Greedy Methods

Problems whose solutions can be "ranked"

	Travel	Investment	Course selection
Feasible solutions	stay on highway, finish in x days	don't spend more than one has	finish in 4 years
Optimal solutions	shortest distance, minimum time	maximum returns, minimum risks	best combination of depth and breadth
Decisions	which highways to take	invest or not in a portfolio	take a course or not

- Decisions can be made
  - one at a time, **without** backtracking
  - Greedy method
  - *Which decisions to make next*?
  - How to guarantee optimality?
- Try many (all) possible combinations and choose one which is the best
  - Dynamic programming
  - How to test multiple solutions efficiently?

The Greedy Method

- Input *n* elements stored in an array *A*(*1:n*)
- Procedure Greedy
  - Solution = *NULL*
  - for *i=1* to *n* do
    - x = SELECT(*A*)
    - if FEASIBLE(Solution, x)
    - then Solution = UNION(Solution, x)
    - endif
  - enddo
  - return (Solution)

- A sequence of *n* decisions w.r.t *n* inputs
- *SELECT:* select one of the remaining decisions to make according to some *optimization* measure
  - once a decision is made, it will ***not*** become invalid at a later time
  - *optimization* should be based on the partial solutions built so far
- *FEASIBLE:* whether the partial solution satisfies some preset constraints

- *Strategy*: construct feasible solutions one step at a time which optimize (minimize or maximize) a certain objective function
- *Make the obvious decisions first*!
- *Then try to show it is indeed optimal!*

Knapsack problem

- Input:
  - a set of n objects (P_i, W_i) $i = 1, \ldots, n$
  - a knapsack of capacity M
- Output: fill the knapsack to maximize the total profit earned
- Feasibility constraint: $\sum_{i=1}^{n} W_i X_i \leq M$
- Objective function: $\max \sum_{i=1}^{n} P_i X_i \quad 0 \leq X_i \leq 1$

Example

$n = 3, M = 20$

$(P_1, P_2, P_3) = (25,24,15)$

$(W_1, W_2, W_3) = (18,15,10)$

(X_1, X_2, X_3)	$\sum_{i=1}^{n} W_i X_i$	$\sum_{i=1}^{n} P_i X_i$	
$(1, \frac{2}{15}, 0)$	20	28.2	largest increase in profit
$(0, \frac{2}{3}, 1)$	20	31	smallest increase in weight
$(0,1, \frac{1}{2})$	20	31.5	largest increase in profit to weight ratio

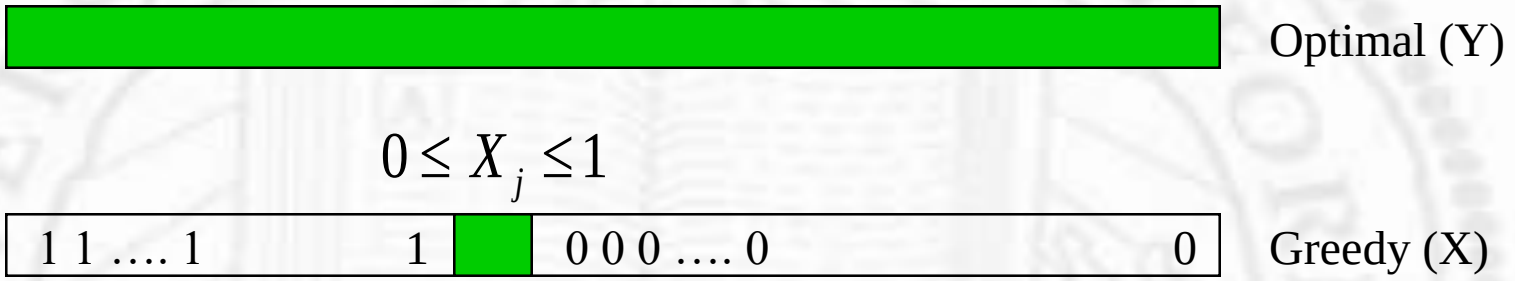
$(\frac{P_1}{W_1}, \frac{P_2}{W_2}, \frac{P_3}{W_3}) = (1.39,1.6,1.5)$

- For all three algorithms
  - decisions are made one object at a time
  - the *ordering* is determined by some optimization measure
    - Largest increase in profit
      - Include the remaining object of the largest profit
    - Smallest increase in weight
      - Include the remaining object of the smallest weight
    - Largest increase in profit/weight
      - Include the remaining object of the largest profit/weight
  - never backtrack
  - all greedy algorithms
  - not all guarantee optimal

- **Proposition**: Greedy selection based on maximizing profit to weight ratio gives the optimal result
- **General proof strategy**:
  - Assume that the greedy solution is $X = (X_1, X_2, ..., X_n)$
  - Assume that the optimal solution is $Y = (Y_1, Y_2, ..., Y_n)$
  - Then they better be different
  - Transform Y into X without decreasing the profit of Y

Proof:

- Assume $\frac{P_1}{W_1} \geq \frac{P_2}{W_2} \geq \ldots \geq \frac{P_n}{W_n}$

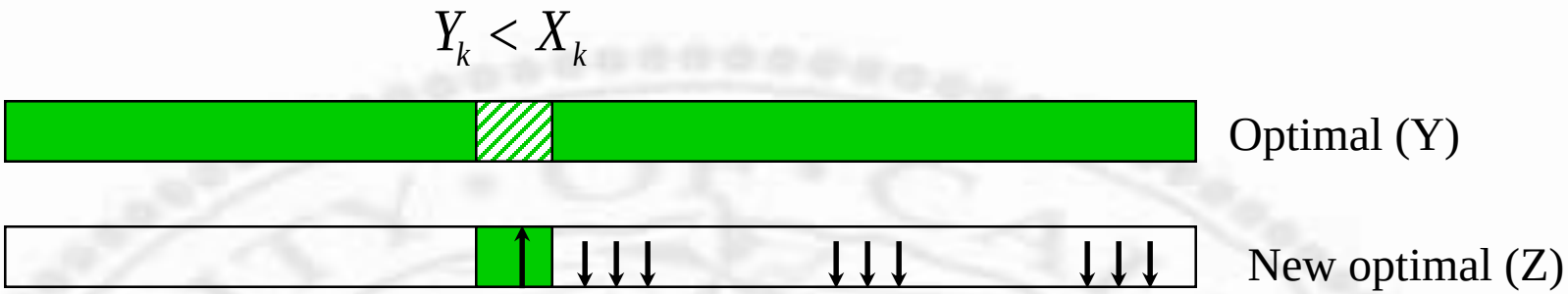


- Let k be the first index where X and Y differ

(i) $k < j$ $X_k = 1 \,\&\, Y_k \neq X_k \Rightarrow Y_k < X_k$

(ii) $k = j$ $if\ Y_k > X_k\ then \sum W_i Y_i > M \Rightarrow Y_k < X_k$

(iii) $k > j$ $X_k = 0 \,\&\, Y_k \neq X_k \Rightarrow Y_k > 0 \,\&\, \sum W_i Y_i > M, not\ possible$



$$\sum_{i=1}^{n} Z_i P_i = \sum_{i=1}^{n} Y_i P_i + (Z_k - Y_k) P_k - \sum_{i=k+1}^{n} (Y_i - Z_i) P_i$$

profit of Y | increase of profit | decrease of profit

$$= \sum_{i=1}^{n} Y_i P_i + (Z_k - Y_k) \frac{P_k}{W_k} W_k - \sum_{i=k+1}^{n} (Y_i - Z_i) \frac{P_i}{W_i} W_i$$

$$\geq \sum_{i=1}^{n} Y_i P_i + \{(Z_k - Y_k) W_k - \sum_{i=k+1}^{n} (Y_i - Z_i) W_i\} \frac{P_k}{W_k}$$

increase in weight | decrease in weight

$$\geq \sum_{i=1}^{n} Y_i P_i$$

- Z is also an optimal solution
- Either $Z=X$ (Done)
- Or not (Repeat the above procedure until $Z=X$)

- Time complexity
  - Sort the *n* objects according to profit to weight ratio *O*(*nlogn*)
  - Scan down the sorted list

$$\textbf{if } W_i \leq \textbf{ remaing capaity then}$$
$$X_i \leftarrow 1$$
$$\textbf{remaining capacity} -= W_i$$
$$\textbf{else}$$
$$X_i \leftarrow \frac{\textbf{remaining capacity}}{W_i}$$
$$\textbf{remaing capacity} \leftarrow 0$$
$$\textbf{endif}$$

– Complexity *O(nlogn)*

Optimal Storage on Tape

- Input:
  - A set of *n* programs of different length
  - A computer tape of length *L*
- Output:
  - A storage pattern which minimizes the total retrieval time (*TRT*)
    - before each retrieval, head is repositioned at the front

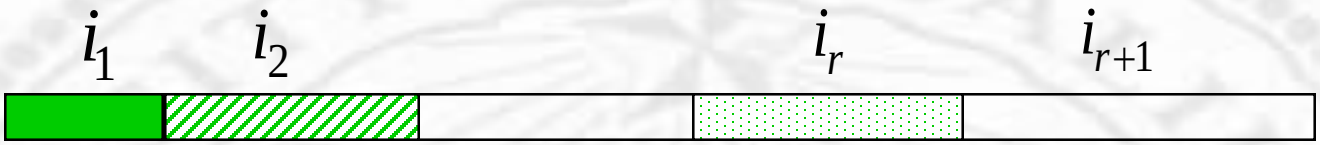
$$TRT = \sum_{1 \le j \le n} \sum_{1 \le k \le j} l_{i_k} \qquad I = i_1, i_2, \ldots i_n$$

- Objective function: minimize *TRT*
- Feasibility constraint: $\sum_{1 \le k \le n} l_{i_k} \le L$
- Example

$n = 3, (l_1, l_2, l_3) = (5,10,3), L = 20$

ordering (i_1, i_2, i_3)			*TRT*	
1,2,3	$5+$	$5+10+$	$5+10+3$	$=38$
1,3,2	$5+$	$5+3+$	$5+3+10$	$=31$
2,1,3	$10+$	$10+5+$	$10+5+3$	$=43$
2,3,1	$10+$	$10+3+$	$10+3+5$	$=41$
3,1,2	$3+$	$3+5+$	$3+5+10$	$=29$
3,2,1	$3+$	$3+10+$	$3+10+5$	$=34$

- *SELECT*: Select the program to store next which minimizes the increase in *TRT*



$$TRT_{old} = \sum_{1 \le j \le r} \sum_{1 \le k \le j} l_{i_k}$$

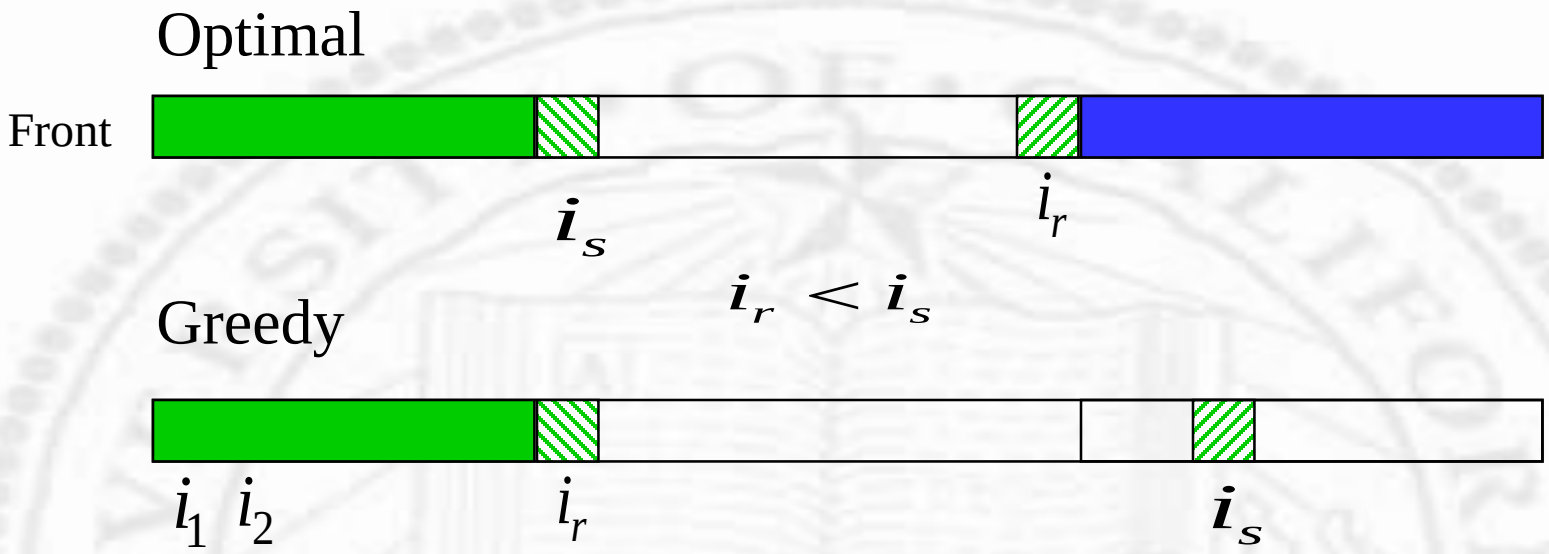
$$TRT_{new} = \sum_{1 \le j \le r+1} \sum_{1 \le k \le j} l_{i_k} = TRT_{old} + \sum_{1 \le k \le r+1} l_{i_k}$$

$$TRT_{new} - TRT_{old} = \sum_{1 \le k \le r+1} l_{i_k} = \sum_{1 \le k \le r} l_{i_k} + l_{r+1}$$

fixed

Currently shortest program

- Proof strategy
  - follow the same principle as in knapsack problem
    - there is a greedy solution
    - there is an optimal solution
    - they are different
    - line them up and they better differ in some storage locations
    - then make them the same (by swapping)
    - prove that the swapping does not reduce the optimality



- Swap i_r and i_s in the optimal solution

- Intuitively

- For programs stored in 
  - retrieval does not scan through either *a* or *b*
  - ordering of *a* and *b* not important
- For programs stored in 
  - retrieval scans through both *a* and *b*
  - ordering of *a* and *b* not important
- For programs stored in 
  - retrieval scans through *a* but not *b*
  - ordering of *a* and *b* is important

- **Proposition**: The storage pattern with nondecreasing length order produces the smallest *TRT*

$$\sum_{1\le j\le n}\sum_{1\le k\le j} l_{i_k}$$

$$\begin{aligned}
&= l_{i_1} \\
&+ l_{i_1} + l_{i_2} \\
&+ l_{i_1} + l_{i_2} + l_{i_3} \\
&\ldots \quad \ldots \quad \ldots \\
&+ l_{i_1} + l_{i_2} + l_{i_3} + \ldots + l_{i_n} \\
&= \mathrm{n} l_{i_1} + (n-1) l_{i_2} + (n-2) l_{i_3} + \ldots + l_{i_n}
\end{aligned}$$

$$= \sum_{1\le k\le n} (n-k+1) l_{i_k}$$

- If prog. *a* and prog. *b* are out of order, then swap them should reduce the *TRT*

$$TRT_{old} = \sum_{\substack{k \\ k \neq a \\ k \neq b}} (n-k+1)l_{i_k} + (n-a+1)l_{i_a} + (n-b+1)l_{i_b}$$

$$TRT_{new} = \sum_{\substack{k \\ k \neq a \\ k \neq b}} (n-k+1)l_{i_k} + (n-a+1)l_{i_b} + (n-b+1)l_{i_a}$$

$$TRT_{old} - TRT_{new} = (n-a+1)(l_{i_a} - l_{i_b}) + (n-b+1)(l_{i_b} - l_{i_a})$$

$$= (b-a)(l_{i_a} - l_{i_b}) > 0$$

- Time complexity: *O(nlogn)* for sorting

Optimal Merge Pattern

- Input: a set of files of different lengths
- Output: an optimal sequence of *two-way* merges to obtain a sorted files

$F_i, 1 \le i \le n,$ of length q_i
merge files F_i and F_j requires $O(q_i + q_j)$ time

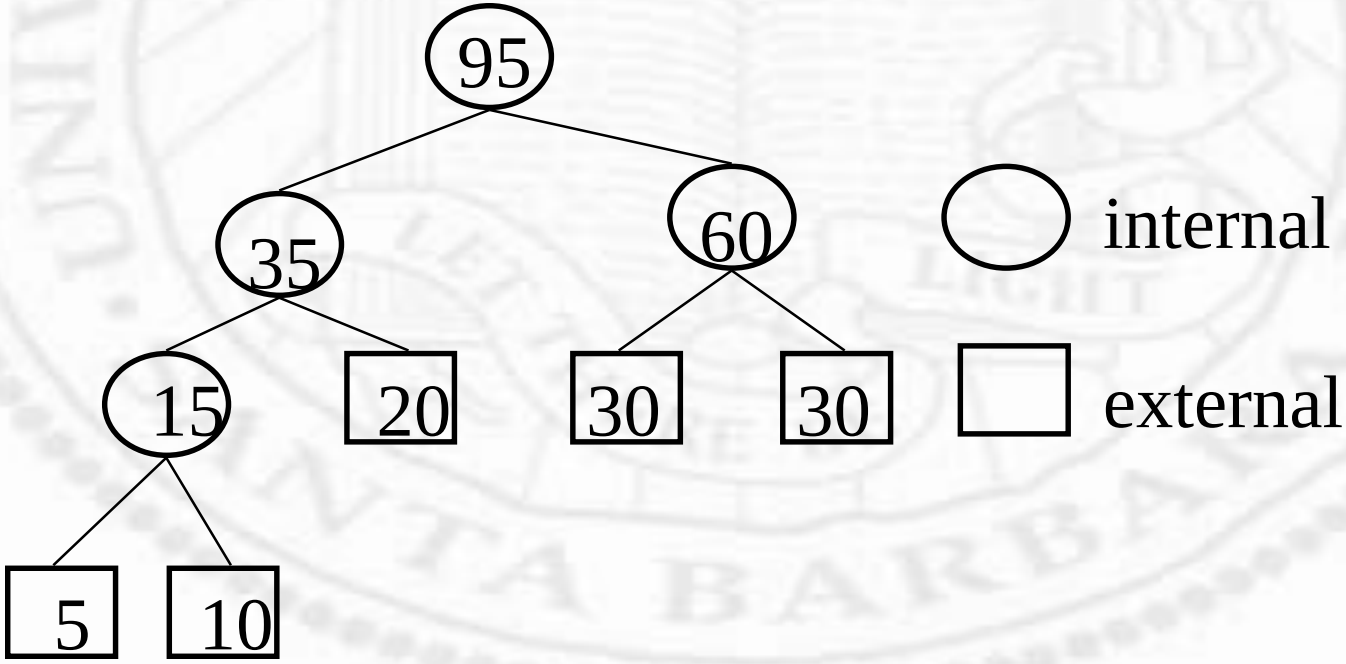
- Example

$n = 3, (q_1, q_2, q_3) = (30, 20, 10)$

ordering	cost
1,2,3	$50 + 60 = 110$
1,3,2	$40 + 60 = 100$
2,1,3	$50 + 60 = 110$
2,3,1	$30 + 60 = 90$
3,1,2	$40 + 60 = 100$
3,2,1	$30 + 60 = 90$

- Programs (files) stored on a tape (already merged together) may affect the access times (the merge times) of new programs (files) to be stored (merged)
- *SELECT*: At each step, merge two smallest files

- Binary merge tree
  - Distance from an external node to root = # of times a file is involved in merging
  - Total # of record moves for file i $d_i q_i$
    - external path length to reach node i
  - Total # of record moves for all files $\sum_{i=1}^{n} d_i q_i$
    - total external path length



Huffman Code

- For data compression to save storage space and transmission bandwidth
- ASCII code uses *fixed*-length 8 bits/character code words, $O(8n)$ for storage and transmission
- Huffman codes uses *variable*-length code words depending on the *frequency of occurrence*

- Example

	a	b	c	d	e	f
Frequency (thousands)	45	13	12	16	9	5
Fixed-length	000	001	010	011	100	101
Variable-length	0	101	100	111	1101	1100

$$\text{fixed}-\text{length}\,100{,}0$$

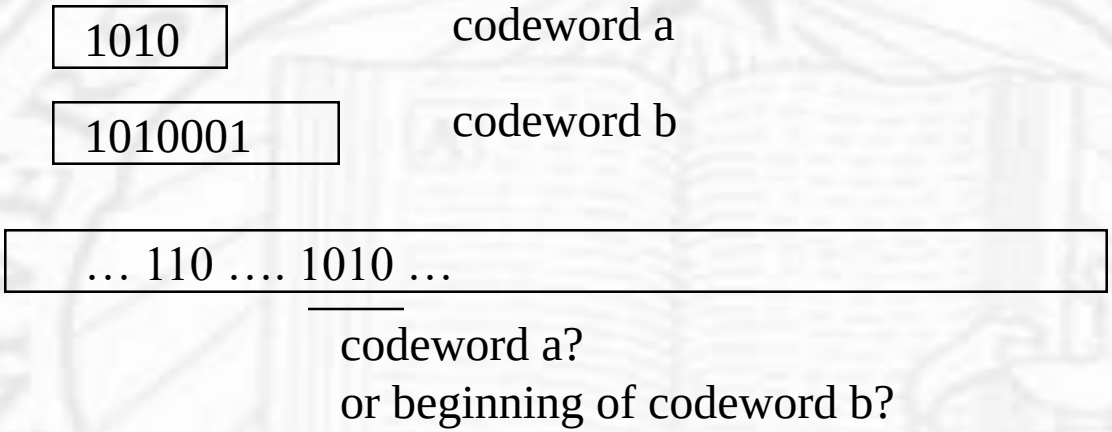
$$00\times 3=300{,}000\text{bits}$$

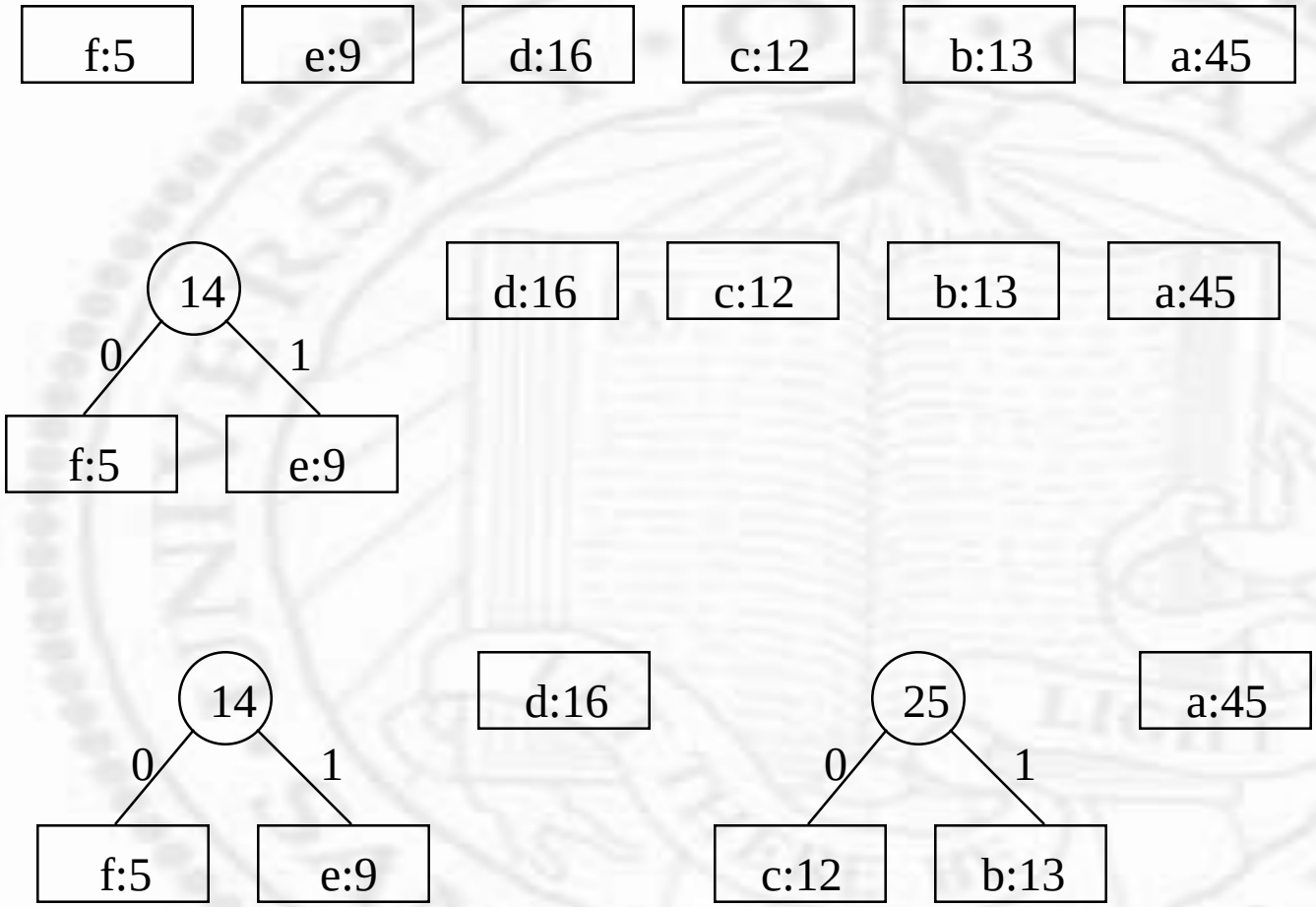
$$\text{variable}\text{-length}\,45{,}000\times 1+13{,}000\times 3+12{,}000\times 3$$

$$+16{,}000\times 3+9{,}000\times 4+5{,}000\times 4=224{,}000\text{bits}$$

Prefix codes

- no codeword is a prefix of some other codeword




f:5
e:9
d:16
c:12
b:13
a:45
14
0
1
f:5
e:9
d:16
c:12
b:13
a:45
14
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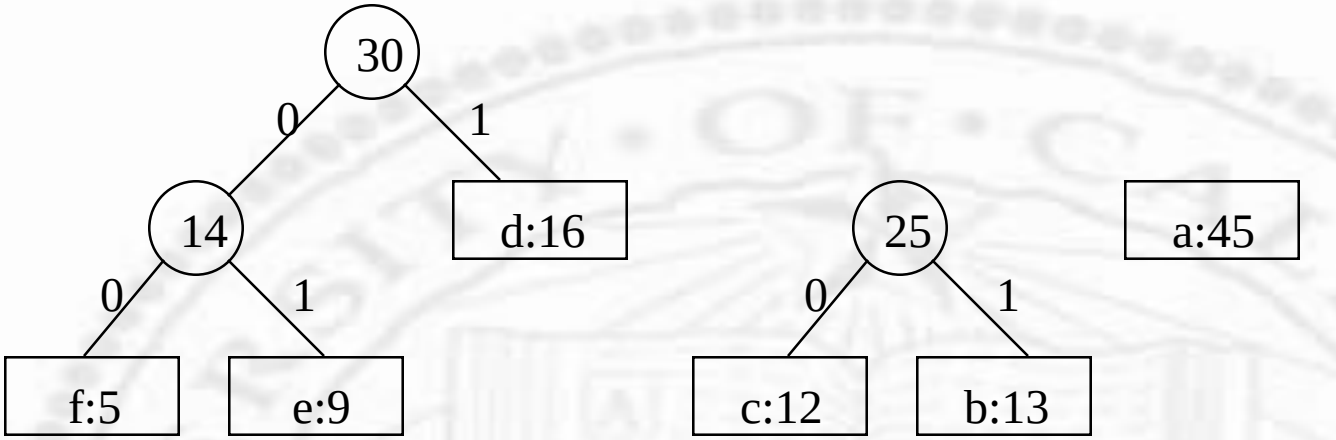
f:5
e:9
d:16
25
0
1
c:12
b:13
a:45


30
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d:16
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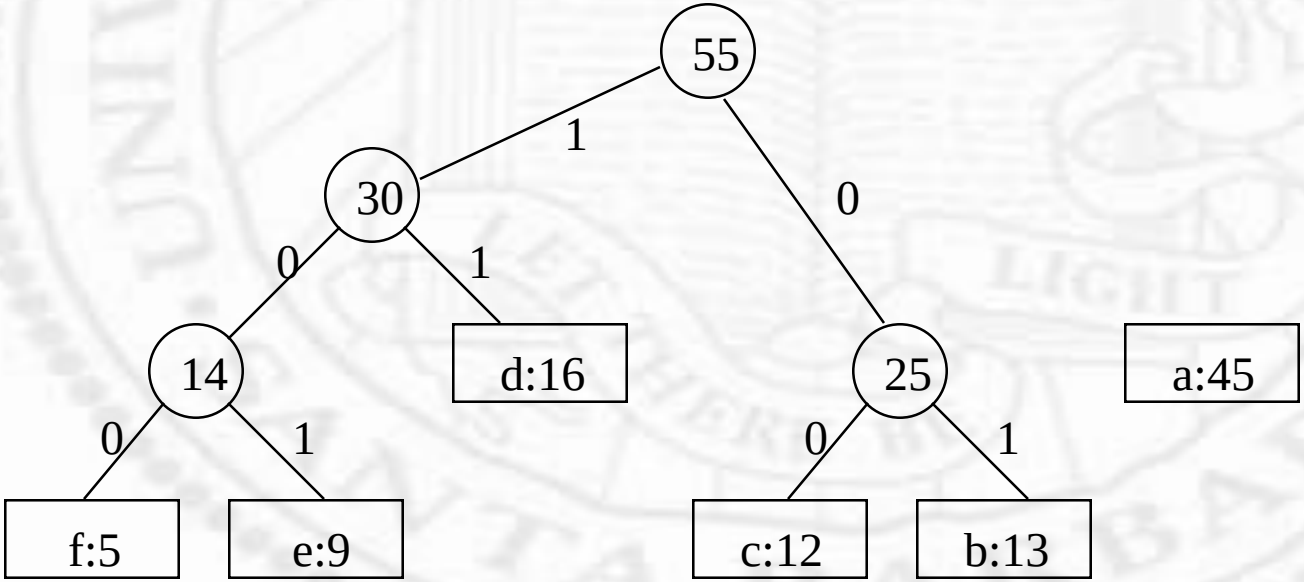
f:5
e:9

25
a:45
0
1
c:12
b:13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a:45
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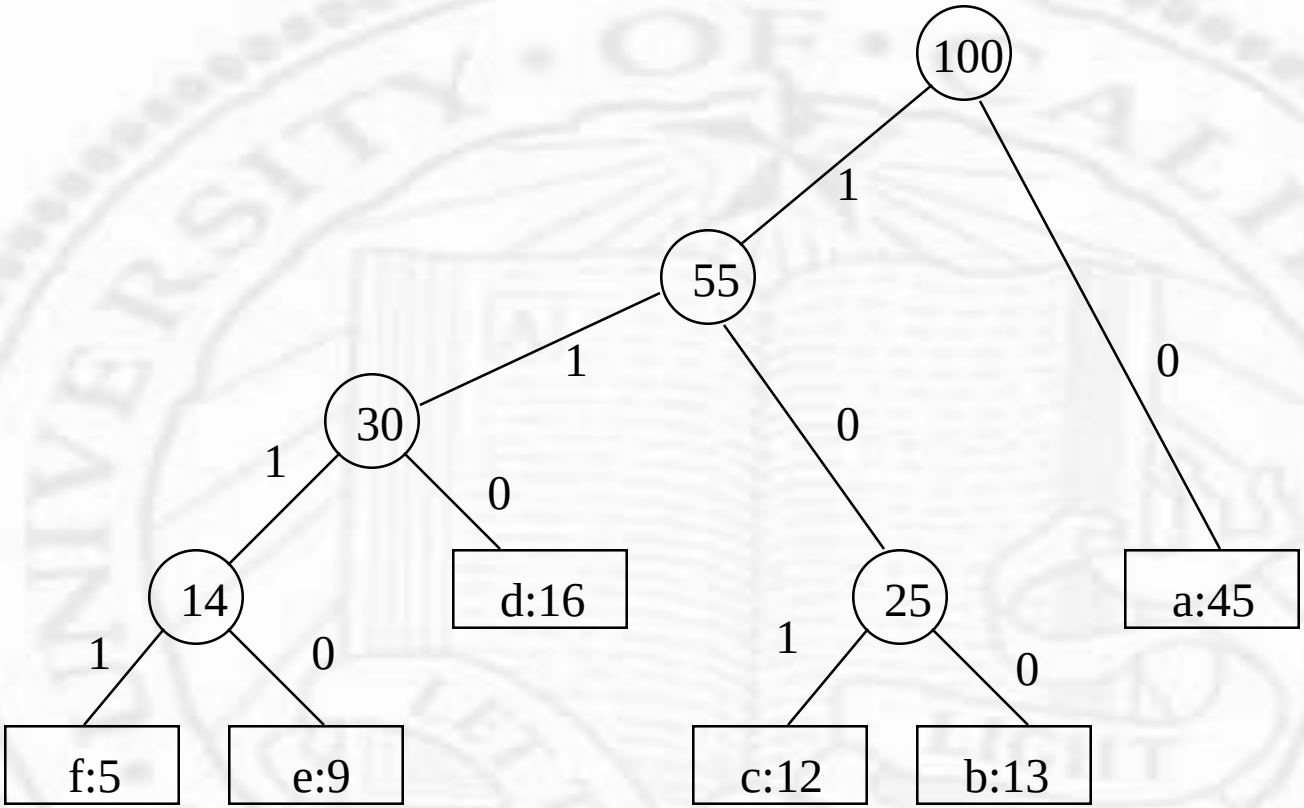
f:5
e:9
c:12
b:13


100
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f:5
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e:9
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d:16
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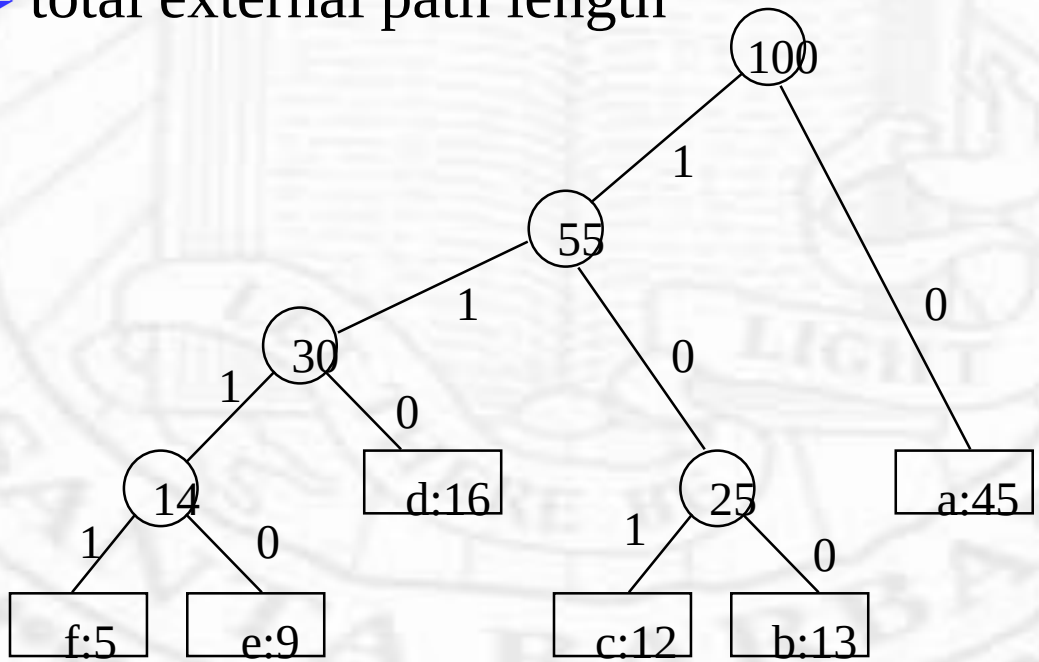
25
1
c:12
0
b:13
0
a:45

Huffman code

- Distance from an external node to root = # of bits in the code word
- Total effort of sending i
  - external path length to reach node *i*
- Total effort of sending all alphabets
  - total external path length

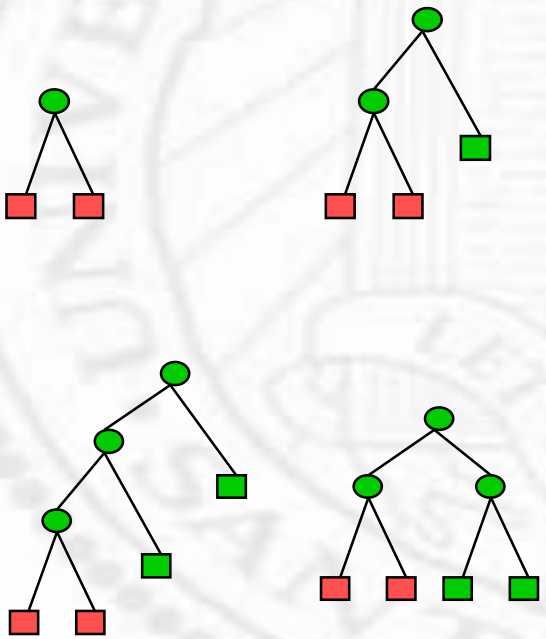
$d_i q_i$

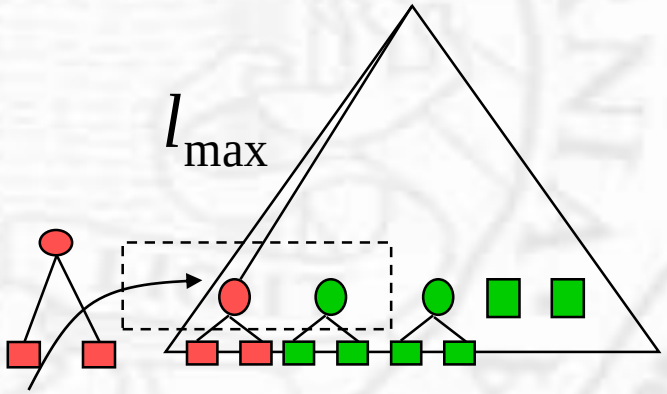
$\sum_{i=1}^{n} d_i q_i$



An Important Fact

- Using Huffman-tree rules, nodes that are merged first must have a longer path to the root than nodes that are merged later





$$p_1 + p_2 \le p_i + p_j, \quad i, j > 2$$

No node can be merged more than once before p1+p2 is involved again

An Important Fact (cont.)

- An iteration:
  - Between two successive merges involve p1
- In an iteration
  - No node can be involved in more than one merge
  - No node can increase in path length more than p1
- Hence, p1 must have the longest path length

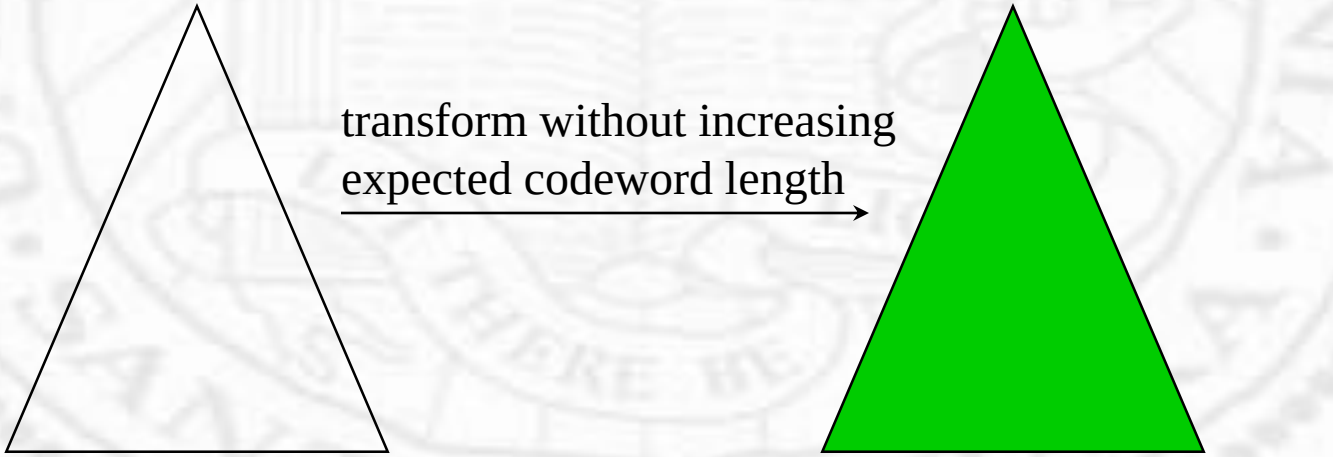
- **Proposition**: Huffman construction minimizes the expected codeword length

$$\sum_{i=1}^{n} p_i l_i$$

p_i probability of occurance

l_i codeword length

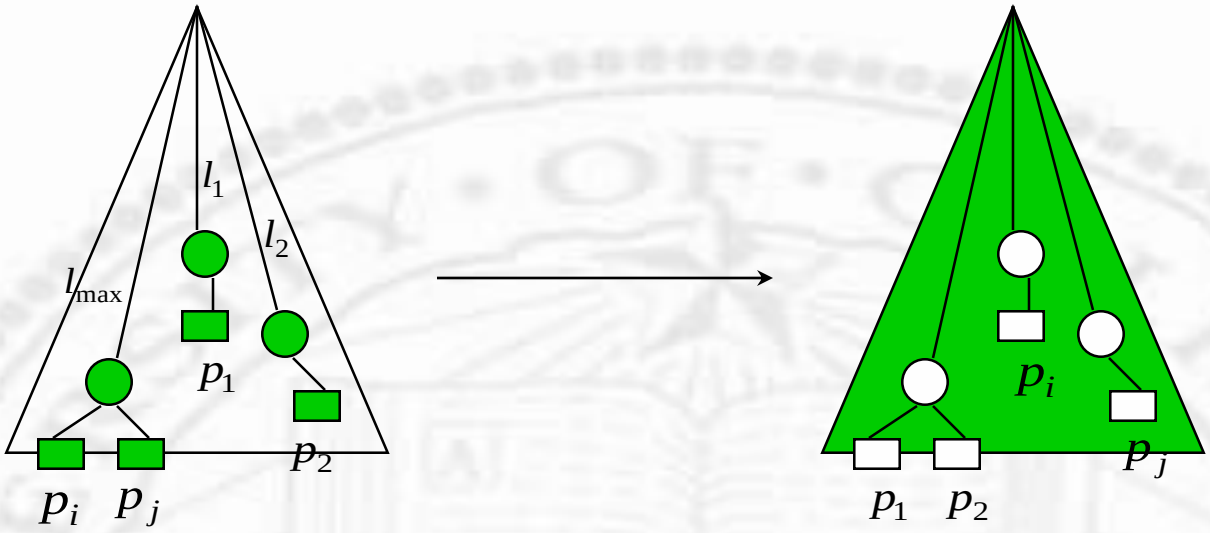
- **Proof:** *Assume that* $p_1 \leq p_2 \leq \ldots \leq p_n$



Optimal prefix-code tree

Huffman prefix-code tree

Optimal prefix-code tree



$$\sum_{k=1}^{n} p_k l_k = \sum_{k\neq 1,2,i,j} p_k l_k + l_{\max} p_i + l_{\max} p_j + l_1 p_1 + l_2 p_2$$

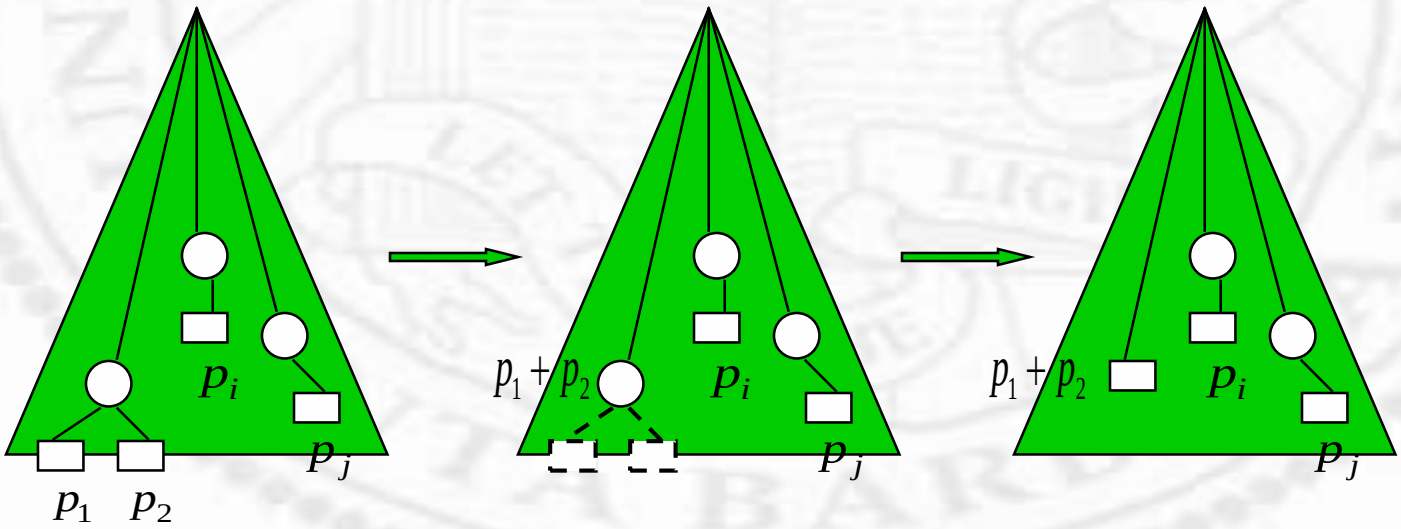
$$\sum_{k=1}^{n} p_k l_k' = \sum_{k\neq 1,2,i,j} p_k l_k + l_{\max} p_1 + l_{\max} p_2 + l_1 p_i + l_2 p_j$$

$$\sum_{k=1}^{n} p_k l_k - \sum_{k=1}^{n} p_k l_k'$$

$$= l_{\max} p_i + l_{\max} p_j + l_1 p_1 + l_2 p_2 - l_{\max} p_1 - l_{\max} p_2 - l_1 p_i - l_2 p_j$$

$$= (l_{\max} - l_1)(p_i - p_1) + (l_{\max} - l_2)(p_j - p_2) \geq 0$$

- Recursion
  - once p1 and p2 are moved to their right locations
  - merge them into a single node of p1+p2
  - now, greedy method will select from p1+p2, p3, …, pn the smallest two to merge
  - if that is not the case for optimal, then ...



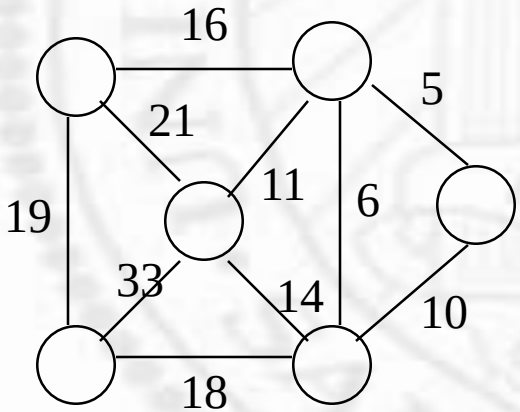
- Time complexity
  - with *n* alphabets to code, exactly *n-1* merges are needed
  - for each merge
    - find an least-frequently-used alphabet
    - find the next least-frequently-used alphabet
    - merge
    - put merged subtrees back into the list of subtrees
  - priority queue (heap) is ideal for this operation
  - $O(n)$ steps of *detelemin* and *insert* ($O(logn)$)
  - $O(nlogn)$

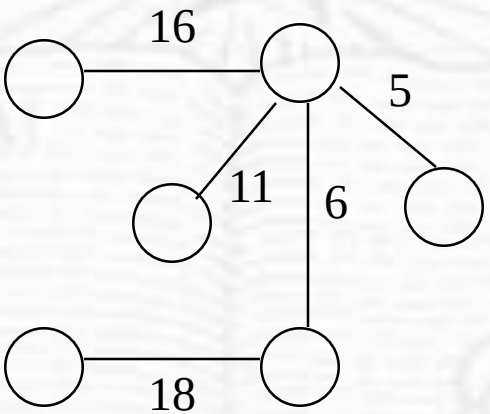
Minimum-Cost Spanning Tree

- Input: *G=(V,E)*, an undirected, labeled graph
- Output: *T=(V,E')*, a subgraph of G
  - includes all the vertices
  - is a tree

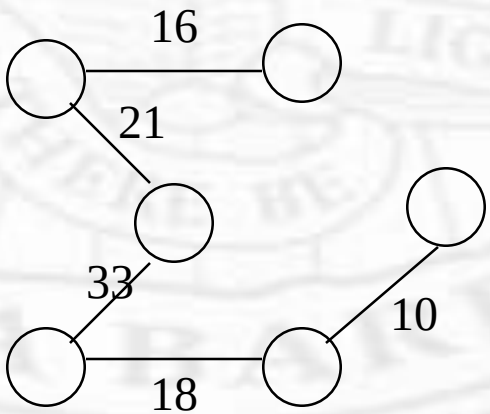
  } (Spanning tree)
  - the sum of labels (costs) of all tree branches is minimum among all spanning trees

- Objective function: $\sum_{i \in SP} cost(e_i)$
- Feasibility constraint: a tree containing all vertices
- Example





$$\sum_{i \in SP} cost(e_i) = 56$$

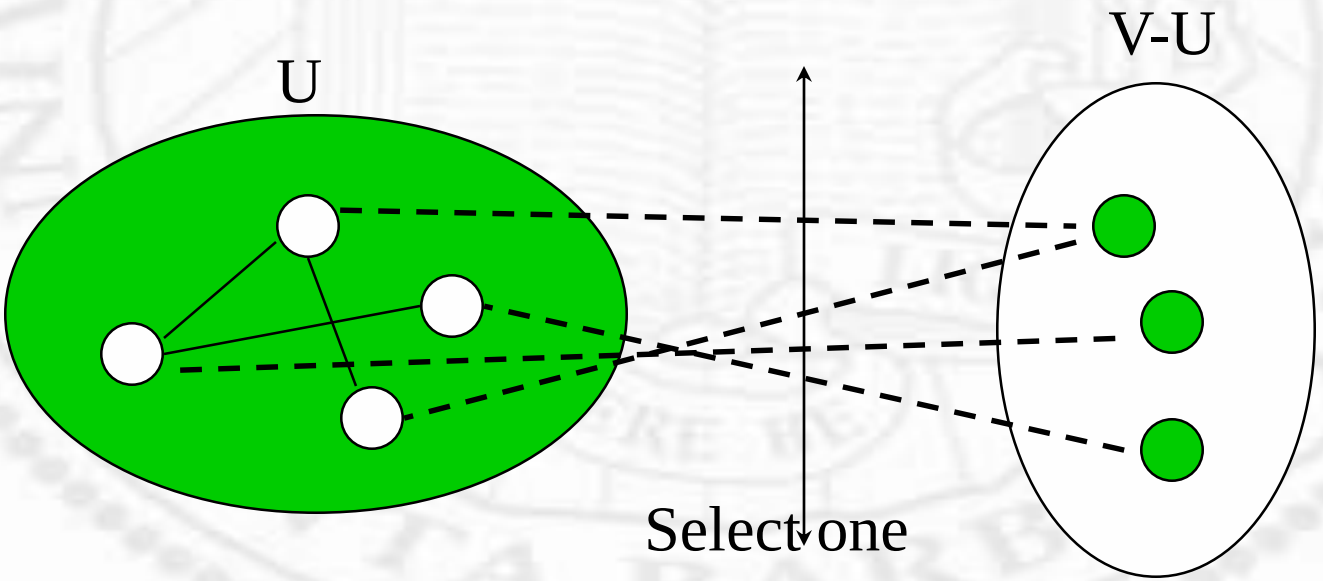


$$\sum_{i \in SP} cost(e_i) = 98$$

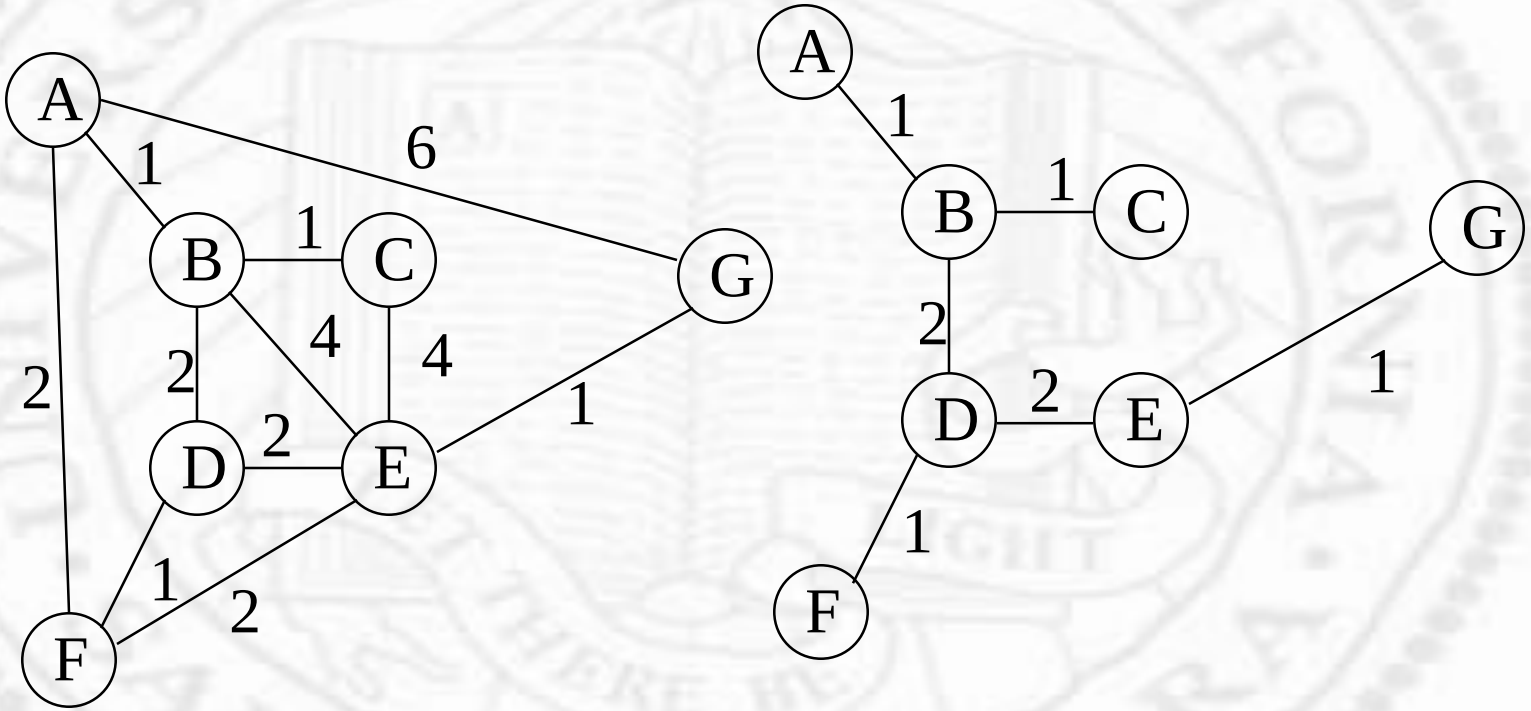
- *SELECT:* At each step, choose an edge with minimum cost (*optimality*) such that (*feasibility*):
  - the partial solution is always a tree (Prim)
  - the partial solution has potential of becoming a tree (no cycles, but not necessarily connected) (Kruskal)

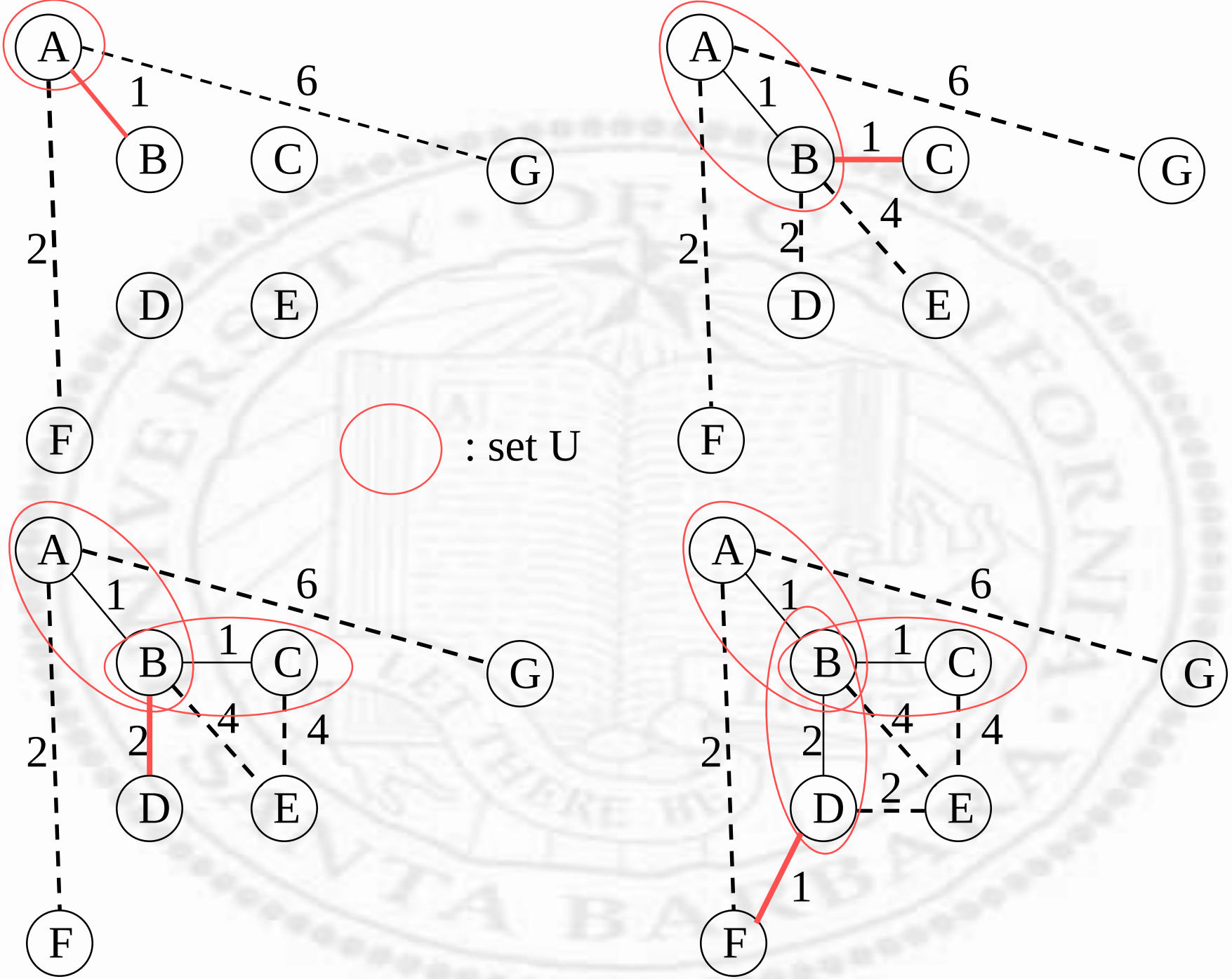
Prim's algorithm

- First step: select a minimum cost edge, include it in the solution
- Other steps: select an edge *(u,v)*, *u* in *U* and *v* in *V-U*, until all vertices are counted for



Example

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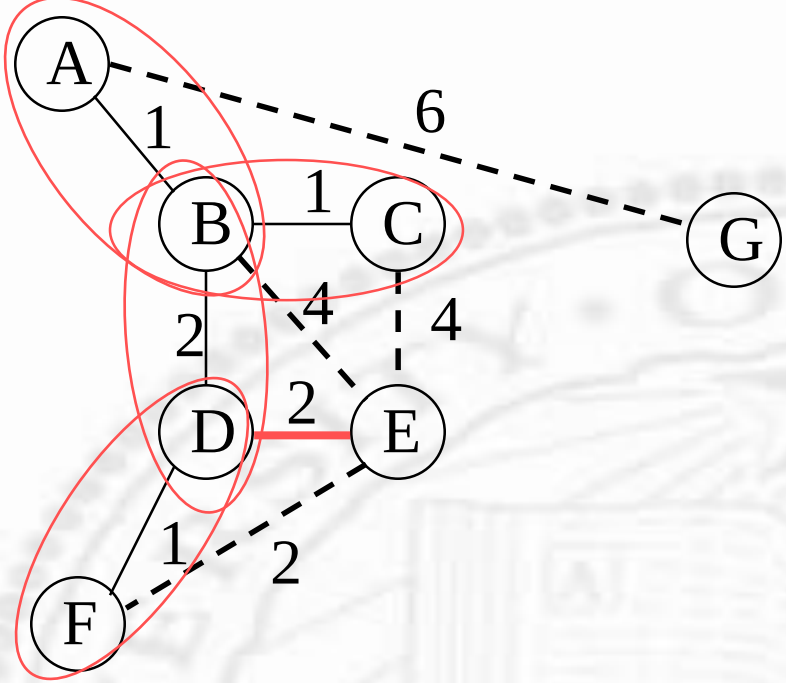
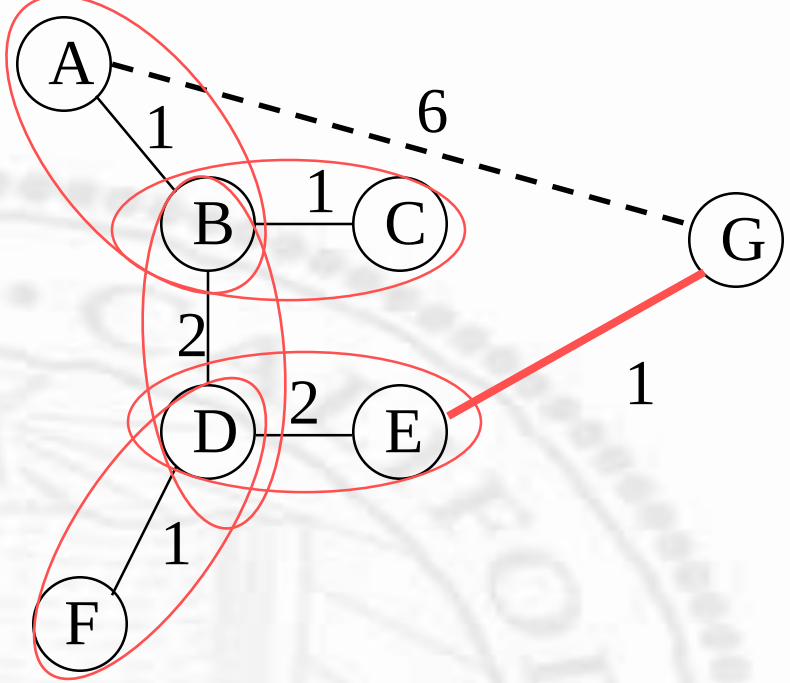
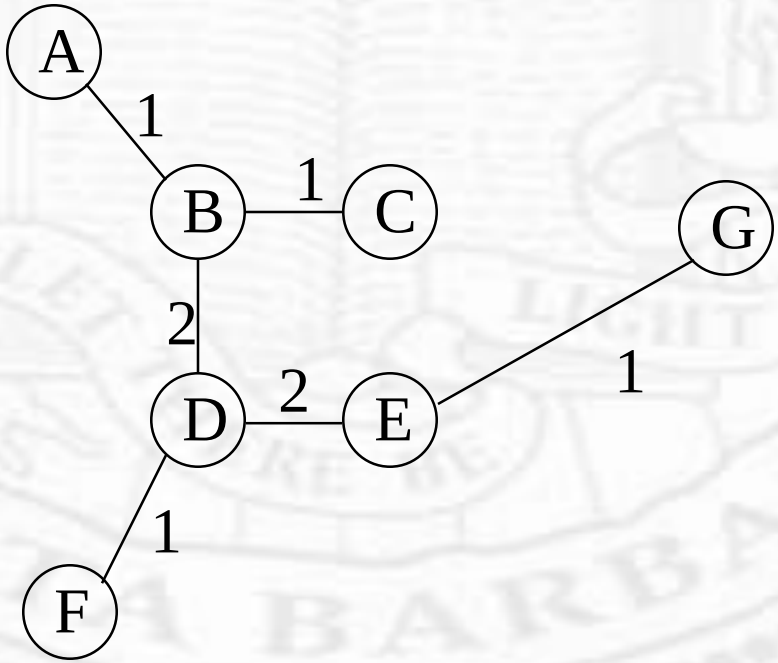
1





: set U

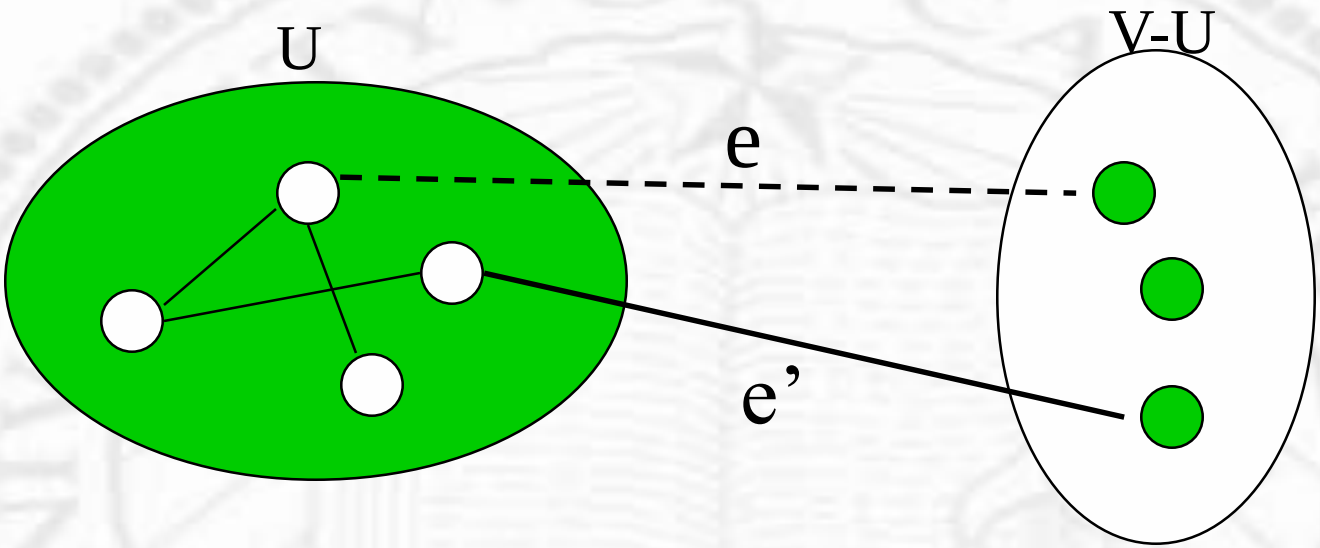


Step	A	B	C	D	E	F	G
1	–	$\underline{(1,A)}$	(∞,A)	(∞,A)	(∞,A)	$(2,A)$	$(6,A)$
2	–	–	$\underline{(1,B)}$	$(2,B)$	$(4,B)$	$(2,A)$	$(6,A)$
3	–	–	–	$\underline{(2,B)}$	$(4,B)$	$(2,A)$	$(6,A)$
4	–	–	–	–	$(2,D)$	$\underline{(1,D)}$	$(6,A)$
5	–	–	–	–	$\underline{(2,D)}$	–	$(6,A)$
6	–	–	–	–	–	–	$(1,E)$

$$Cost_i = \min(Cost_i, Cost(new,i))$$ Cost update

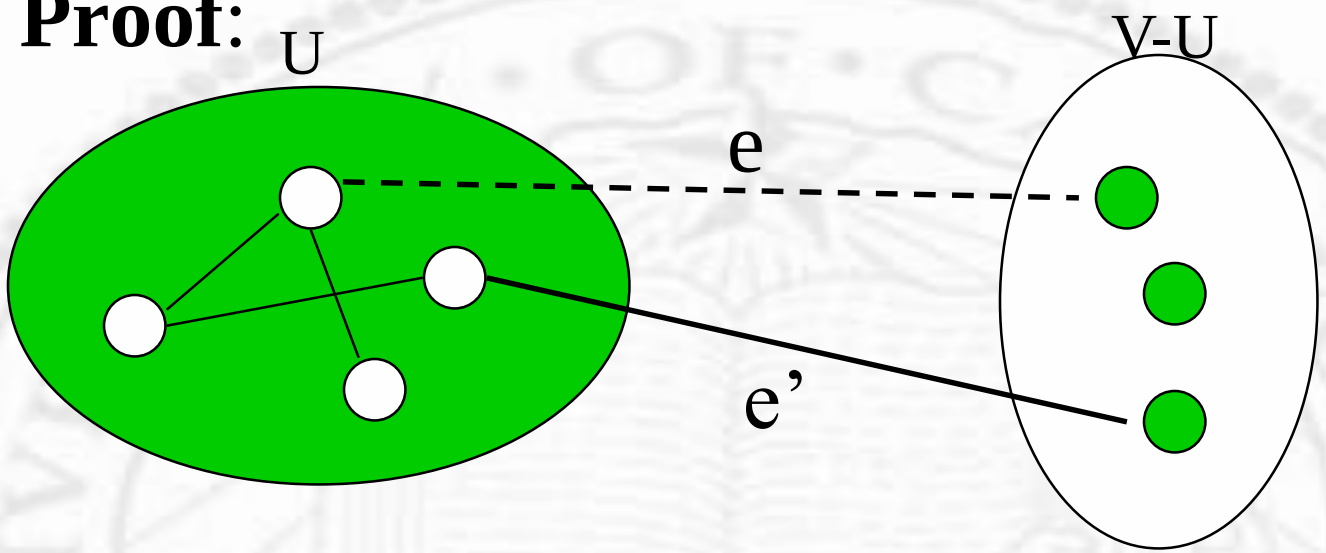
$$Closest_i = (Cost_i == Cost(new,i))?new:Closet_i$$ Nearest neighbor update

- **Proposition**: Prim's algorithm finds MCST
- **Proof**:

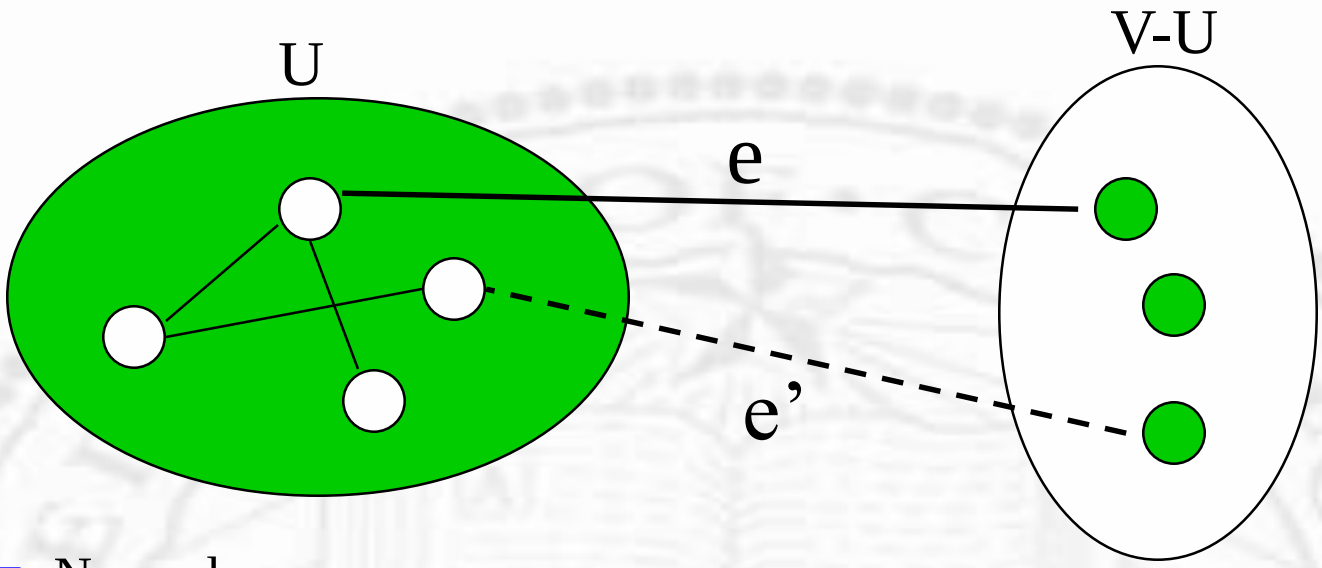


  - Again, there are two solutions, PRIM and MCST
  - They better differ, and MCST has a lower cost
  - In the construction of PRIM, if an edge e is considered
    - It is in MCST, ok, continue (cannot be forever)
    - If it is not in MCST, then ….

- **Proposition**: Prim's algorithm finds MCST
- **Proof**:



  - Let U be the subgraph (tree) considered so far
  - Let V-U be the remaining part, then
  - There must be at least one edge (e') chosen between U and V-U in MCST
    - Prim's algorithm selects the minimum cost one (*e*)
    - *e'* can be replaced by *e* in the MCST



- No cycle
  - U has no cycle
  - V-U has no cycle
  - Between U and V-U cannot has cycle w. a single path e
- Still connected
  - U is connected
  - V-U is connected
  - U and V-U connected through either e or e'
- The same number of edges => it is a spanning tree
- A tree of a smaller cost

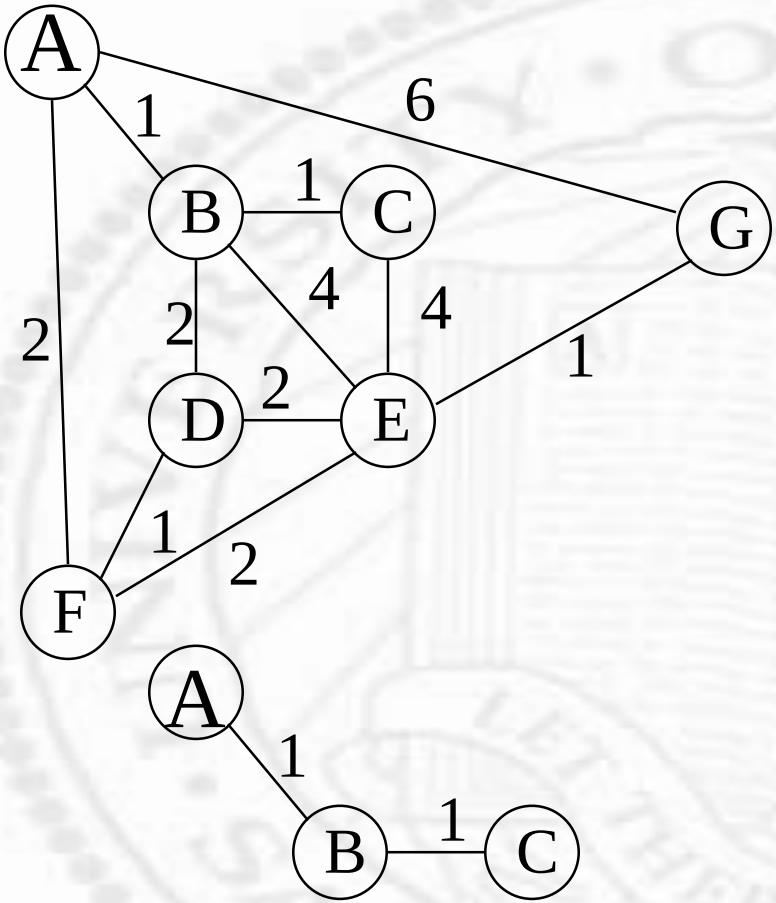
- Time complexity
  - Totally *n* vertices have to be connected
  - Each time an edge is added, one additional vertex is accounted for
    - Loop through *n-1* times
  - Through each loop

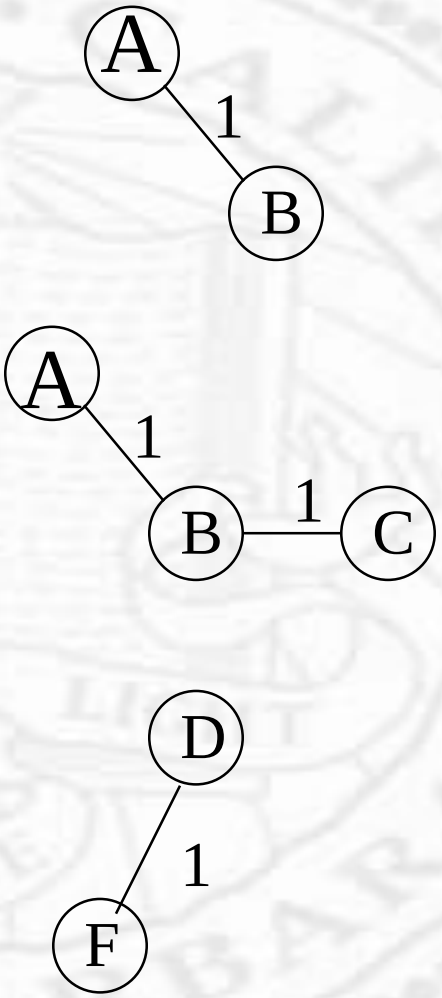
O(n-i) - Select the edge of a minimum cost from *U to V-U*

O(n-i) - Update the nearest vertex and cost for vertices in *V-U'*

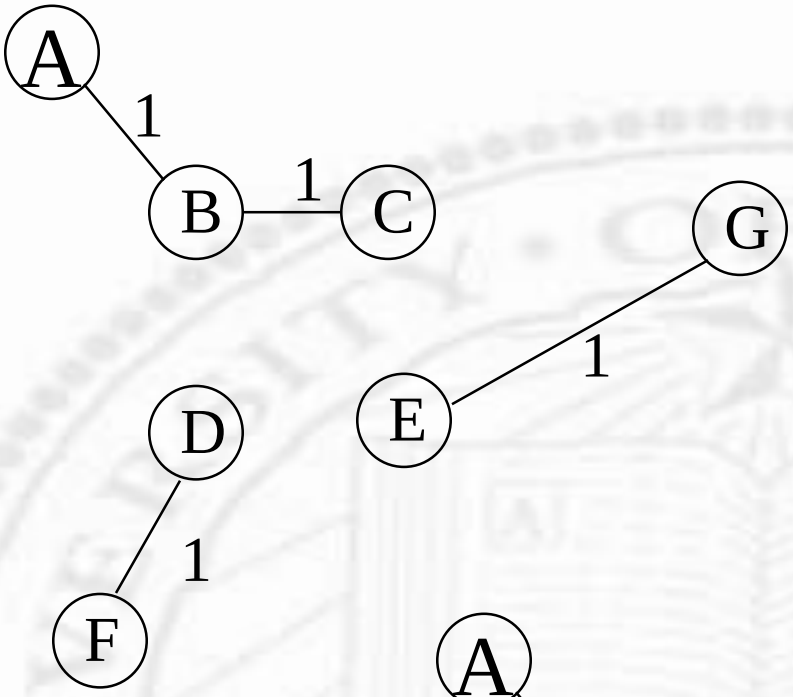
$$\sum_{i=1}^{n-1}(n-i) = O(n^2)$$

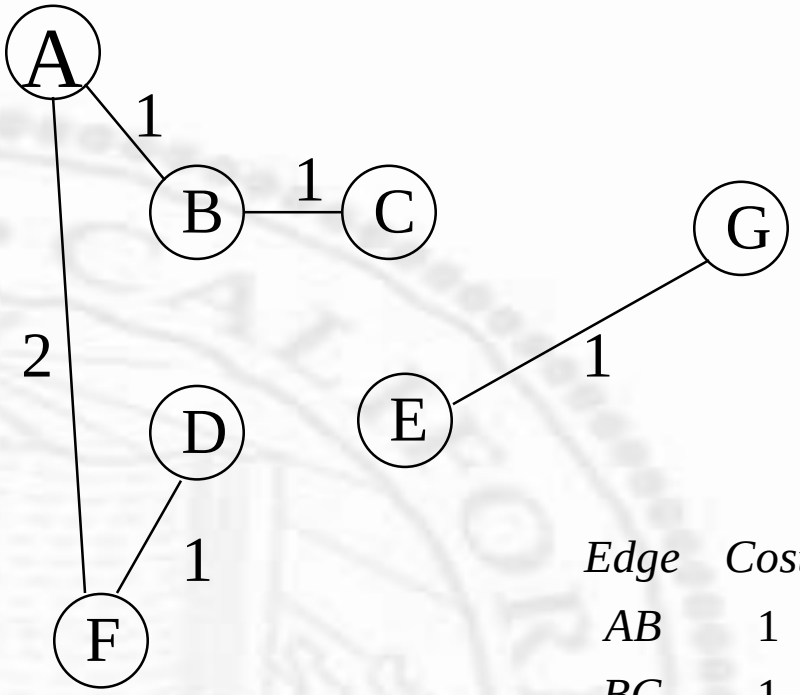
Kruskal's algorithm

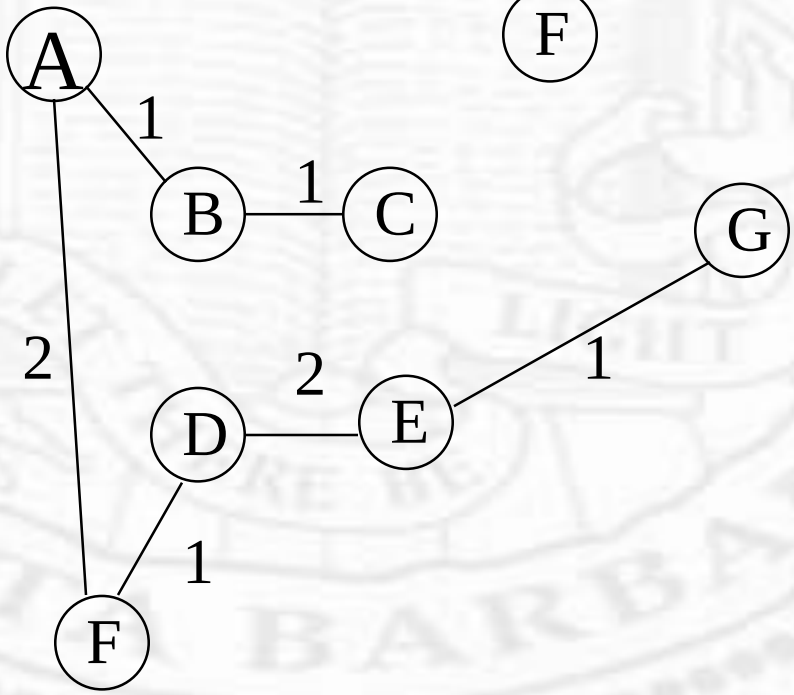




Edge	*Cost*
AB	1
BC	1
DF	1
EG	1
AF	2
BD	2(×)
DE	2
EF	2(×)
BE	4(×)
CE	4(×)
AG	6(×)







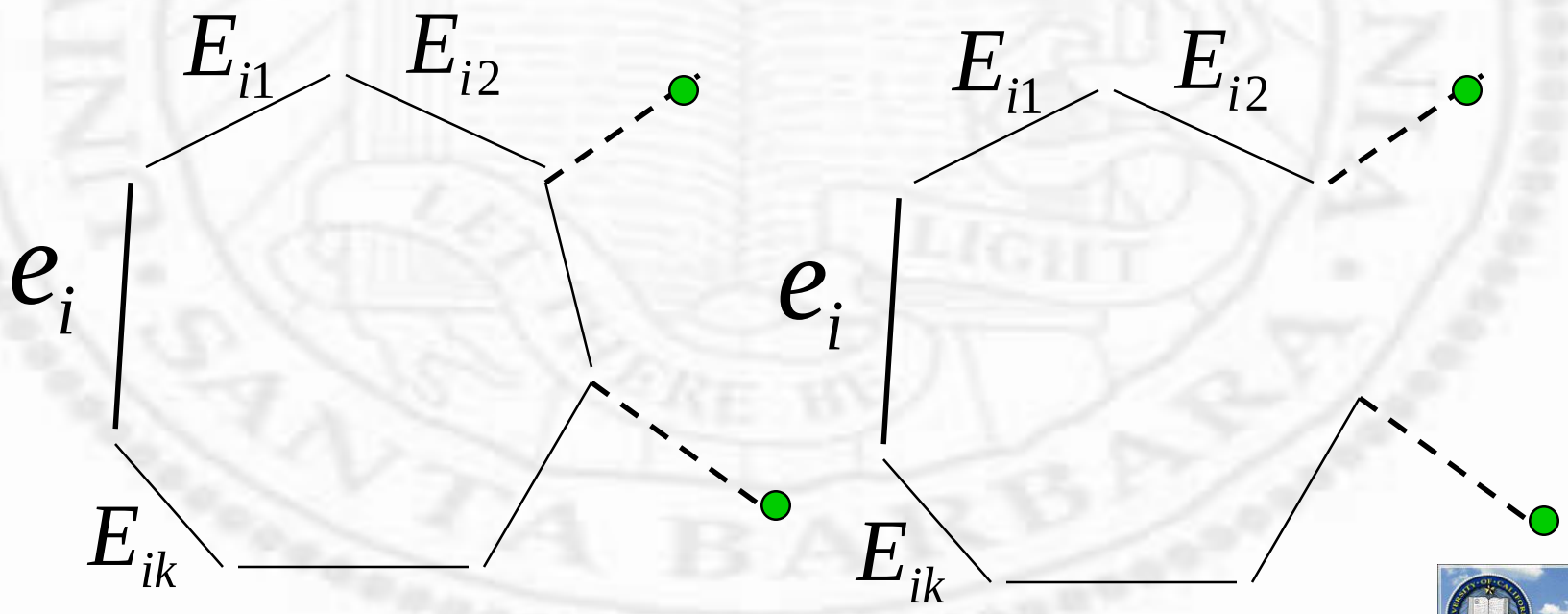
Edge	*Cost*
AB	1
BC	1
DF	1
EG	1
AF	2
BD	2(×)
DE	2
EF	2(×)
BE	4(×)
CE	4(×)
AG	6(×)

- **Proposition**: Kruskal's algorithm find MCST
- **Proof**:

T	T'
Kruskal's	*MST*
e_1	E_1
e_2	E_2
e_3	E_3
$\vdots$	$\vdots$
e_i	E_i ⟵ first index the two solutions differ
$\vdots$	$\vdots$
e_e	E_e

$$Cost(e_i) \le Cost(E_j) \quad i \le j \le e$$

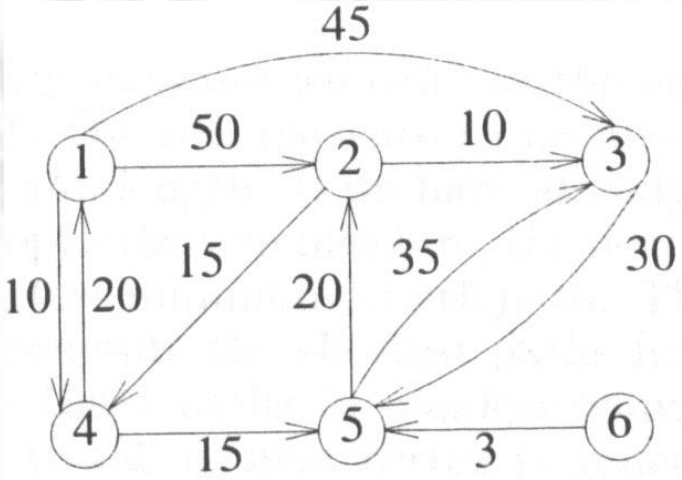
- Including e_i in MCST creates a cycle
  - Not all edges in the cycle belongs to T (Kruskal's)
  - At least one of them must have a higher costs
  - Remove that high-cost edge breaks the cycle and maintain the tree structure



- Time complexity
  - Total *e* edges are considered in order of nondecreasing cost
    - Use partially-ordered tree (heap) to represent edges
      - Construction *O(e log e)*
      - Deletemin *O(e log e)*
  - At each step, remove edge with a minimum cost and check to see whether it creates a cycle if included
    - Use Union-and-Find tree
      - Initially each vertex in a set by itself
      - Inclusion of an edge, join the sets containing the edge's two end points
      - Edges are not included if the two end points are in the same set
  - *O(e log e)*

Single-Source Shortest Path

- Input:
  - *G=(V,E)*, an directed, labeled graph
  - A source vertex
- Output:
  - The shortest path, from source to every other vertices in the graph, if one exists



(a) Graph

	Path	*Length*
1)	1, 4	10
2)	1, 4, 5	25
3)	1, 4, 5, 2	45
4)	1, 3	45

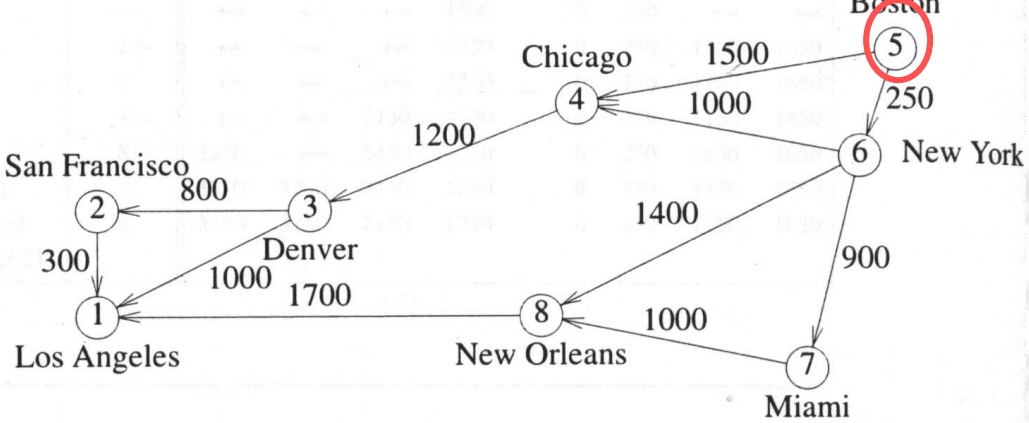
(b) Shortest paths from 1

Possible Greedy Strategies

- Exploring a maze where you cannot see beyond the first turn
- Extremely greedy: with no memory, go where the path leads you (good paths can turn bad at any instance)
- Cautiously greedy: with memory, go where the shortest path encountered so far (backtracking to the path necessary)

Greedy Selection

1. Visited set = {*s*}
2. From visited set, find all 1-distance (direct edge) neighbors
3. Visit the one with the shortest distance: *n*
4. Enlarge visited set = visited set U {*n*}
5. Update distances to the remaining vertices
    1. Go through original visited set
    2. Go through *n*
6. Go back to 2



(a) Digraph

```
If (dist(w)>dist(n)+cost(n,w)) {
    dist(w) = dist(n)+cost(n,w);
    previous_neighbor = n;
}
```

Iteration	S	Vertex selected	Distance							
			LA	SF	DEN	CHI	BOST	NY	MIA	NO
			[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]
Initial	--	----	+∞	+∞	+∞	1500	0	250	+∞	+∞
1	{5}	6	+∞	+∞	+∞	1250	0	250	1150	1650
2	{5,6}	7	+∞	+∞	+∞	1250	0	250	1150	1650
3	{5,6,7}	4	+∞	+∞	2450	1250	0	250	1150	1650
4	{5,6,7,4}	8	3350	+∞	2450	1250	0	250	1150	1650
5	{5,6,7,4,8}	3	3350	3250	2450	1250	0	250	1150	1650
6	{5,6,7,4,8,3}	2	3350	3250	2450	1250	0	250	1150	1650
	{5,6,7,4,8,3,2}									

Complexity: $O(n^2)$

Comparison

- Prim's MCST
- Two groups
  - Already in ST (U)
  - Not yet in ST (V-U)
- Update
  - Find the min edge from U to V-U

$$Cost_i = \min(Cost_i, Cost(new, i))$$
$$Closest_i = (Cost_i == Cost(new, i))?new:Closet_i$$

- Build table of partial solutions: O(n) steps, <O(n) updates $O(n^2)$

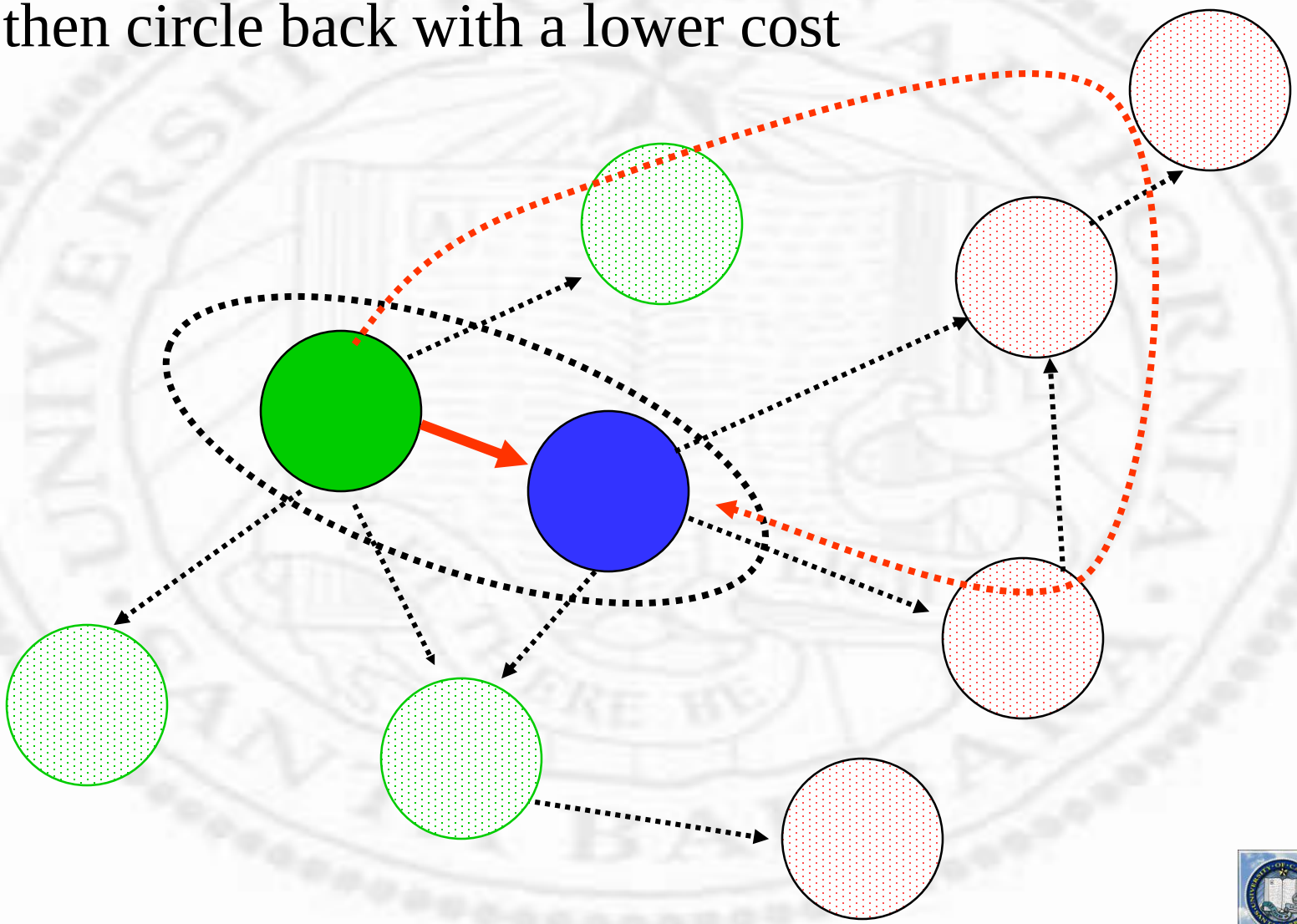
- Dijkstra's shortest path
- Two groups
  - Already found path to
  - Not yet found path to
- Update
  - Find the shortest path from U to V-U

```
If (dist(w)>dist(n)+cost(n,w)) {
   dist(w) = dist(n)+cost(n,w);
   previous_neighbor = n;
}
```

- Build table of partial solutions: O(n) steps, <O(n) updates $O(n^2)$

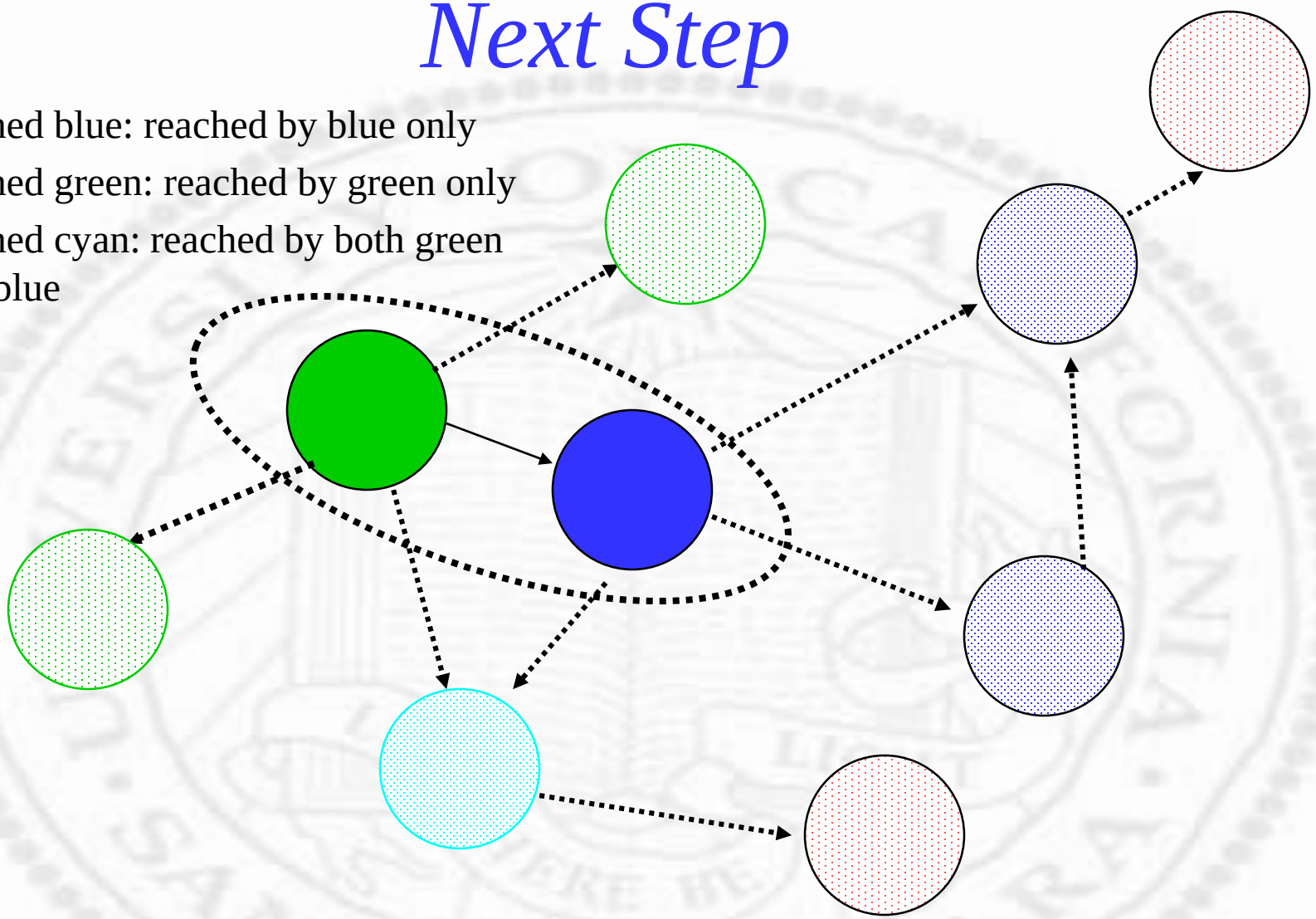
Initially

- You cannot go to blue through dashed green and then circle back with a lower cost



Next Step

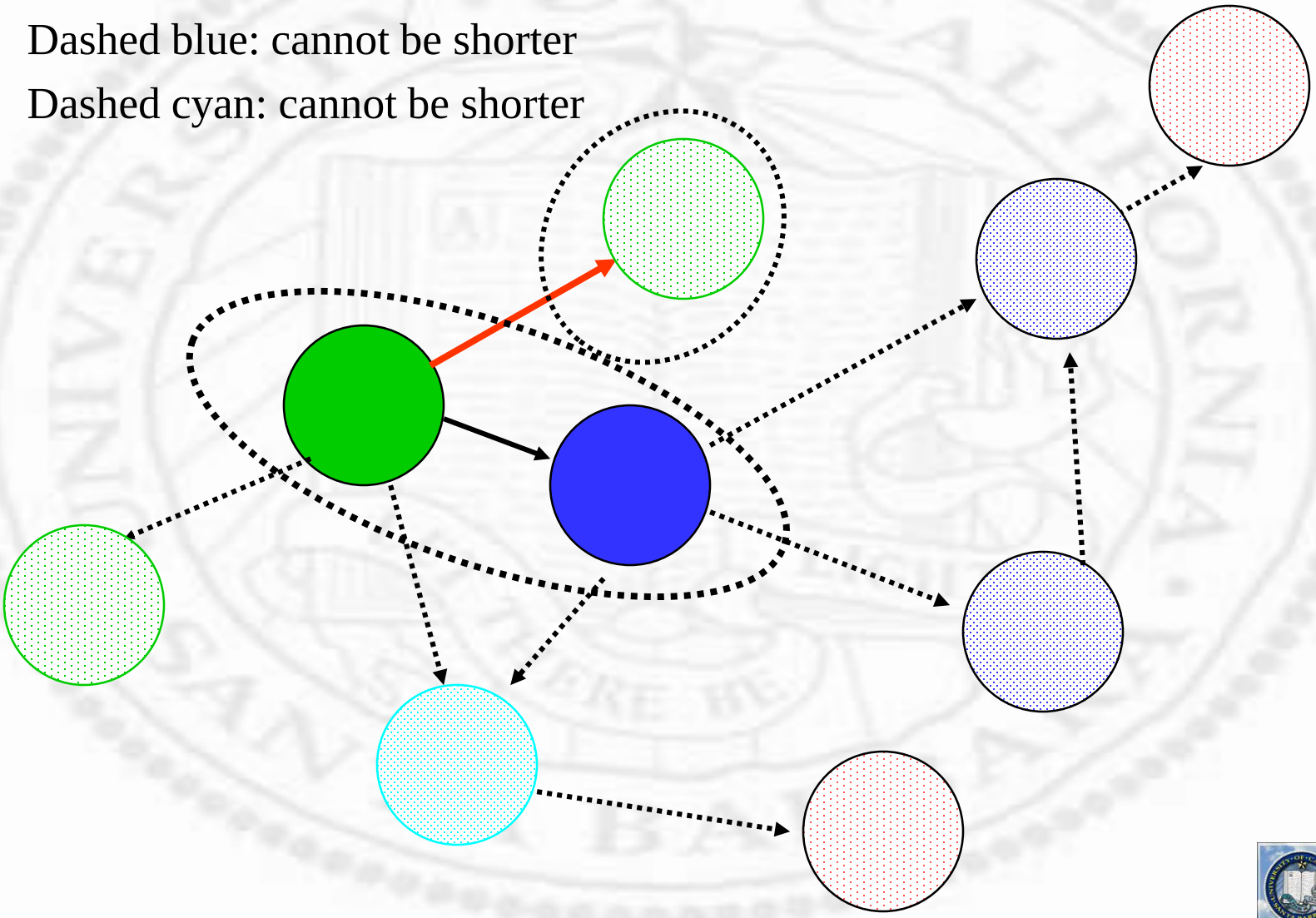
- Dashed blue: reached by blue only
- Dashed green: reached by green only
- Dashed cyan: reached by both green and blue



- One of the dashed blue, green, or cyan will be visited next (i.e., the shortest path to the visited node is determined greedily)
- Is that possible to go through other dashed blue, green, or cyan and circle back to the visited node with a shorter path?

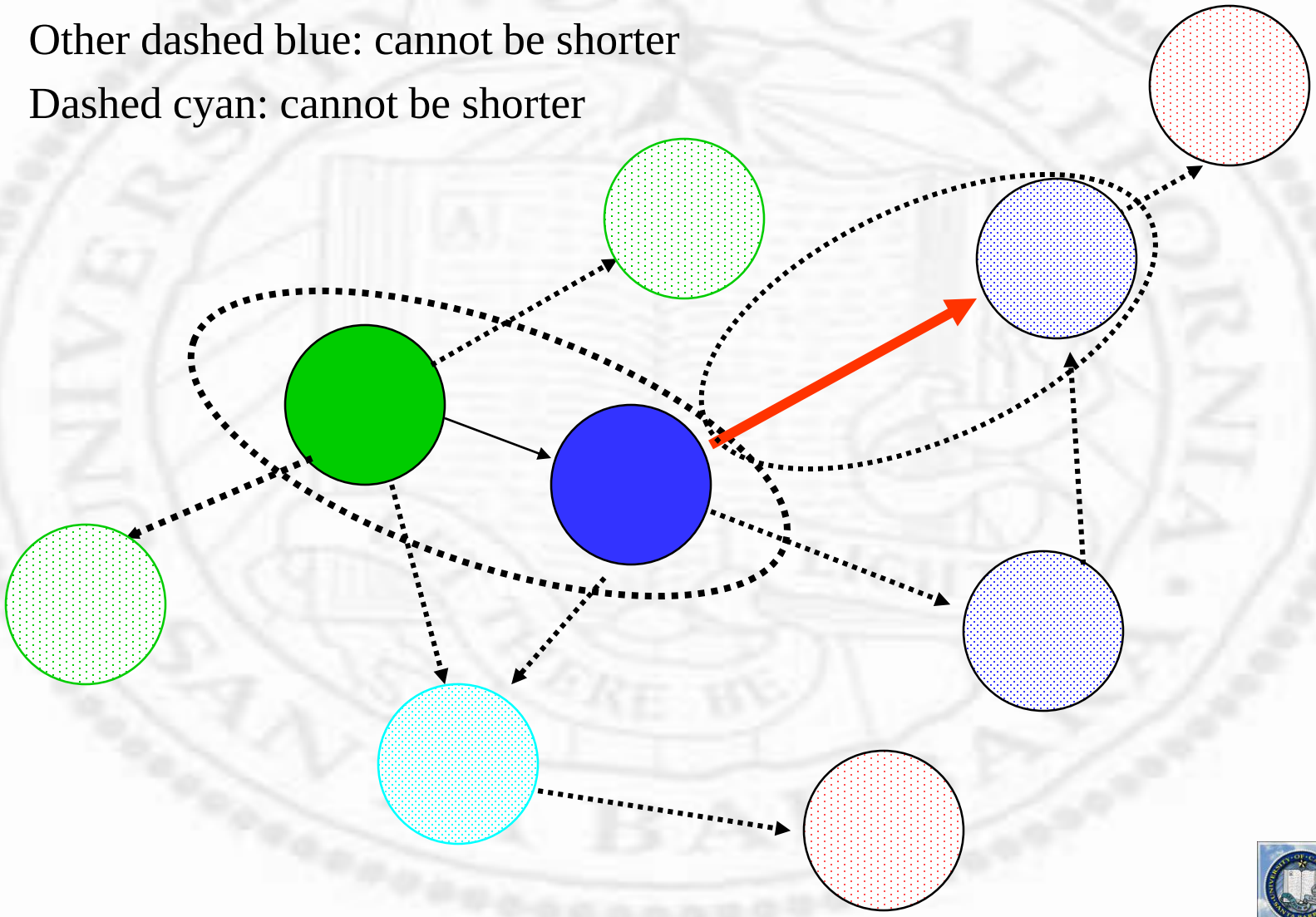
Case one: Dashed green is selected

- Other dashed green: cannot be shorter
- Dashed blue: cannot be shorter
- Dashed cyan: cannot be shorter



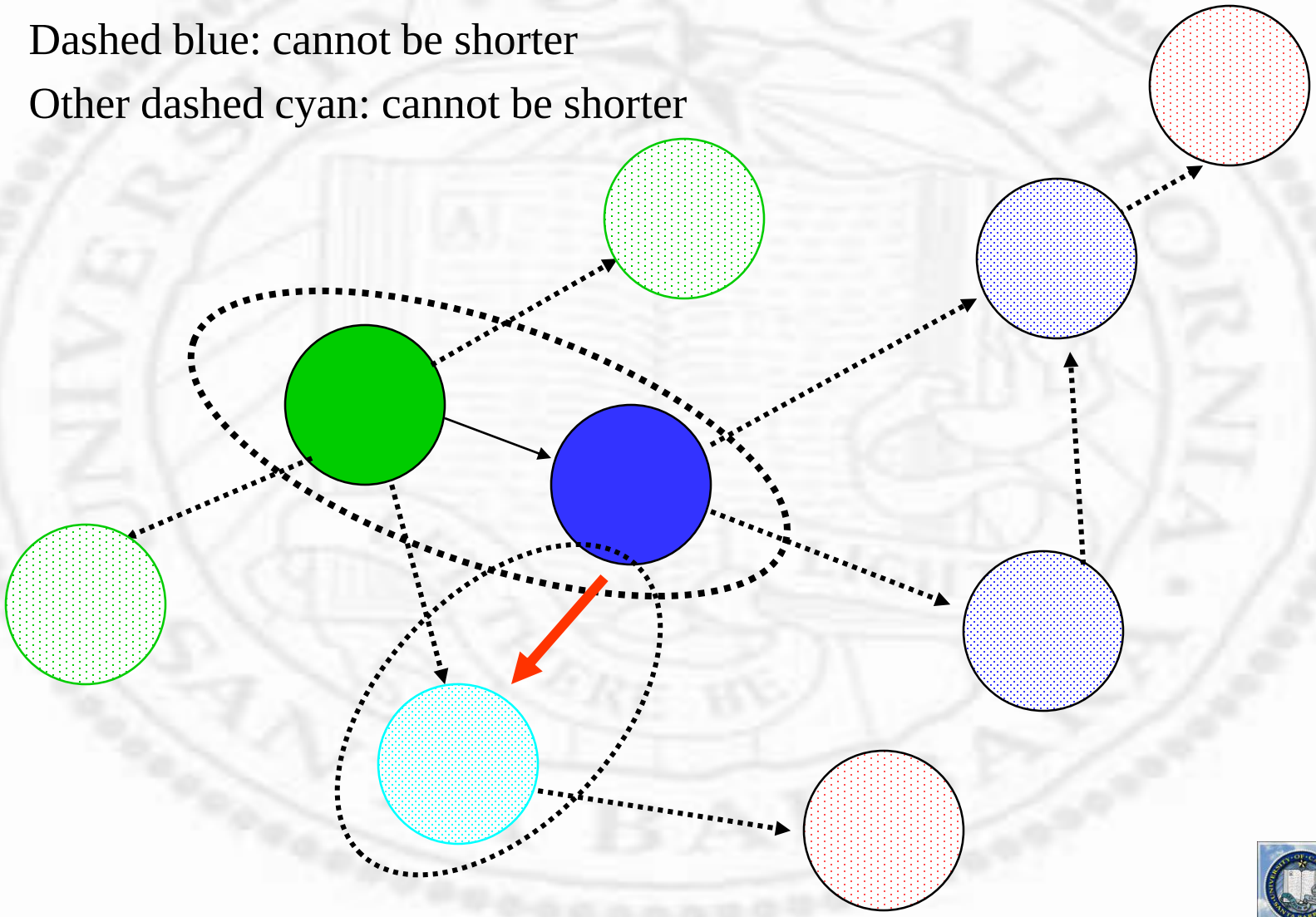
Case two: Dashed blue is selected

- Dashed green: cannot be shorter
- Other dashed blue: cannot be shorter
- Dashed cyan: cannot be shorter

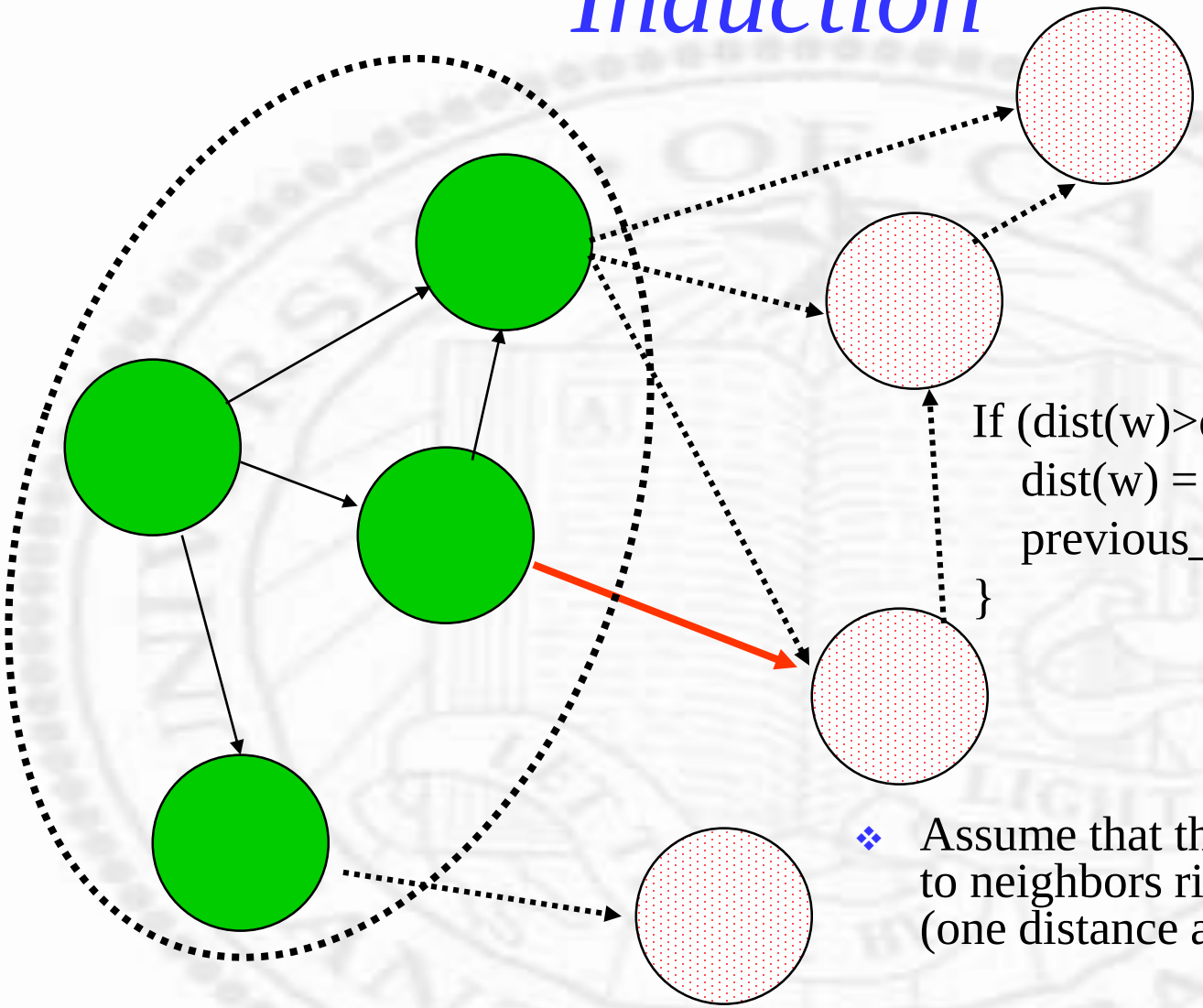


Case three: Dashed cyan is selected

- Dashed green: cannot be shorter
- Dashed blue: cannot be shorter
- Other dashed cyan: cannot be shorter



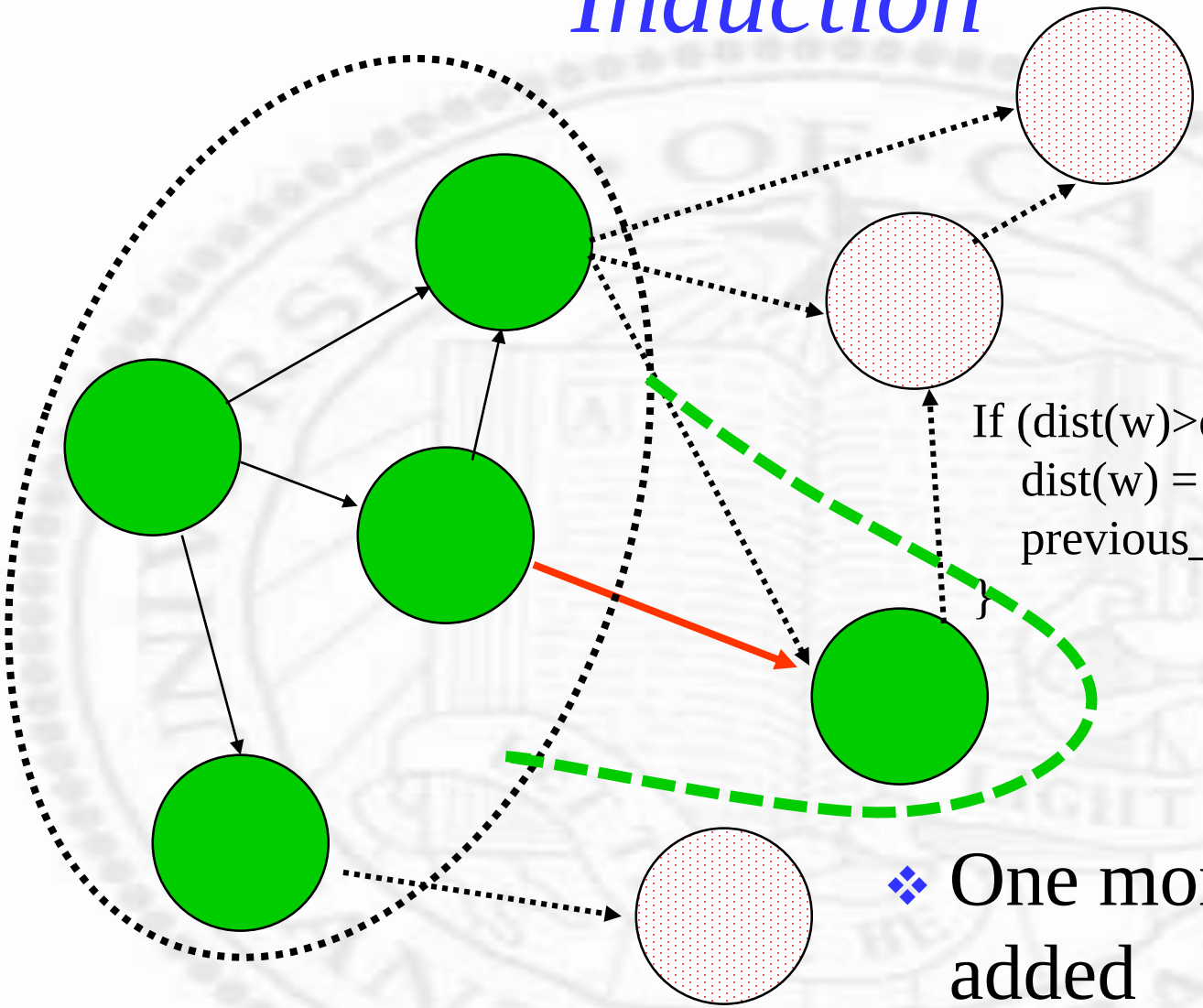
Induction



```
If (dist(w)>dist(n)+cost(n,w)) {
    dist(w) = dist(n)+cost(n,w);
    previous_neighbor = n;
}
```

- Assume that the (current) shortest path to neighbors right outside the wall (one distance away) has been found

Induction



```
If (dist(w)>dist(n)+cost(n,w)) {
    dist(w) = dist(n)+cost(n,w);
    previous_neighbor = n;
}
```

- One more node is added

- Three things can happen for a node still outside the wall (the envelop) after a new node is added
  - Not reached by the new node
    - The current best path didn't change
  - Reached by the new node but not any node in the previous envelop
    - The current best path must be the one via the new node
  - Reached by the new node and also nodes in the previous envelop
    - The update process should record the best between the two
- Hence, when "the best of the best" is chosen to go out the wall, one cannot jump through other paths on the wall and circle back to get a better result

Job Sequencing with Deadlines

- Input:
  - a set of *n* jobs, each with a deadline and a profit if completed before deadline
  - one machine to execute all the jobs
  - each job takes one unit of time
- Output:
  - a subset of jobs, each completed before deadline, with maximum profit

- Objective function: $\max \sum_{i \in J} P_i$
- Feasibility constraint:
- Example:

$n = 4, (P_1, P_2, P_3, P_4) = (100,10,15,27)$

$(d_1, d_2, d_3, d_4) = (2,1,2,1$

feasible	schedule	profit
(1,2)	2,1	110
(1,3)	1,3 or 3,1	115
(1,4)	4,1	127
(2,3)	2,3	25
(3,4)	4,3	42
(1)	1	100
(2)	2	10
(3)	3	15
(4)	4	27

- *SELECT:* select the job with maximum profit subject to the constraint that the resulting schedule is still feasible

	J	$\sum_{i \in J} P_i$	
Initially	Φ	0	
1	(1)	100	
4	(1,4)	127	
3	(1,4)	127	(1,4,3) *not feasible*
2	(1,4)	127	(1,4,2) *not feasible*

(Q1) How to determine if J is feasible?

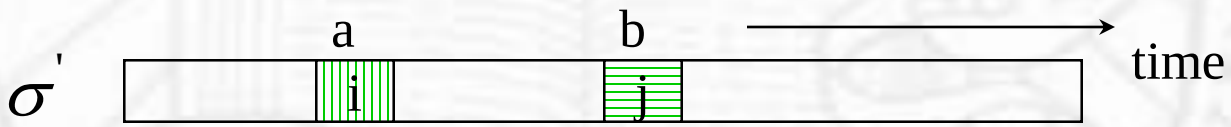
(Q2) Is greedy algorithm optimal?

(Q1) If $J=\{1,2,3,\ldots,k\}$

- try all possible ($k!$) permutations (schedules) and see whether at least one of them allows all jobs to be finished before their deadlines
- intuitively, jobs with earlier deadline (more urgent) should be performed first
- check the permutation $\sigma^* = (i_1, i_2, \ldots, i_k)$

$$d_{i_1} \le d_{i_2} \le \ldots \le d_{i_k}$$

- **Proposition**: $J=\{1,2,\ldots,k\}$ is feasible if and only if σ^* is feasible
- **Proof**:
  - If σ^* is feasible, then $J=\{1,2,\ldots,k\}$ is feasible (by definition)
  - If $J=\{1,2,\ldots,k\}$ is feasible, then



$d_i \geq a$ job completed before deadline

$d_j \geq b$

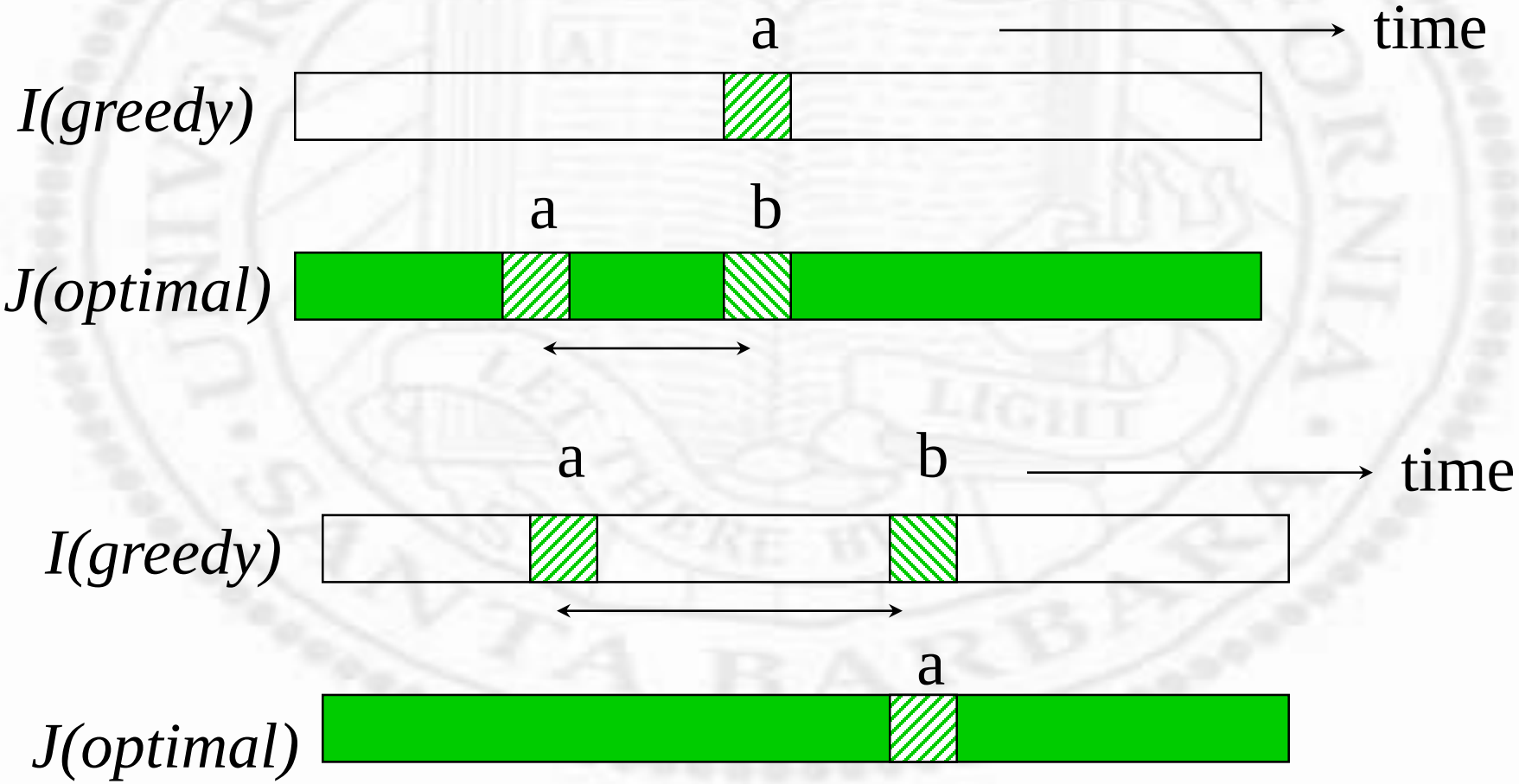
$d_i \geq d_j$ out of order

$d_j \geq b > a$ j can be moved forward

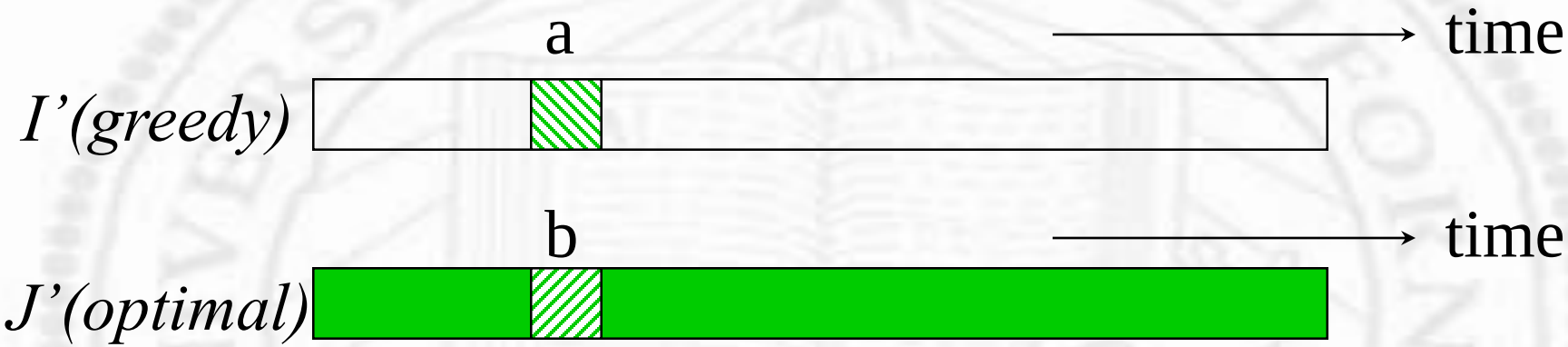
$d_i \geq d_j \geq b$ i can be moved backward

- **Proposition**: The greedy method produces a schedule with the maximum profit
- **Proof**:
  - Two *different* solutions: optimal and greedy
  - Jobs that are in both optimal and greedy
    - make sure that they are scheduled at the same time
  - Jobs that are in one but not the other
    - change them into ones in greedy without decreasing profit
  - The process continues until two solutions are equal

- For jobs that are in both
  - the job is scheduled the same in both
  - the job is scheduled earlier in optimal
  - the job is scheduled earlier in greedy
  - Again, change optimal to greedy



- For jobs that are different
  - *I'* and *J'* are such that jobs common to both are scheduled at the same slot



$P_a \geq P_b$ ∵ if b has a larger profit and is feasible, it will appear in the greedy solution

  - Replace *b* with *a* in the optimal solution will not decrease the profit

Finally

- Can it be that greedy solution still does more jobs than optimal?
  - No, optimal will not be optimal then
- Can it be that optimal solution does more jobs than greedy?
  - No, if such a job is feasible, how come greedy solution doesn't include it?

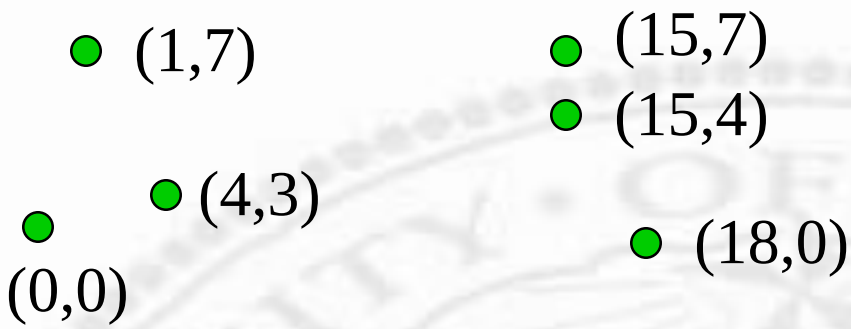
- Time complexity
  - Sort jobs according to nondecreasing profit $O(nlogn)$
  - Consider n jobs in turn
    - for each job, insert the job into the partial solution using its deadline $O(i)$
    - check whether the new solution is still feasible $O(i)$

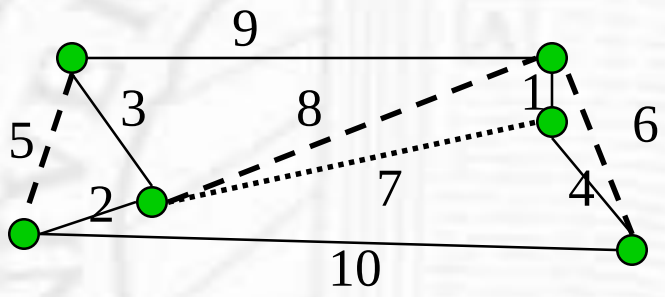
$$O(n^2)$$

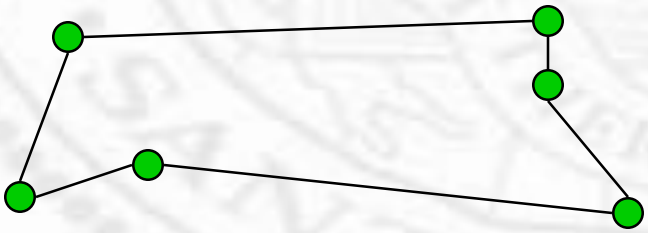
Greedy Method as Heuristics

- For problems whose solutions are found by "try-all-possibilities," an optimal solution is difficult to compute for large problem size
- Greedy method can usually produce a "very good" solution at a fraction of the cost

- Example: Traveling salesperson's problem
  - Input: a fully connected, labeled undirected graph
  - Output: a tour (a simple cycle including all vertices) whose edge weights are minimum.
  - Greedy method:
    - A variant of Kruskal's algorithm
    - Consider edges in nondecreasing cost
    - The edge under consideration, together with all edges already selected:
      - do not cause a vertex to have a degree of three or more
      - do not form a cycle, unless the number of edges equals to that of vertices







- Greedy solution
  - 5,6 rejected: cycle
  - 7,8 rejected: vertex degree larger than 2
  - cost = 49.73
- Optimal solution
  - cost = 48.39