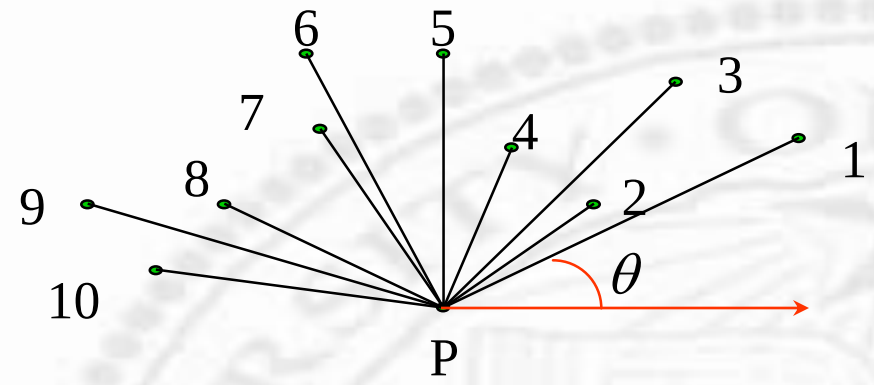
Convex Hull

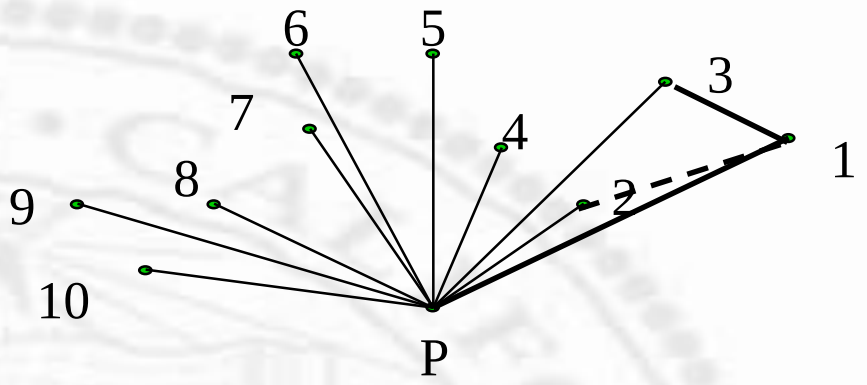
- Input: a collection of *n* points
- Output: the smallest convex polygon that encloses the set of points
  - 2D case: points as nails sticking out on a table, put a rubberband around them

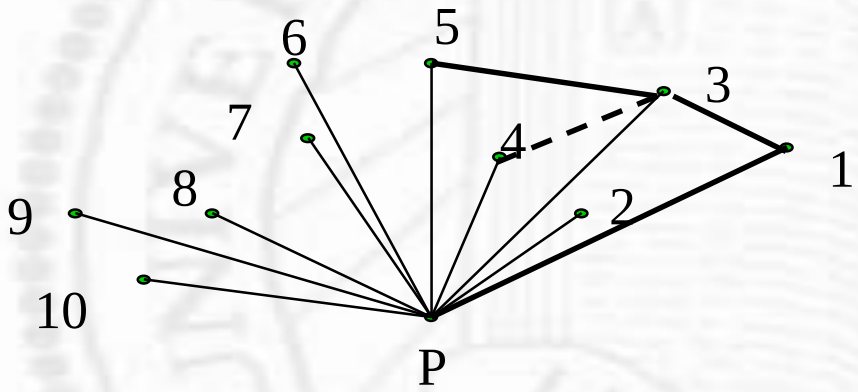


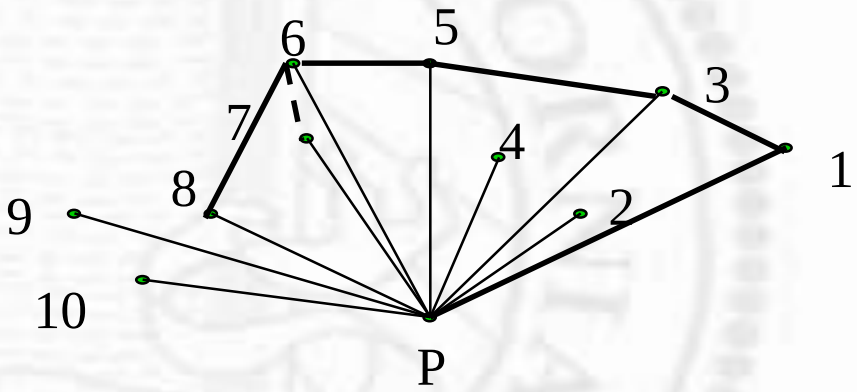
- Properties
  - Use given points as vertices
  - Contain all extreme points in the set
  - Points of smallest and largest x and y coordinates are included
  - Traverse the edge of the hull
    - counterclockwise, all points must be on the left
    - clockwise, all points must be on the right

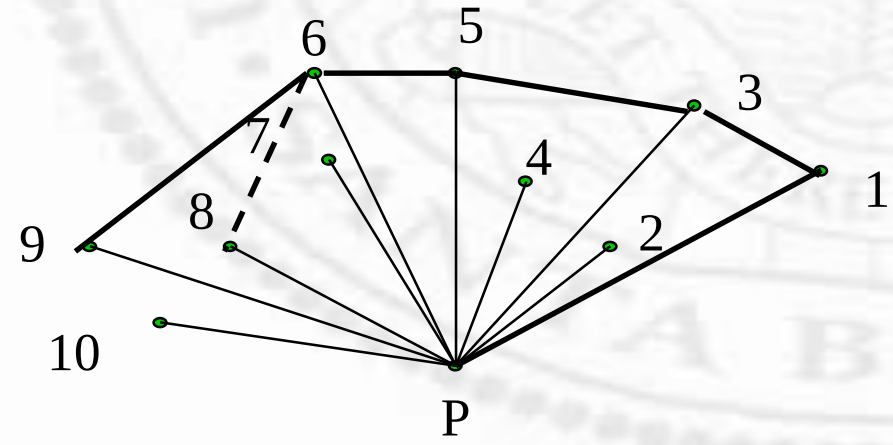
Graham's Scan (package wrapping)

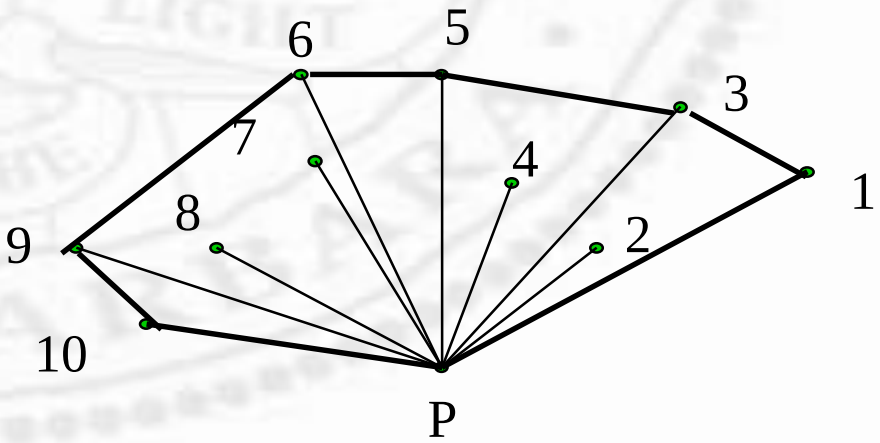






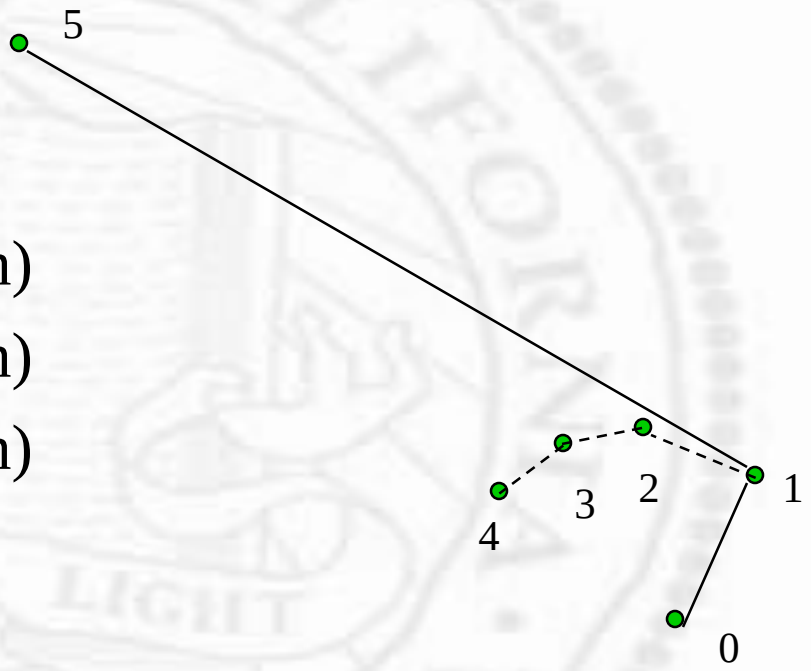






- Start at some point that guaranteed to be on the convex hull (e.g., point with smallest y coordinate)
- From that point, compute theta (see previous slide) for all remaining points
- Sort by theta and consider each point in turn
- After examining *i-1* points
  - p[1..M] are on the convex hull
- After examining *i* points
  - p[M] is *recursively* eliminated if p[M], p[M-1] and P[i] make the wrong turn

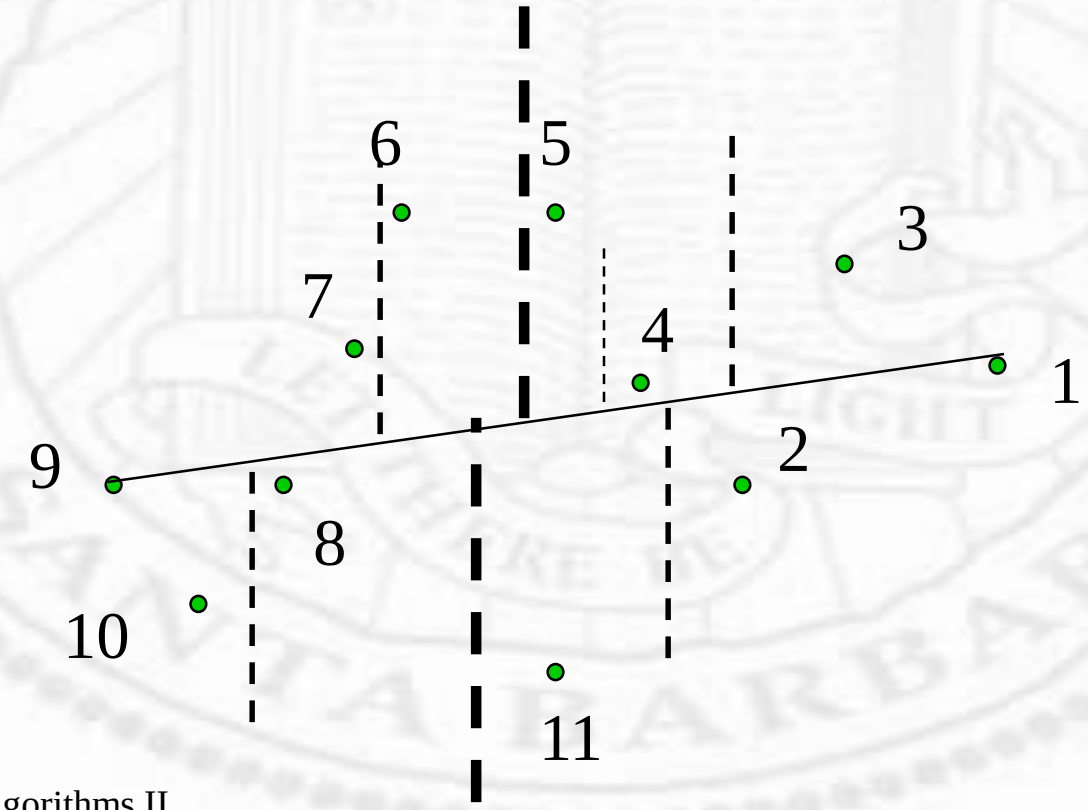
- Example
  - 0 is the base
  - 1,2,3,4 will be included in the hull (all make left turns)
  - when 5 is considered
    - 4 is eliminated (3,4,5 right turn)
    - 3 is eliminated (2,3,5 right turn)
    - 2 is eliminated (1,2,5 right turn)
    - 1 is kept (0, 1,5 left turn)
    - 5 is added



Complexity

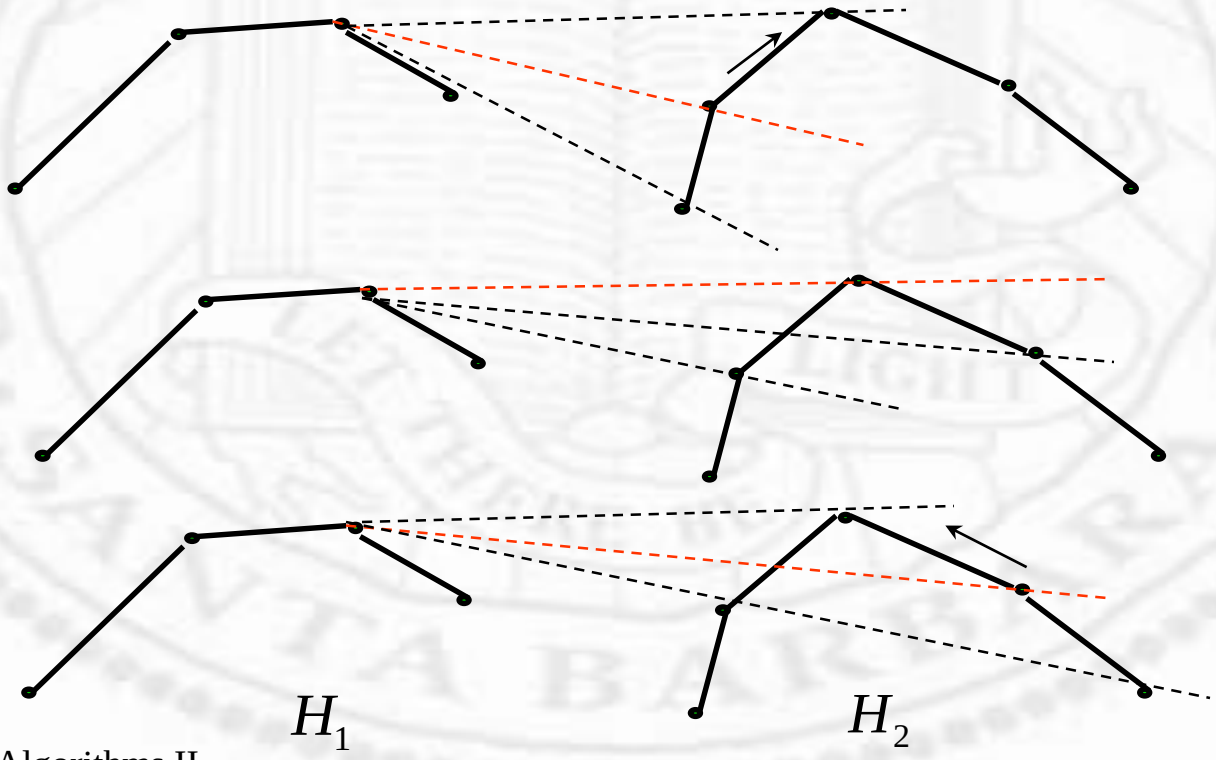
- Angular sorting O(nlogn)
- With n vertices
  - Loop: add vertices to the CH
  - Loop: delete vertices from the CH
  - Each vertex can be added and/or deleted only once
  - Each add/delete operation takes constant time (inner product)
  - O(n) total
- Whole operation: O(nlogn)

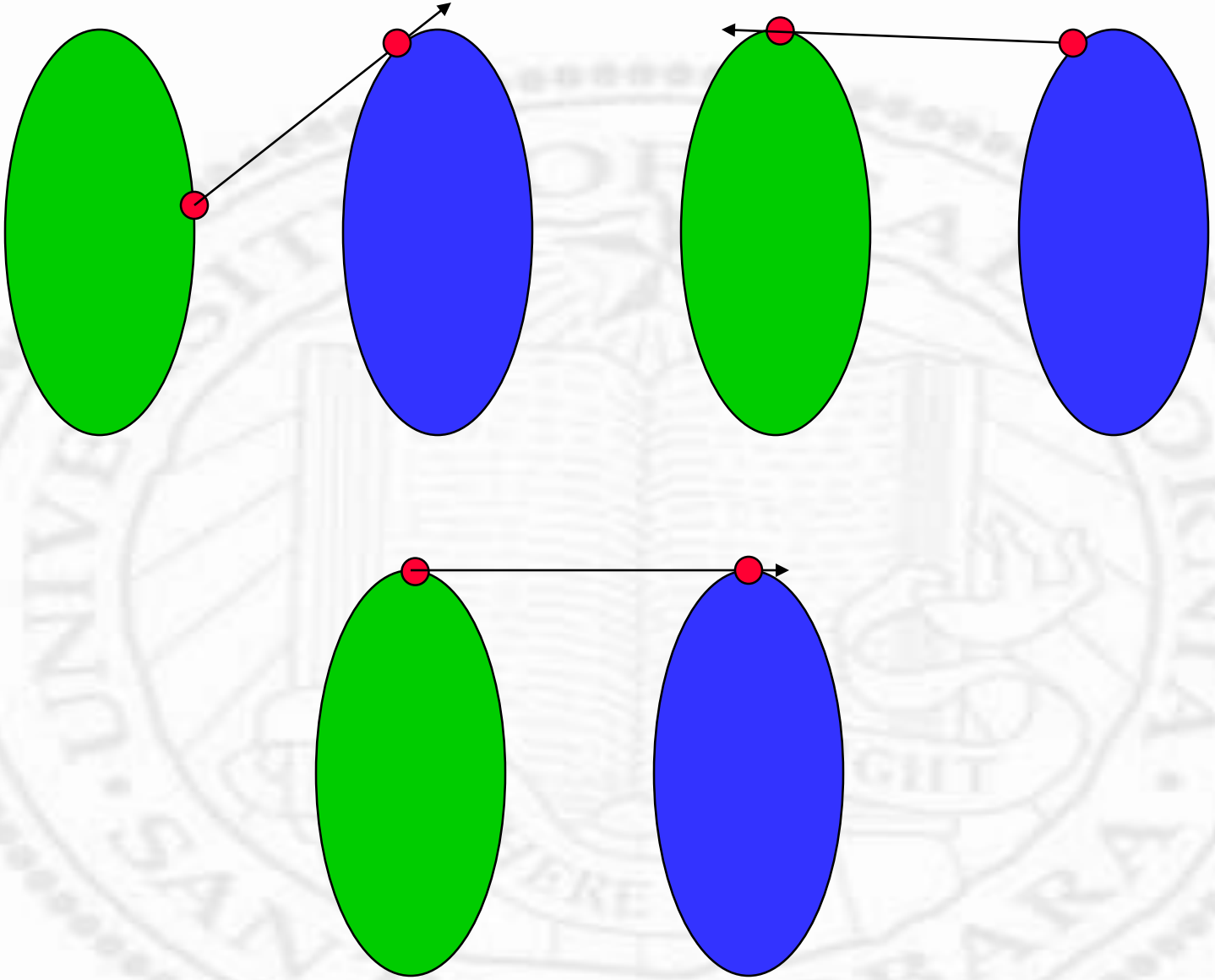
- Divide-and-Conquer
  - Upper hull and lower hull division (not essential)
  - Recursive division



Merge

- Intuition: connecting extreme points (points with the largest *y* coordinate on two hulls)
- Or more precisely, move the connecting lines are high (low) as possible for upper hull (lower hull)
- sort by *y*, too expensive (*O(nlogn)*)
- hill climbing (binary search on sorted *x*)





- If H1 and H2 are two upper hulls with at most m points each. If *p* is any point on H1, its point of tangency, *q*, with H2 can be found on O(*logm*) time
- If H1 and H2 are two upper hulls with at most m points each, their common tangent can be found on O(*log^2 m*) time
- The Divide-and-Conquer convex hull algorithm has a complexity of O(*nlogn*)

UpperTangent(HA ; HB) :

(1)Let a be the rightmost point of HA .

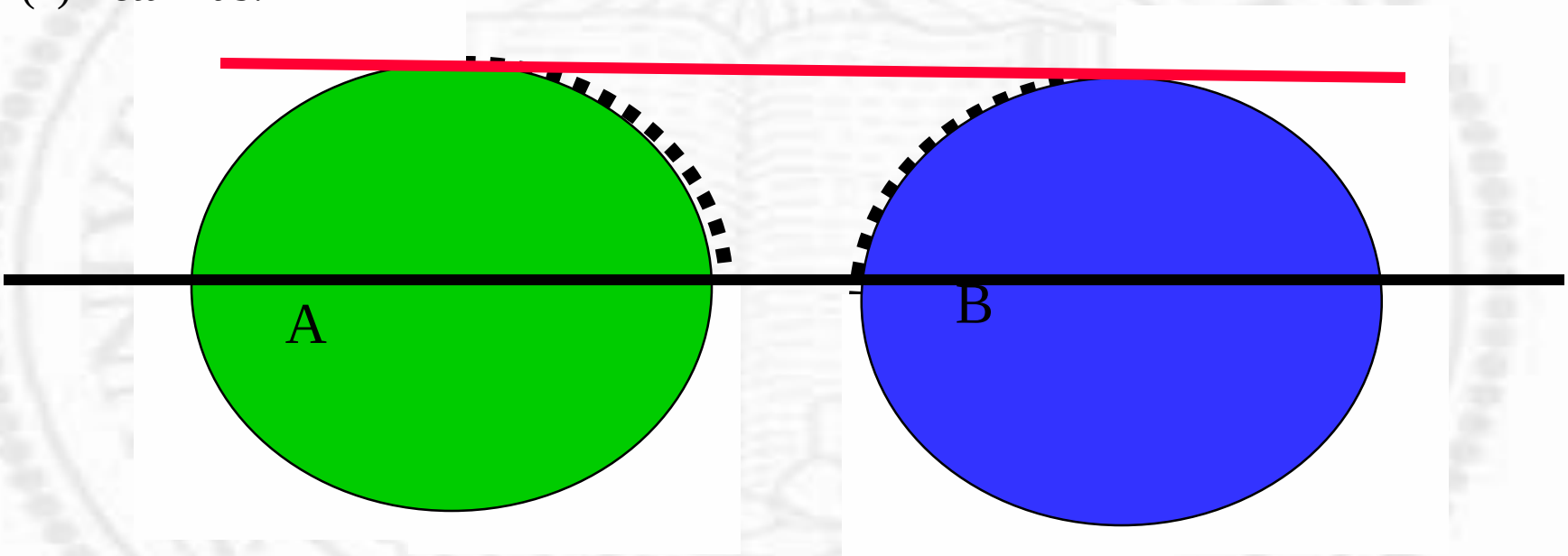
(2)Let b be the leftmost point of HB .

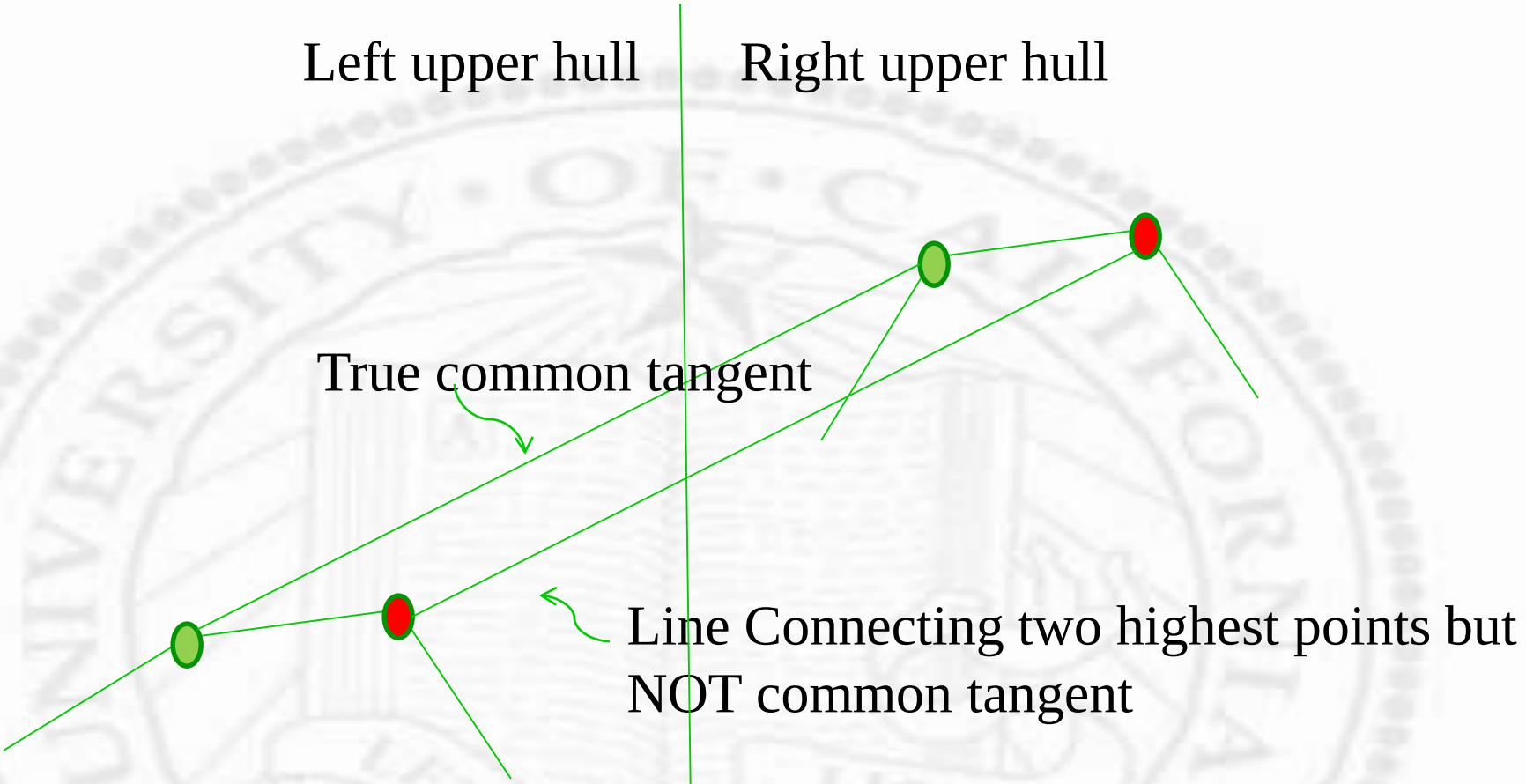
(3)While ab is not a upper tangent for HA and HB do

(a) While ab is not a upper tangent to HA do a = a - 1 (move a counterclockwise).

(b) While ab is not a upper tangent to HB do b = b + 1 (move b clockwise).

(4) Return ab.



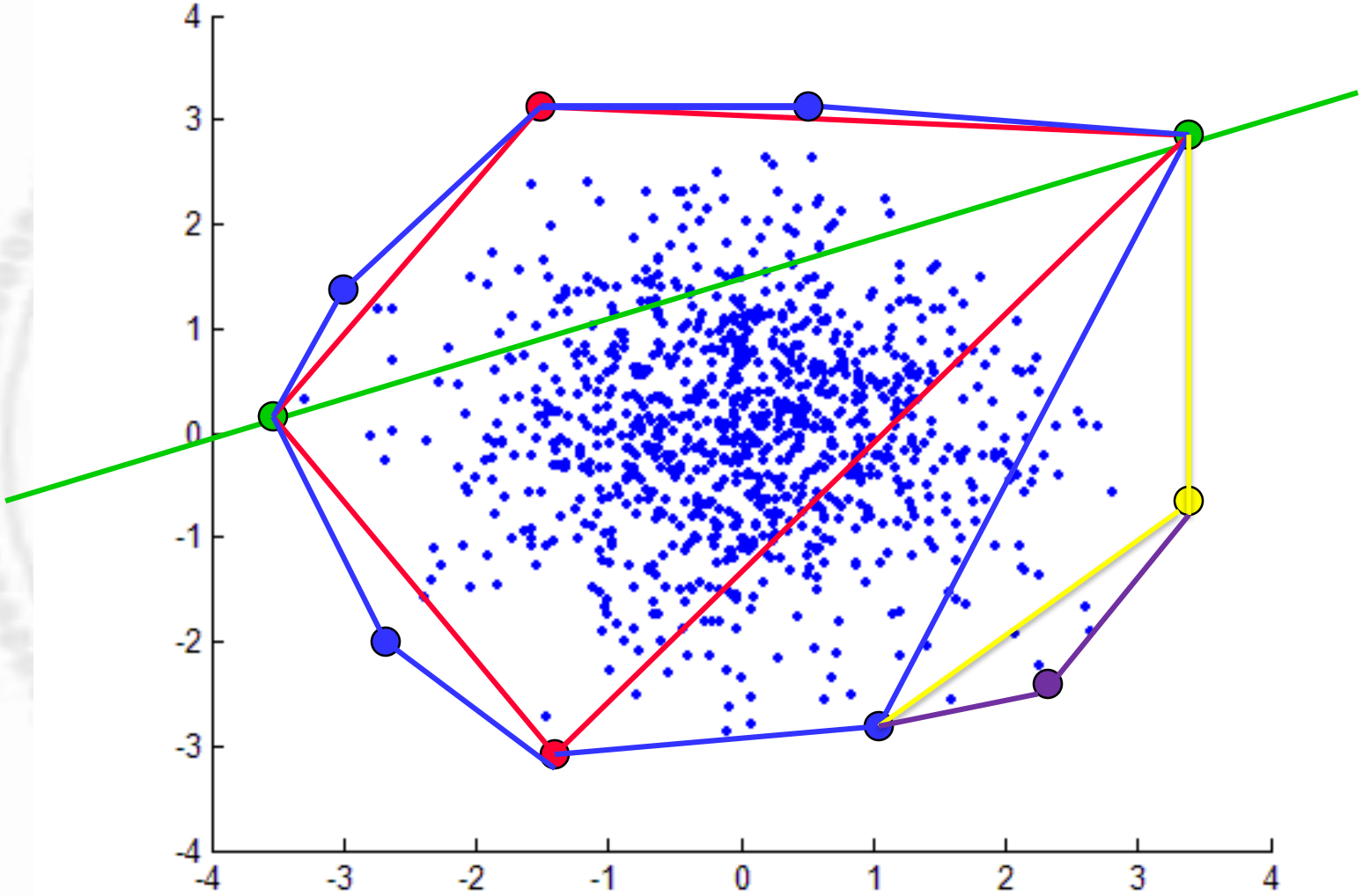


- Nitty-Gritty Details
  - Line connecting two highest points in component hulls is NOT necessarily the common tangent

- Time complexity
  - Upper and lower hulls division
    - largest and smallest x points *O(n)*
    - partition points into two halves *O(n)*
  - Recursive division
    - sort points by x *O(nlogn)*
    - main step

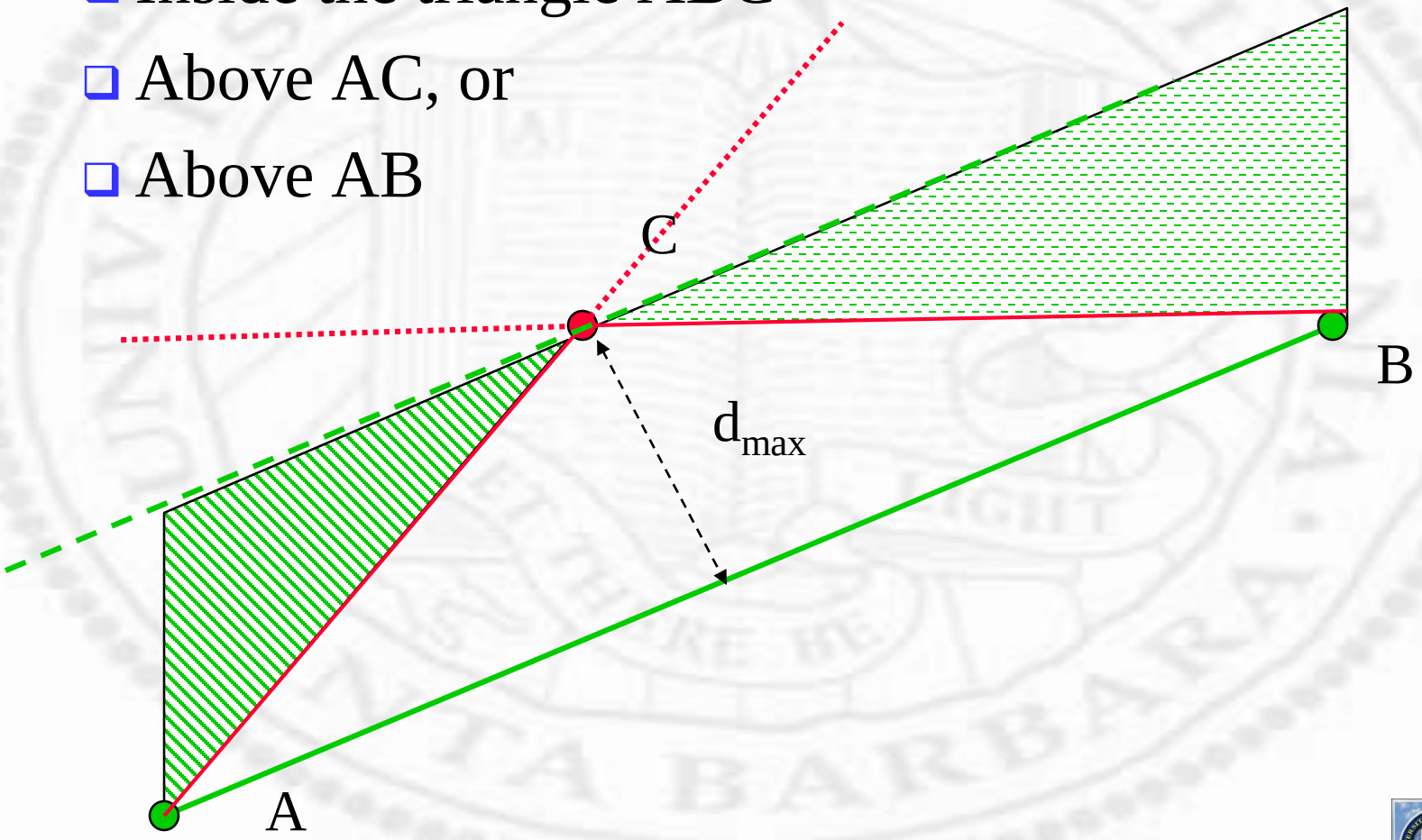
$$T(n) = 2T(\frac{n}{2}) + merge = 2T(\frac{n}{2}) + O(\log^2 n)$$
$$= O(n \log n)$$

Yet Another Divide-and-Conquer Algorithm (QuickHull)



Graphical Illustration

- Three possibilities O(n) time:
  - Inside the triangle ABC
  - Above AC, or
  - Above AB



Complexity

- If points are uniformly distributed in a unit square, expected # of points on the hull is O(logn)
- Quickhull discards interior points very quickly and narrows in peripheral points
- Like Quicksort, average time is O(nlogn) but worst case performance is O(n^2)

Complexity

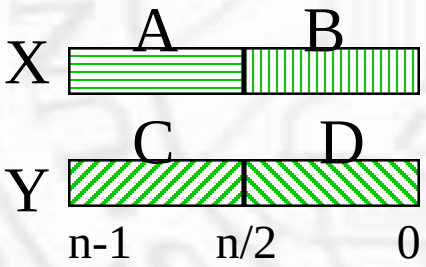
- Quick sort
  - Select pivot (O(1))
  - Partition into two parts O(n)
  - Recursive division
  - Trivial concatenation
  - T(n) = T(i) + T(n-i) + O(n)
- Quick hull
  - Select furthest point (O(n))
  - Partition into three parts O(n)
  - Recursive division
  - Trivial concatenation
  - T(n) = T(i) + T(n-i) + O(n)

Moral of the story

- Algorithm design is an art. We have seen three different convex hull algorithms
  - One based on domain knowledge only
  - Two based on divide-and-conquer

Multiplying Long Integers

- Input: two *n*-bit integer *x* and *y*
- Output: a *2n*-bit integer $x \times y$
- Divide-and-Conquer strategy



$$x = A2^{\frac{n}{2}} + B$$

$$y = C2^{\frac{n}{2}} + D$$

$$x \times y = (A2^{\frac{n}{2}} + B)(C2^{\frac{n}{2}} + D)$$

$$= AC2^{n} + (AD + BC)2^{\frac{n}{2}} + BD$$

- *Divide:* multiply 2 *n*-bit integers
  - = 4 multiplies of 2 *n/2*-bit integers
  - + 3 additions of integers (*2n* bits)
  - + 2 shifts
- *Small(p,q):* when the length becomes *1*
- *G(p,q): 1*-bit AND
- *Combine:* shift and addition

$$\underset{A}{0\,1} \times \underset{C}{0\,1} \times 2^4 + \underset{A}{0\,1} \times \underset{D}{1\,1} + \underset{B}{1\,0} \times \underset{C}{0\,1} \times 2^2 + \underset{B}{1\,0} \times \underset{D}{1\,1}$$

$$0 \times 0 \times 2^2 + 0 \times 1 + 1 \times 0 \times 2 + 1 \times 1$$

$$0 \times 1 \times 2^2 + 0 \times 1 + 1 \times 1 \times 2 + 1 \times 1$$

$$1 \times 0 \times 2^2 + 1 \times 1 + 0 \times 0 \times 2 + 1 \times 0$$

$$1 \times 1 \times 2^2 + 1 \times 1 + 0 \times 1 \times 2 + 1 \times 0$$

$$0\,0\,0\,1 \times 2^4 + 0\,0\,1\,1 + 0\,0\,1\,0 \times 2^2 + 0\,1\,1\,0$$

00010000 + 00010100 + 00000110 = 00101010

=42

Time complexity

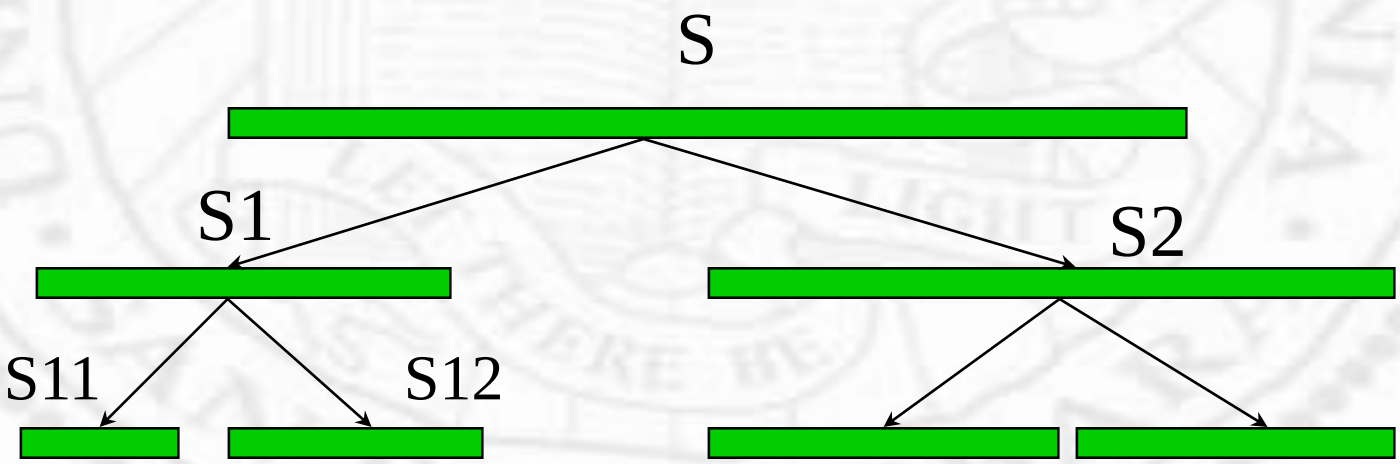
$$x \times y = (A2^{\frac{n}{2}} + B)(C2^{\frac{n}{2}} + D)$$

$$= AC2^n + (AD + BC)2^{\frac{n}{2}} + BD$$

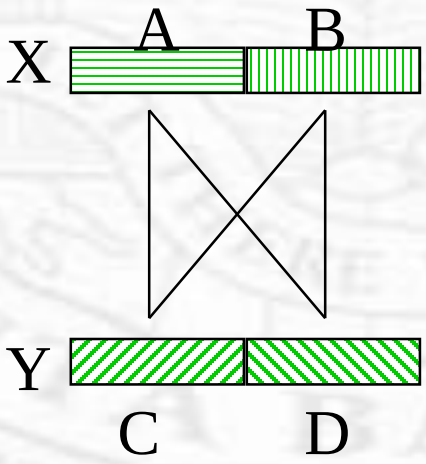
$$\begin{aligned} T(n) &= 4T(\frac{n}{2}) + cn \\ &= 4(4T(\frac{n}{4}) + c\frac{n}{2}) + cn = 4^2 T(\frac{n}{4}) + cn(1+2) \\ &\dots \\ &= 4^k T(1) + cn(1+2+\dots+2^{k-1}) \\ &= 4^{\log n} a + cn(2^{\log n} - 1) \\ &= O(n^2) \end{aligned}$$

- cf. brute force method $O(n^2)$

- Why no improvement using divide-and-conquer?
  - in quick sort
    - elements in *S1* do not compare with those in *S2*
    - elements in *S11* do not compare with those in *S12*
    - *problems are **decomposable** and **independent***



- In integer multiplication
  - problems are *decomposable* ***but not*** i*ndependent*
  - the number of multiplications ***is not*** reduced
  - The fancy way of decomposing the solution still requires every digit in one number to "touch" every digit in the other number (*no* sharing, *no* reuse)



- Possible improvements: through sharing

$$x \times y = (A2^{\frac{n}{2}} + B)(C2^{\frac{n}{2}} + D)$$

$$= AC2^{n} + (AD + BC)2^{\frac{n}{2}} + BD \qquad 4\times, 3+, 2 \leftarrow$$

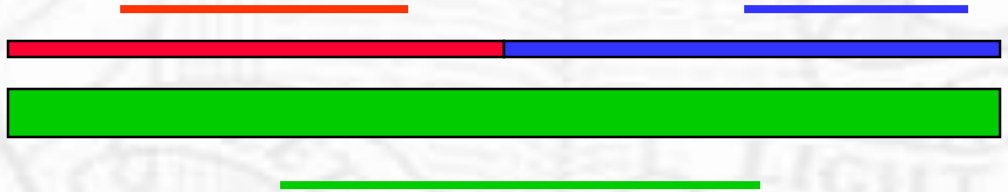
$$= AC2^{n} + \{(A - B)(D - C) + AC + BD\}2^{\frac{n}{2}} + BD$$

$$T(n) = 3T(\frac{n}{2}) + cn = O(n^{\log_2 3}) = O(n^{1.59}) \qquad 3\times, 6+, 2 \leftarrow$$

Maximum Sum

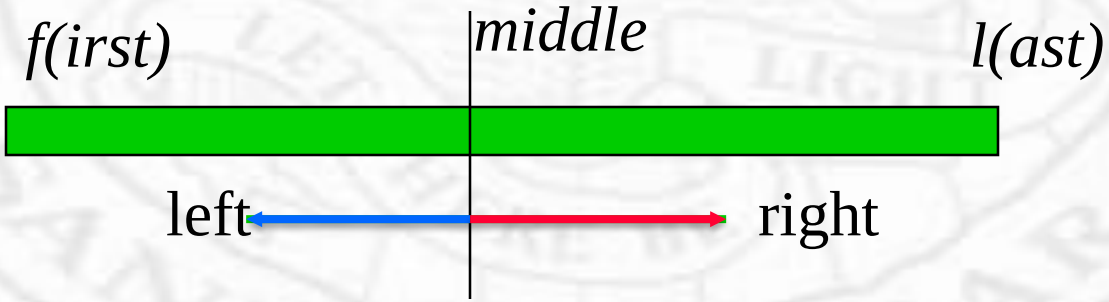
- Just to confuse you more, it is not to say that the subproblems must be totally independent for divide-and-conquer to work
- Given: an array of *n* numbers, possibly negative
- Find: maximum subsequence sum (if all numbers are negative, then the maximum sum is 0)
- -4, 10, 12, -5, -7, 8, 3, 1 is 22

- How does divide-and-conquer work?
  - Divide the array into two parts
  - Compute the maximum sum in each part
  - The global maximum sum is the largest of the two
  - but …
- What happens if the maximum sum sequence straddles the boundary?

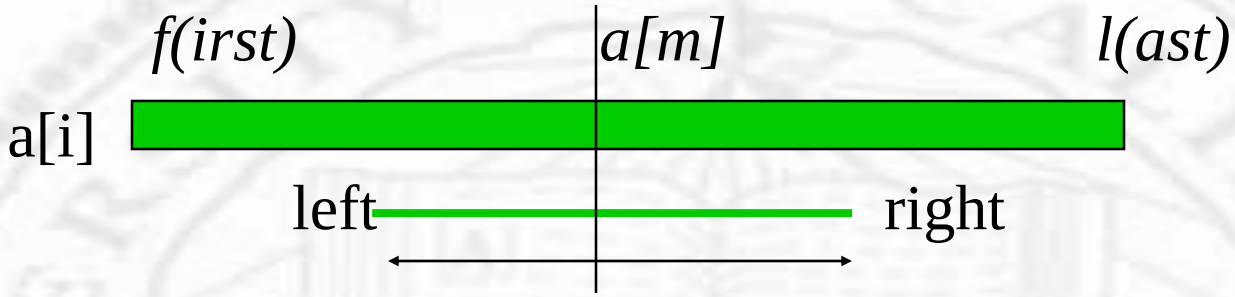


- 4, -3, 5, -2, -1, 2, 6, -2

- Start from the middle
  - Accumulate from middle moving leftward, keep the largest sum
  - Accumulate from middle moving rightward, keep the largest sum
  - The largest partial sum across two parts must be the sum of the above two



- Need a third term which captures the maximum sum of straddling sequence



$b_l = b[0] = a[m];$

$for(i = m-1; i >= f; i--)$

$b[m-i] = b[m-i-1] + a[i];$

$if\ (b[m-i] > b_l)\ then$

$b_l = b[m-i]$

$b_r = b[0] = a[m+1];$

$for(i = m+2; i <= l; i++)$

$b[i-m-1] = b[i-m-2] + a[i];$

$if\ (b[i-m-1] > b_r)\ then$

$b_r = b[i-m-1]$

Retained largest partial sums

- Then the maximum sum is the largest of three terms: two from left and right, one *bl+br*

- Complexity

$$T(n) = 2T(\frac{n}{2}) + n = O(n \log n)$$

- Bruteforce method

```
for (f=1; f<=n; f++)      <-- All possible first pos
    for (l=f; l<=n; l++)      <-- All possible last pos
        for (k=f; k<=l; k++)      <-- Sum from first to last
            Add up all the a[k]
    will be O(n^3)
```

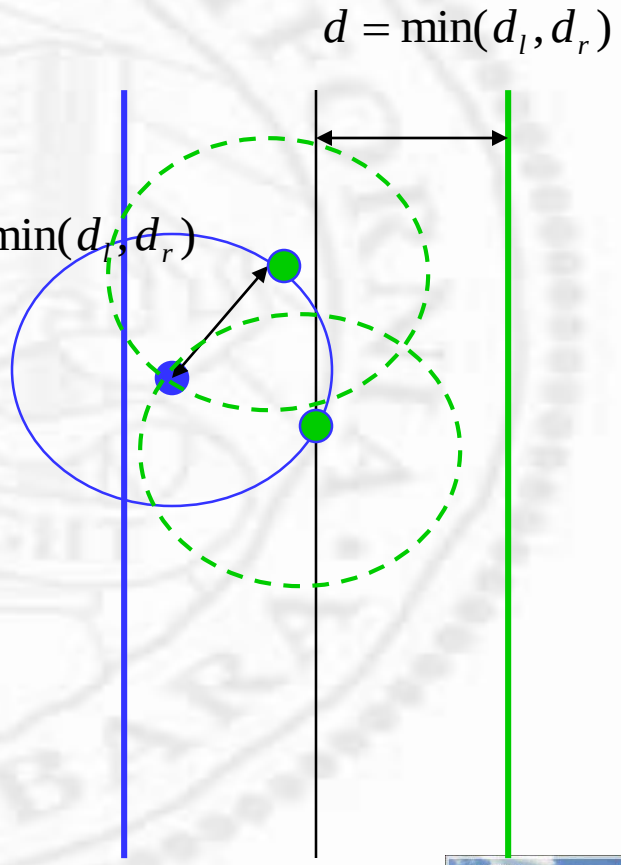
Closest Pair of Points

- Given a set of points on a plane, find the two points which are closest to each other
- Brute force method is O(n^2)
- Can divide-and-conquer do better?
- Obvious solution:
  - partition data sets into two halves (recursively)
  - closest pair of points are in
    - the left half or right half
    - one each in each half

- The closest points in the left and right halves can be found recursively
- But how to find points across boundary?
  - Obvious solution: check each *n/2* points in the left against each *n/2* points in the right
  - The solution will be O(n^2), no better than brute force method
- Again, the problem is that two problems are not entirely independent and combining subsolutions can be tricky.

- Goal: if we want an O(nlogn) solution, then the combination step must be of O(n)
- What is the linear solution in combination?
- A clever trick

- Q: How many points do you have to check?
- A: No blue (green) point can lie inside the circle of radius *d* around another blue (green) point



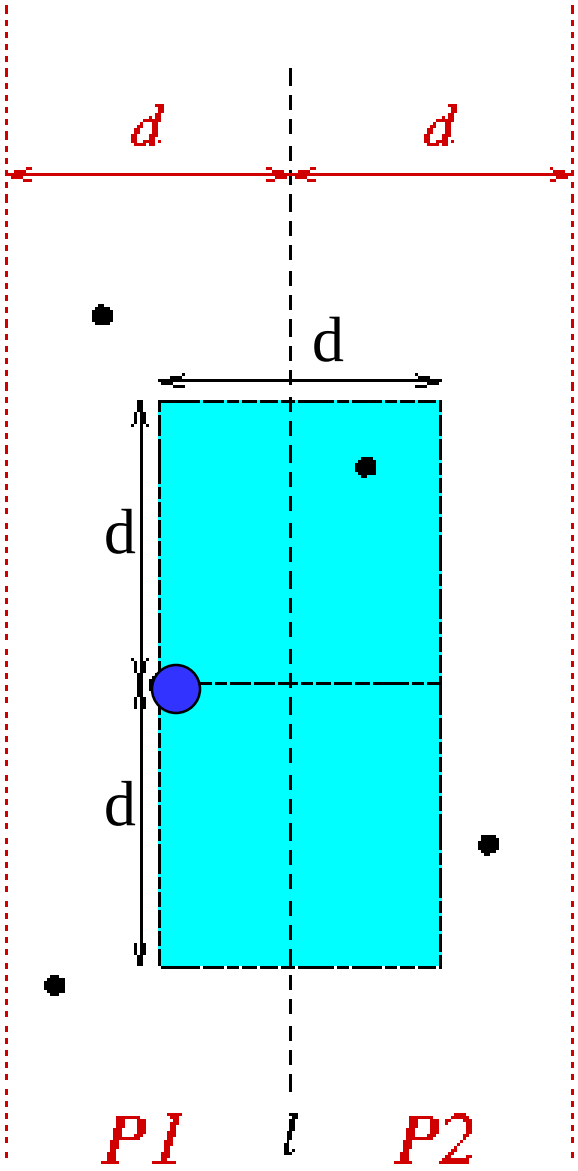
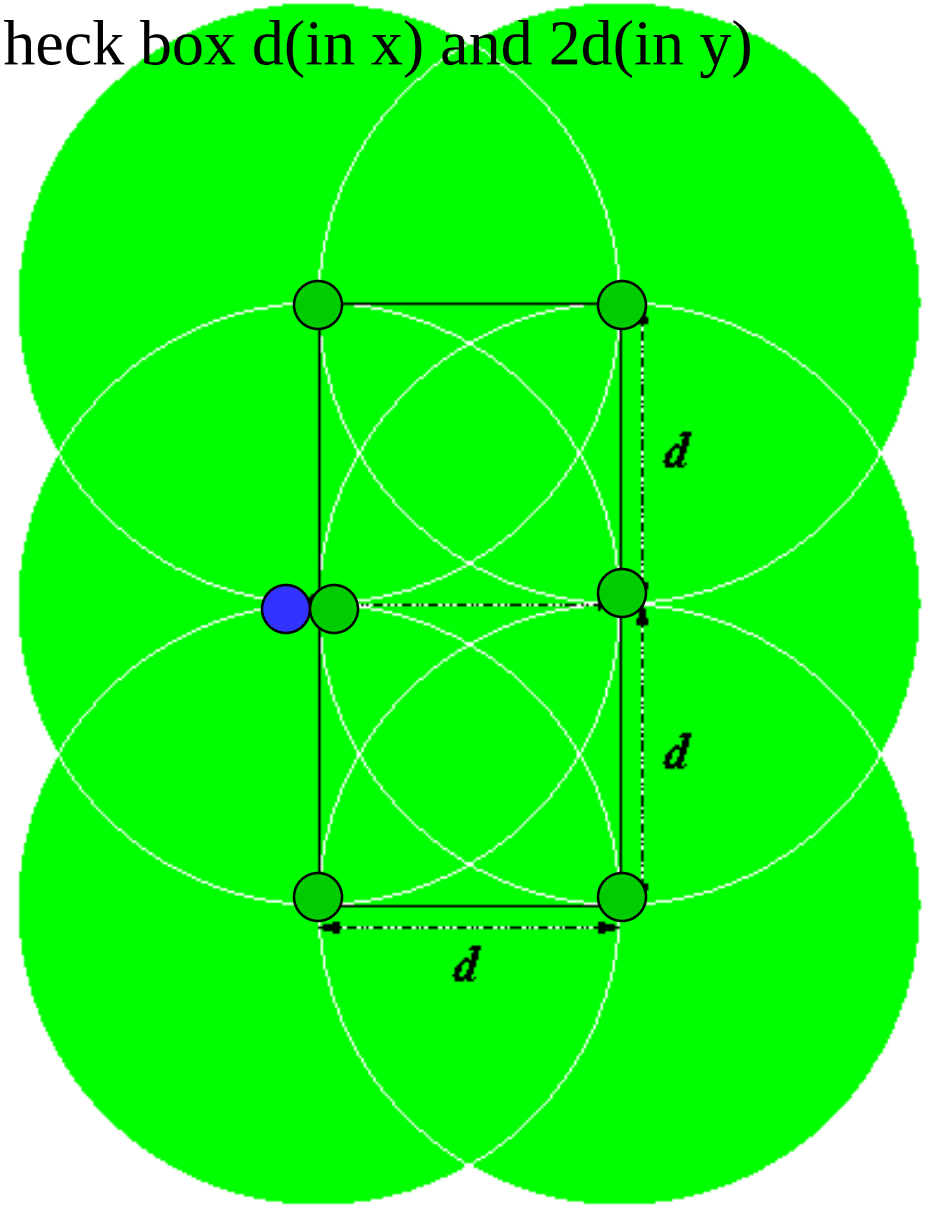


Figure 3.3

Check box d(in x) and 2d(in y)

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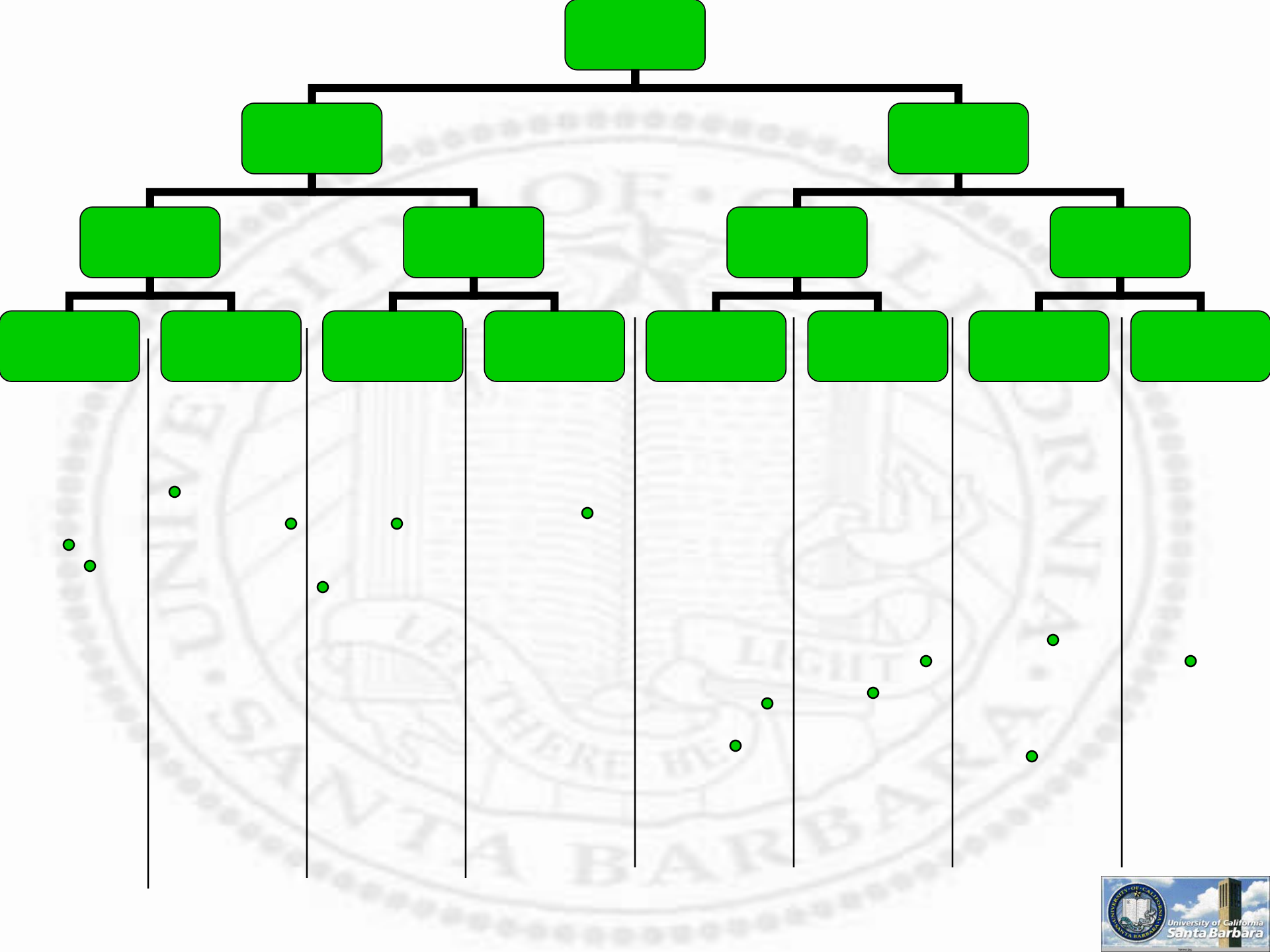
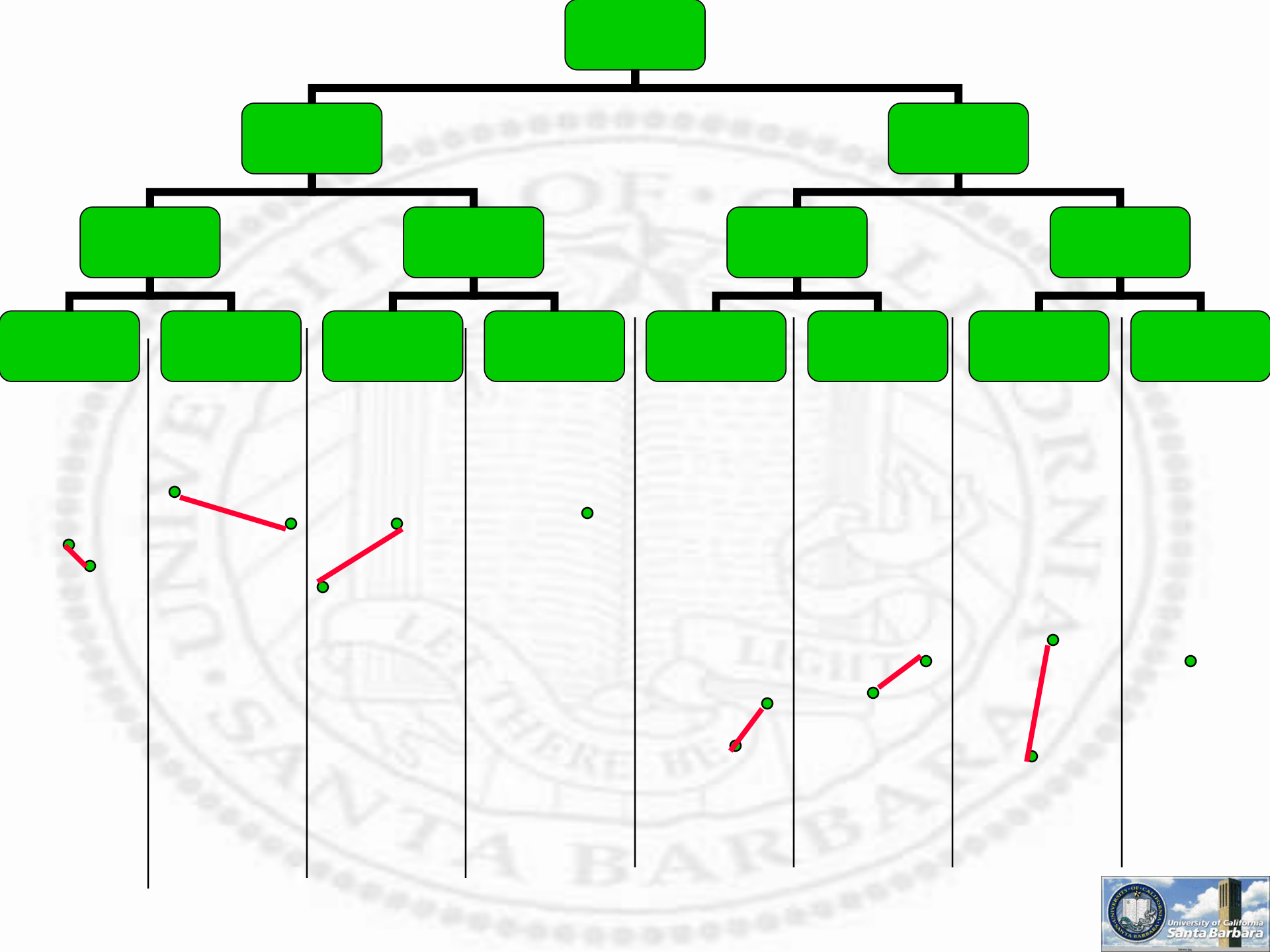
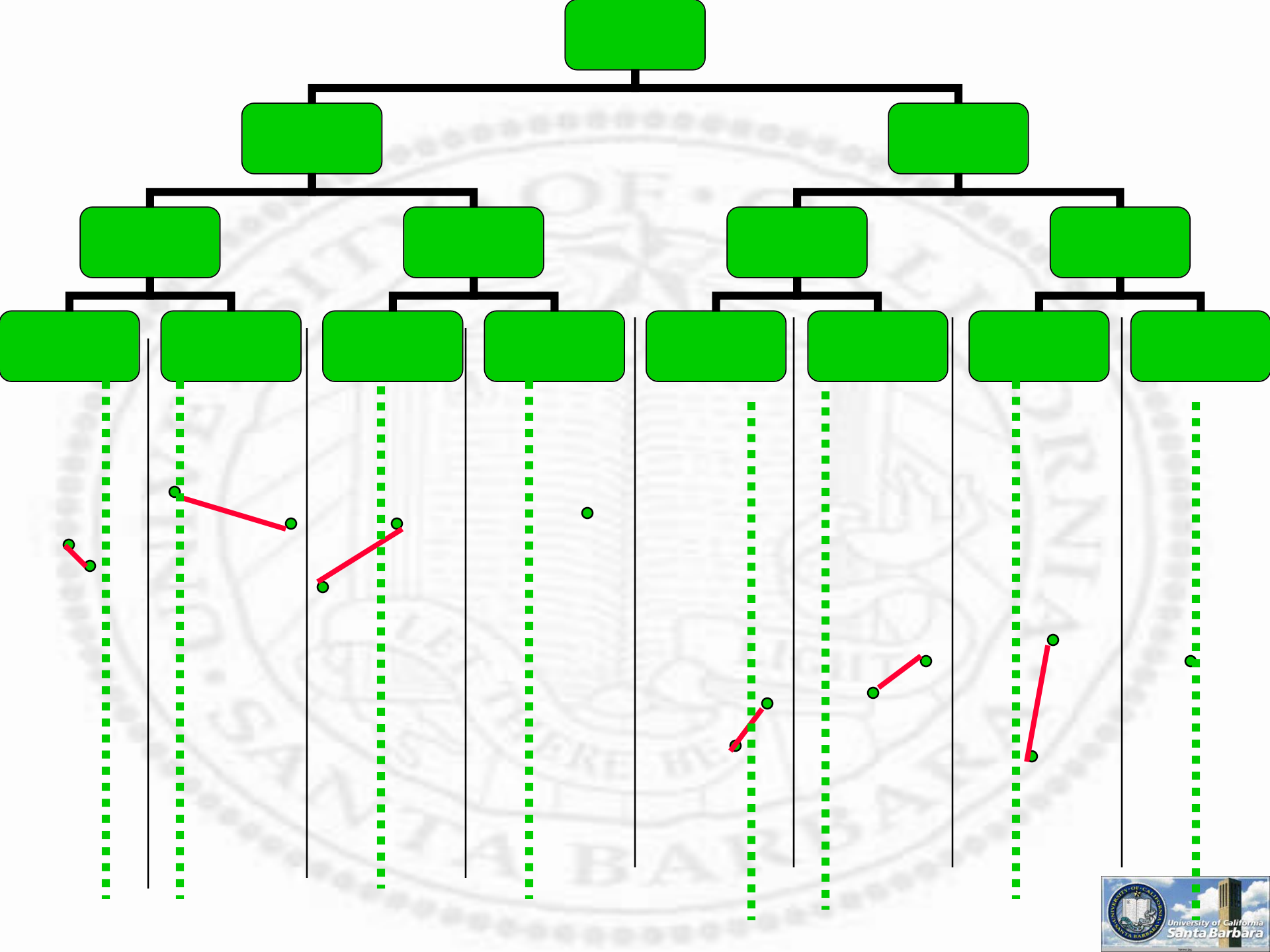
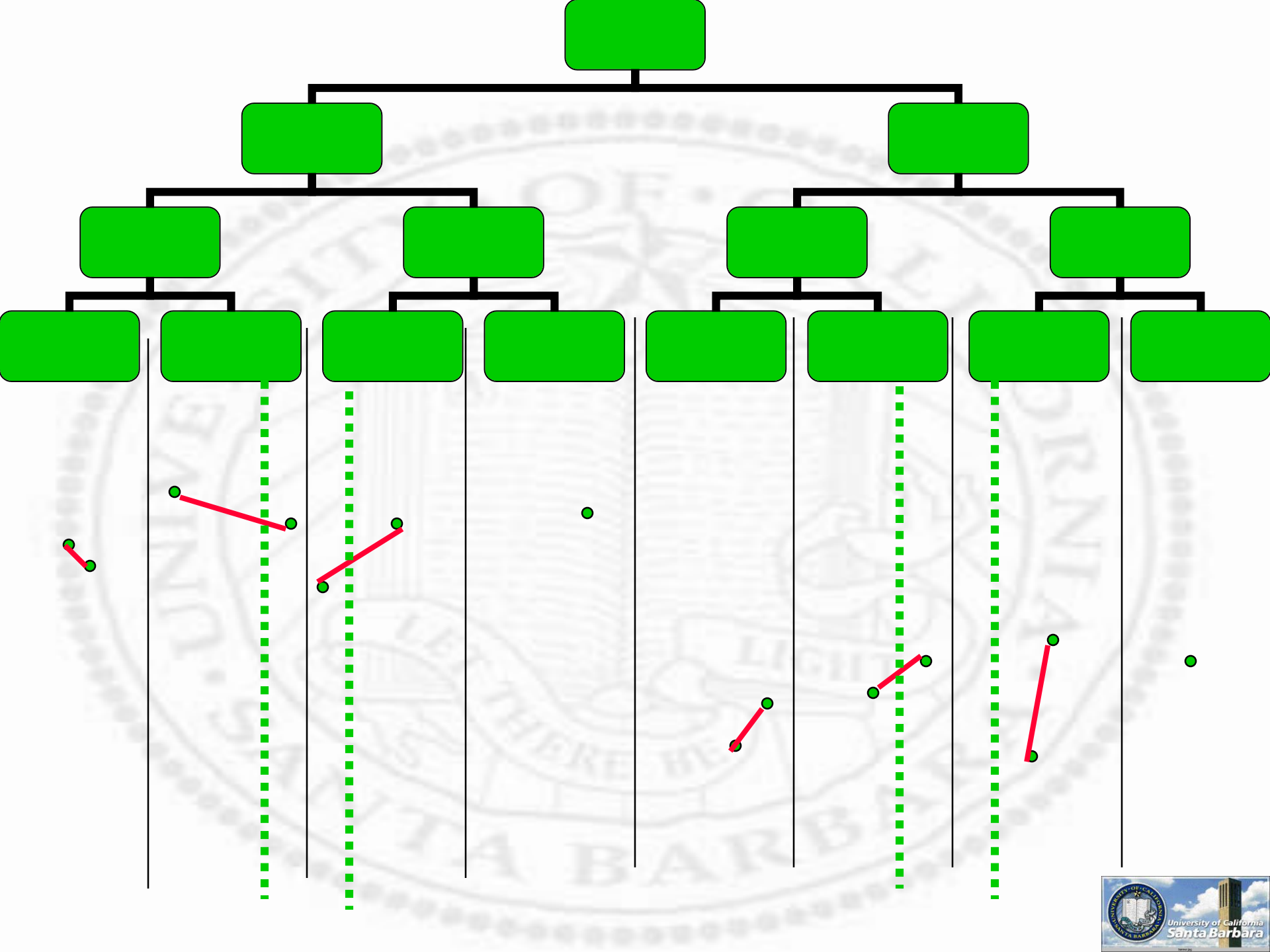
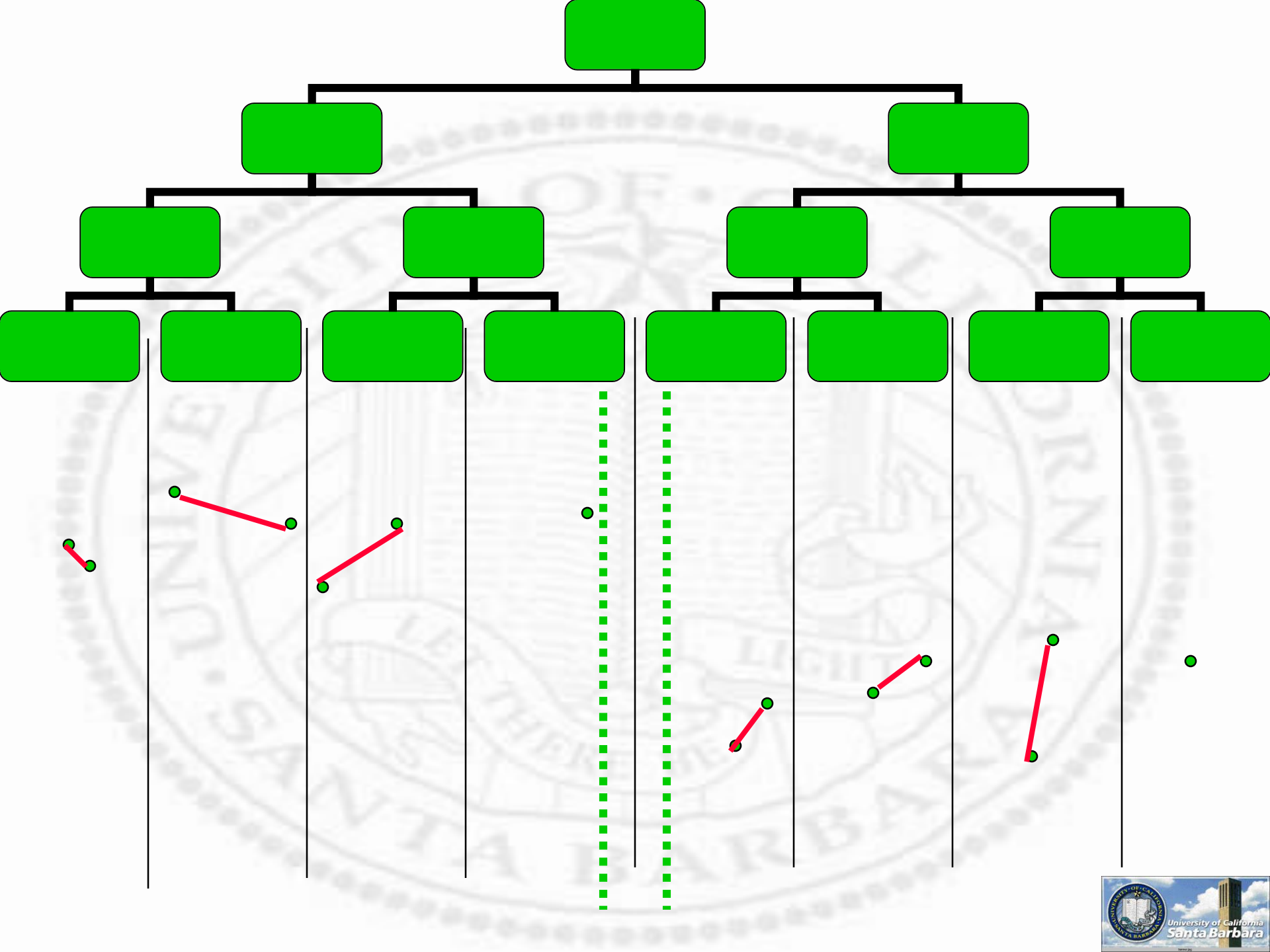










Fourier Transform

- Decompose a (time, space) signal into its frequency components (bases)
- Fourier bases e^{-iwt} are
  - Orthogonal
  - Complete
  - Convenient

Fourier Transform Intuition

- 3D vector space
- Bases **x** =(1,0,0), **y** =(0,1,0), **z** =(0,0,1)
- **v**= (a,b,c)
- a = **v.x**, b=**v.y**, c=**v.z**
- **v**= a**x**+b**y**+c**z**

- Inf-D Function space
- Bases

$$e^{-iwt} = \cos(wt) - i\sin(wt)$$

- Any function f

$$F(w) = \int f(t)e^{-iwt}\,dt$$

$$f(t) = \int F(w)e^{iwt}\,dt$$

Fourier Transform Properties

- Too many, only 2 will be presented here
- Convolution <-> multiplication
- (narrowed-spaced) pulse train <-> (widely-spaced) pulse train
- Why?
  - These two properties alone can explain continuous transform into discrete transform

System Theory

- What is a system?
- How to study a system?

Linear Systems

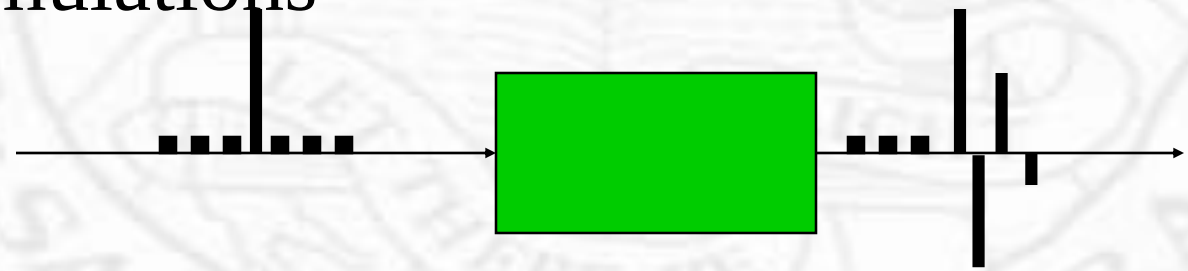
- Three important properties of *linear*, *shift-invariant systems:*
  - Superposition
    - $R(f + g) = R(f) + R(g)$
  - Scaling
    - $R(kf) = k\, R(f)$
    - $R(af + bg) = aR(f) + bR(g)$

  Linear system

  - Shift invariance
    - Translation of stimulus → translation of response
    - $h(t) = R(g(t)) \rightarrow h(t+k) = R(g(t+k))$

Impulse Response

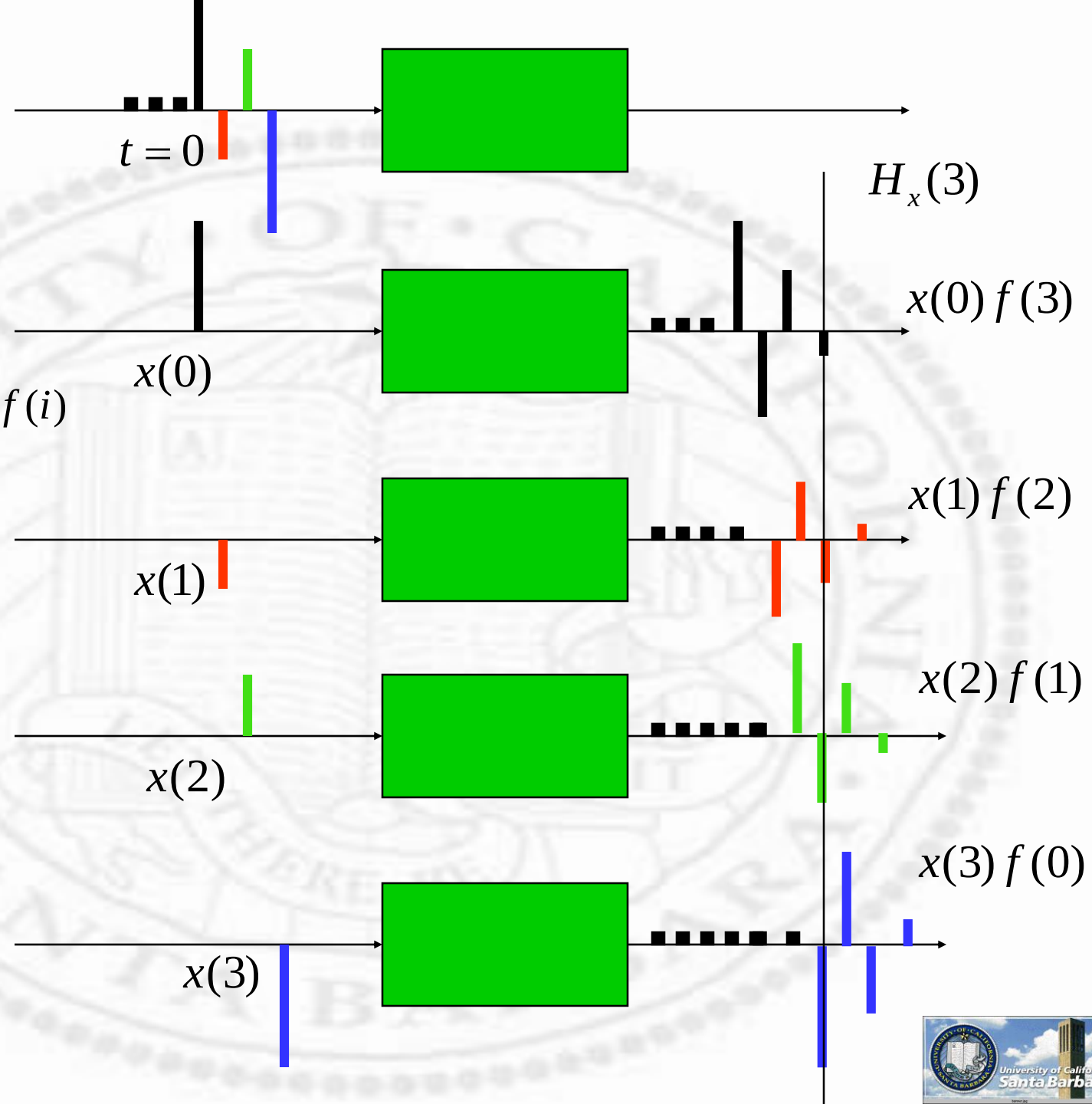
- The response to the simplest stimulation (an impulse)
- Using impulse response and linear, time invariant behavior, one can predict exactly what the system will do to any *arbitrary* stimulations



$$H_x(3) = \sum_{i=0}^{n-1} x(3-i)\, f(i)$$

$$= \sum_{i=3-n+1}^{3} x(i)\, f(3-i)$$

n : filter length



In General

- The system response is a "convolution" of input and system's impulse response function

$$H_x(j) = \sum_{i=0}^{n-1} x(j-i)\, f(i) = \sum_{i=j-n+1}^{j} x(i)\, f(j-i)$$

- In continuous domain, summation becomes integration

$$H_x(t) = \int_{u=0}^{u=w} x(t-u)\, f(u)du = \int_{u=t-w}^{u=t} x(u)\, f(t-u)du$$

Fourier Transform of Convolution

$$H_x(t) = \int_{u=0}^{u=w} x(t-u)\, f(u)du = \int_{u=t-w}^{u=t} x(u)\, f(t-u)du$$

$$H(w) = \int [\int\ f(t-u)g(u)du]e^{-iwt}dt$$

$$= \int\int\ f(t-u)e^{-iwt}dtg(u)du$$

$$= \int\int\ f(t-u)e^{-iw(t-u)}d(t-u)g(u)e^{-iwu}du$$

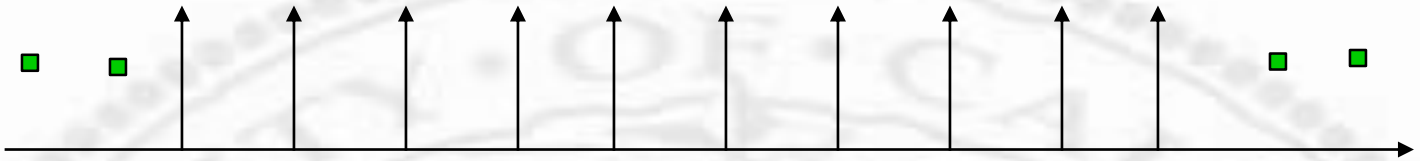
$$= [\int\ f(t-u)e^{-iw(t-u)}d(t-u)][\int g(u)e^{-iwu}du]$$

$$= [\int\ f(t')e^{-iwt'}dt'][\int g(u)e^{-iwu}du]$$

$$= F(w)G(w)$$

$$f(t) \otimes g(t) \Leftrightarrow F(w)G(w)$$

Pulse Train



$$F(w) = \int f(t)e^{-iwt}\,dt = \int \sum \delta(k\Delta t)e^{-iwt}\,dt$$

$$= \sum \int \delta(k\Delta t)e^{-iwt}\,dt = \sum e^{-iwk\Delta t} = \sum e^{-\frac{iwk}{T}}$$

$$= \sum e^{-\frac{i2\pi fk}{T}} = \begin{cases} 1 & f = nT\ (\frac{1}{\mathrm{T}}: \text{fundamenta l freq}) \\ 0 & otherwise \end{cases}$$

- Fast t train -> small Dt -> large T -> large f-> slow f train
- Slow train-> large Dt -> small T -> small f -> fast f train

Combination

- Multiply a signal (f) with a pulse train (sampling or discretization) in the space domain ⇔Convolve signal's spectrum (F) with another pulse train
- Fast changing signal ⇔fast sampling
- Sampling above Nyquist frequency ensures no signal loss

Discrete FT

- If signal is periodical
- Component frequencies must be multiple of the 1/period
- Spectrum is discrete

- If signal is discrete (sampling with a pulse train)
- Component frequencies are results of convolution
- Spectrum is periodical

Time	Frequency
Periodical	Discrete
Discrete	Periodical
Periodical & discrete	Discrete & periodical

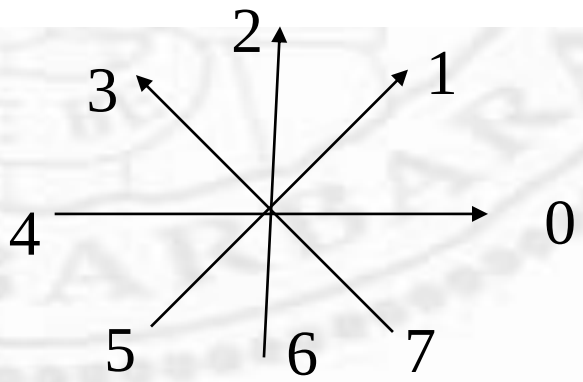
Mathematical Formula

$$F[n] = \sum_{k=0}^{N-1} f[k] e^{-j\frac{2\pi}{N}nk} \qquad (n = 0 : N-1)$$

$$\begin{pmatrix} F[0] \\ F[1] \\ F[2] \\ \vdots \\ F[N-1] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & W & W^2 & W^3 & \dots & W^{N-1} \\ 1 & W^2 & W^4 & W^6 & \dots & W^{N-2} \\ 1 & W^3 & W^6 & W^9 & \dots & W^{N-3} \\ \vdots & & & & & \\ 1 & W^{N-1} & W^{N-2} & W^{N-3} & \dots & W \end{pmatrix} \begin{pmatrix} f[0] \\ f[1] \\ f[2] \\ \vdots \\ f[N-1] \end{pmatrix}$$

$$W = \exp(-j2\pi/N)$$

$W = W^{2n}$



Computation Complexity

- Obviously multiplying n by n matrix with n by 1 vector is $O(n^2)$
- Fast Fourier transform O(nlogn)
- VERY significant as FFT is an important operation of many image and signal processing algorithms
- Based on Divide and Conquer!

Why?

$$F[n] = \sum_{k=0}^{N-1} f[k] e^{-j\frac{2\pi}{N}nk}$$

$$F[n] = \sum_{k=0}^{N-1} f[k] W_N^{nk}$$

$$W = \exp(-j2\pi/N)$$

$$F[n] = \sum_{m=0}^{\frac{N}{2}-1} f[2m] W_N^{2mn} + \sum_{m=0}^{\frac{N}{2}-1} f[2m+1] W_N^{(2m+1)n}$$

$$F[n] = \sum_{m=0}^{\frac{N}{2}-1} f[2m] W_{\frac{N}{2}}^{mn} + W_N^m \sum_{m=0}^{\frac{N}{2}-1} f[2m+1] W_{\frac{N}{2}}^{mn}$$

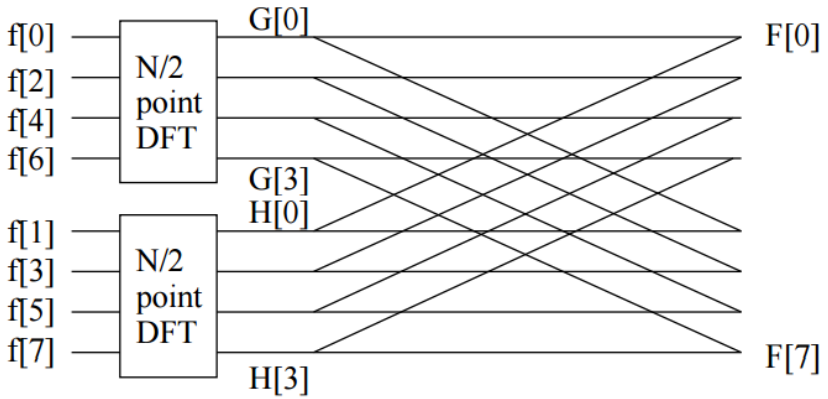
$$W_8^4 = -W_8^0$$
$$W_8^5 = -W_8^1$$
$$W_8^6 = -W_8^2$$
$$W_8^7 = -W_8^3$$

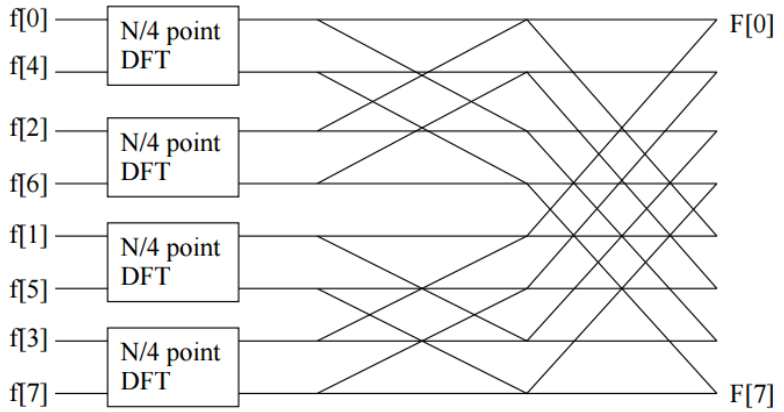
$$F[n] = G[n] + W_N^m H[n]$$

8-point example

$$
\begin{aligned}
F[0] &= G[0] + W_8^0 H[0] \\
F[1] &= G[1] + W_8^1 H[1] \\
F[2] &= G[2] + W_8^2 H[2] \\
F[3] &= G[3] + W_8^3 H[3] \\
F[4] &= G[0] + W_8^4 H[0] = G[0] - W_8^0 H[0] \\
F[5] &= G[1] + W_8^5 H[1] = G[1] - W_8^1 H[1] \\
F[6] &= G[2] + W_8^6 H[2] = G[2] - W_8^2 H[2] \\
F[7] &= G[3] + W_8^7 H[3] = G[3] - W_8^3 H[3]
\end{aligned}
$$

$$
\begin{aligned}
W_8^4 &= -W_8^0 \\
W_8^5 &= -W_8^1 \\
W_8^6 &= -W_8^2 \\
W_8^7 &= -W_8^3
\end{aligned}
$$





Complexity

- O(nlogn) because
- T(n) = 2 T(n/2) + cn

Summary

- How to divide?
  - 1 to 2
    - equal size, e.g. merge sort
    - unequal size, e.g. quick sort
  - 1 to many
    - binary search, Tower of Hanoi (1 to 3)
    - integer multiply, matrix multiply (1 to many)

Summary (cont.)

- When to terminate recursion?
  - depend on the problem at hand
    - simple comparison (binary search)
    - simple move (Hanoi tower)
- How to combine partial results?
  - nothing (binary search)
  - concatenation (quick sort)
  - merge (merge sort)
  - addition and shift (integer multiplication)