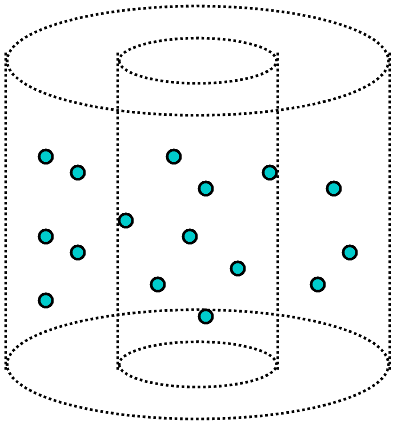
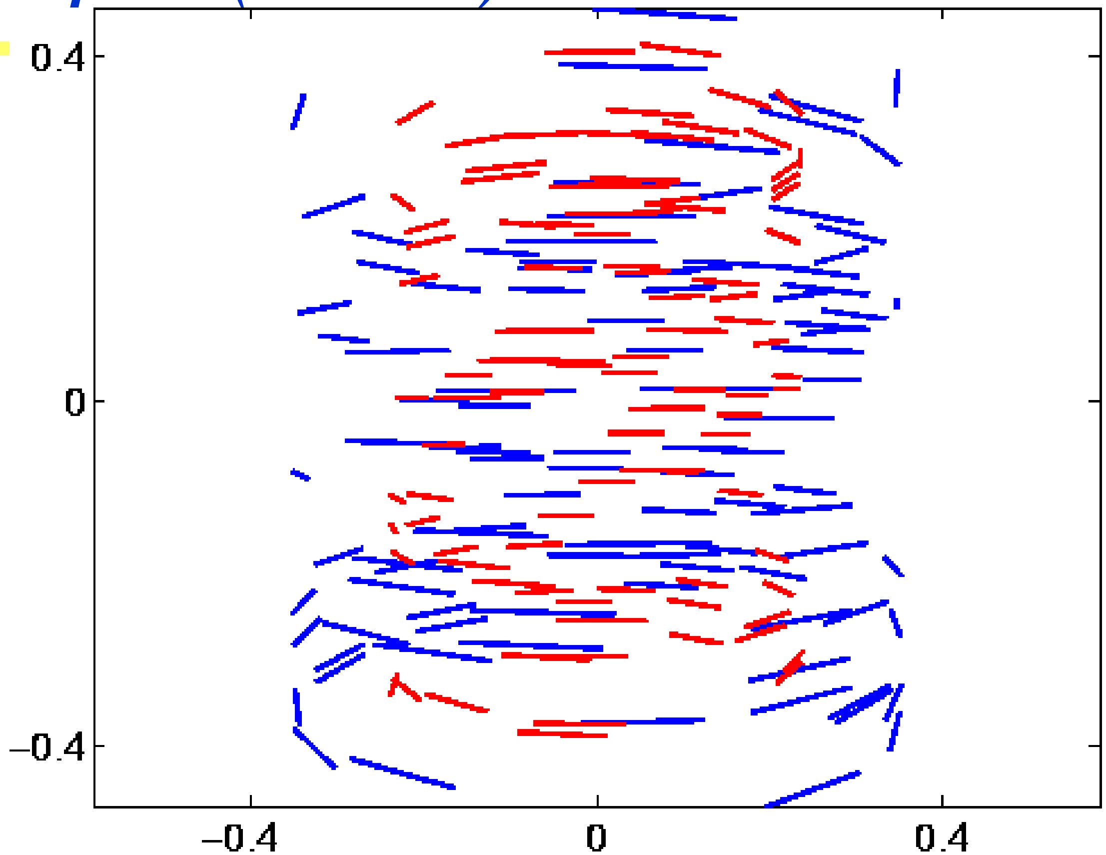
Visual Motion Analysis and Representation

Example

- Ullman's concentric counter-rotating cylinder experiment
- Two concentric cylinders of different radii
- W. a random dot pattern on both surfaces (cylinder surfaces and boundaries are not displayed)
- Stationary: not able to tell them apart
- Counter-rotating: structures apparent

Example (cont.)





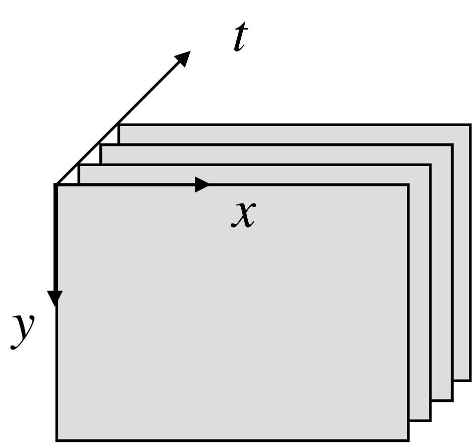
- Motion helps in
  - segmentation (two structures)
  - identification (two cylinders)

Classes of Techniques

- ***Feature-based methods***
  - Extract visual features (corners, textured areas) and track them
  - Sparse motion fields, but possibly robust tracking
  - Suitable especially when image motion is large (10s of pixels)
- ***Direct-methods (Pixel-based methods)***
  - Directly recover image motion from spatio-temporal image brightness variations
  - Global motion parameters directly recovered without an intermediate feature motion calculation
  - Dense motion fields, but more sensitive to appearance variations
  - Suitable for video and when image motion is small (< 10 pixels)

Optical flow and motion analysis

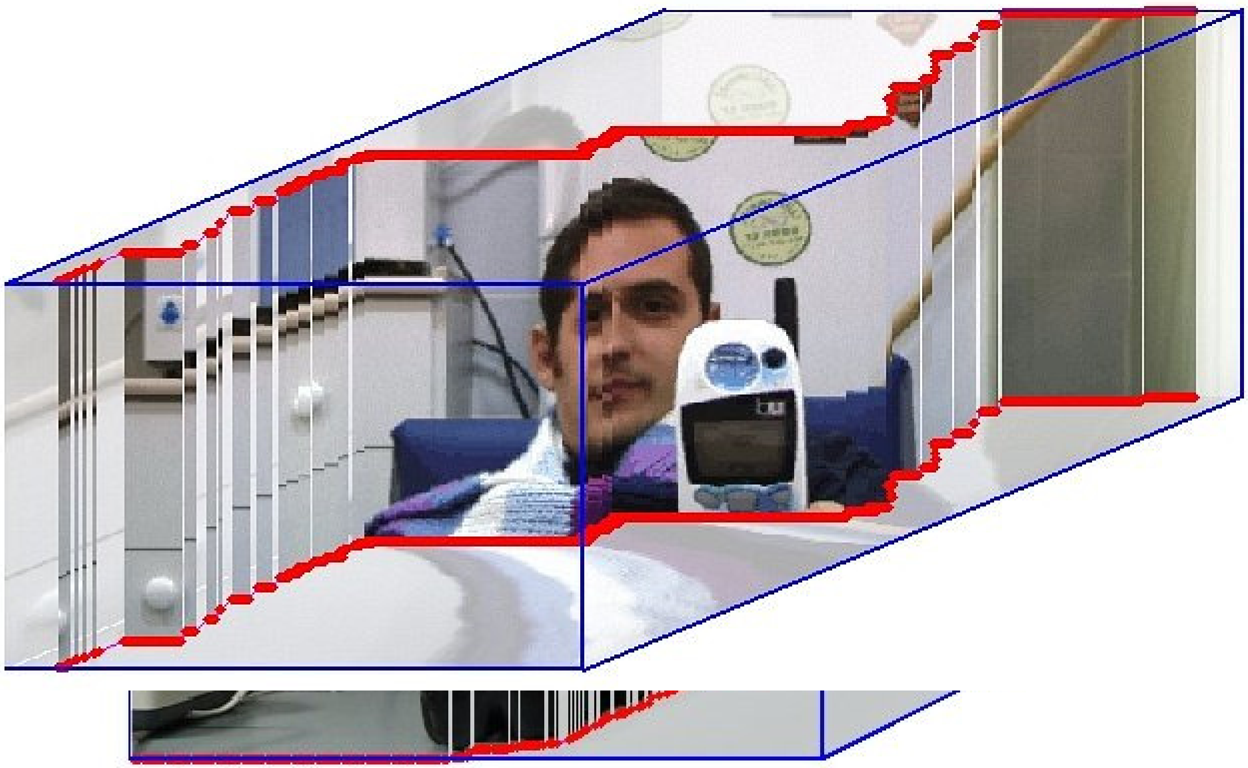
- Now we move to considering images that vary over time – image sequences
  - Typical case is video – images captured at 30 frames/second (or 15, or 60, or ...)
  - $I(x, y, t) \rightarrow I_1(x,y) = I(x, y, t_1)$, $I_2(x,y) = I(x, y, t_2)$, etc.
  - “Spatial-temporal space” describes (x, y, t)



What can change between I_t and I_{t+1}?

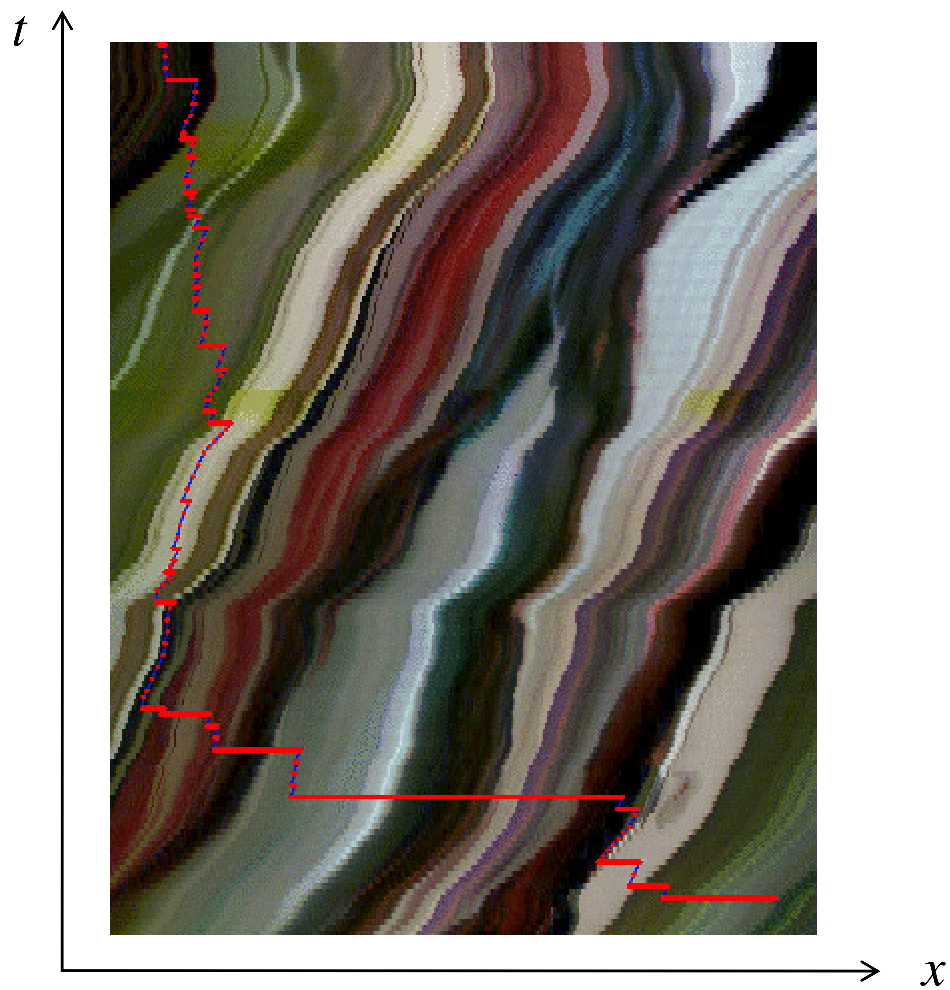
What do images close in time have in common?

Spatio-temporal image data (examples)



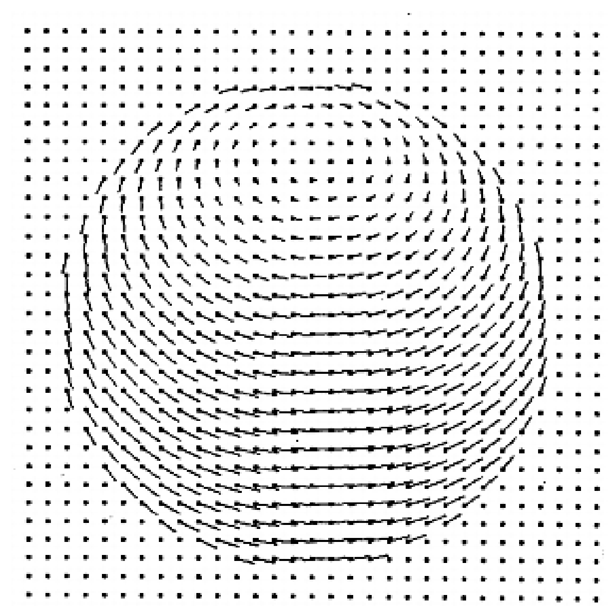
Frames

x-*t* slice



Optical flow and motion analysis

- Optical flow is the *apparent motion* of brightness patterns in the image sequence
  - A 2D vector at each point – a vector field
- The motion field is the *true motion* (3D) at each point, mapped onto the 2D image
  - A vector field
- They are not always the same
  - E.g., white, featureless ball?



In general, we estimate the *motion field* by computing the *optical flow*

The motion field is not *directly* observed

Example

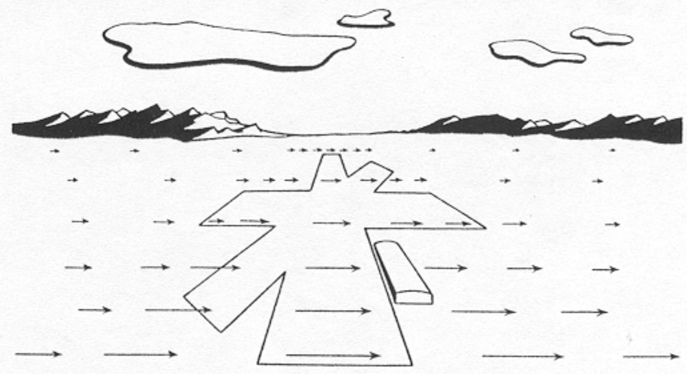
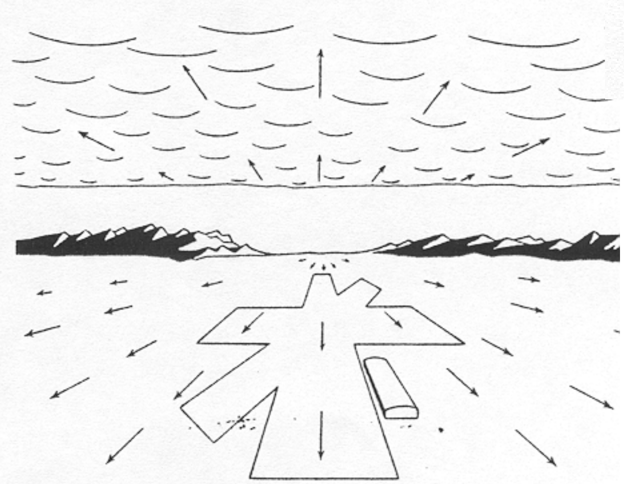
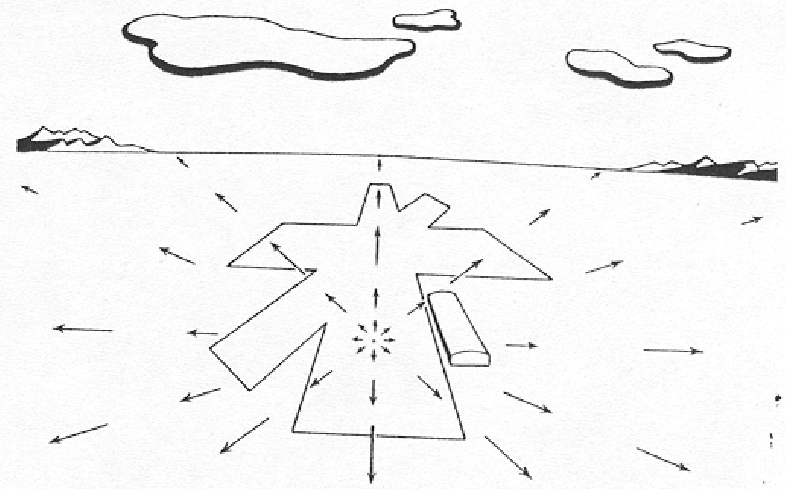
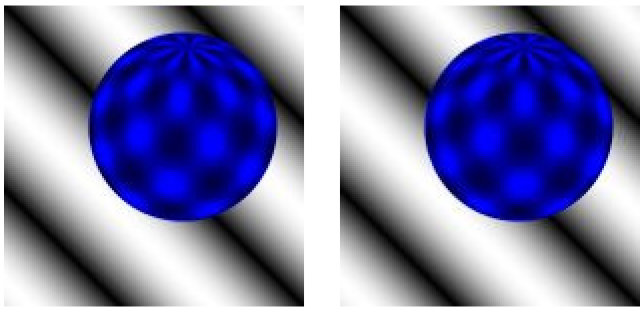


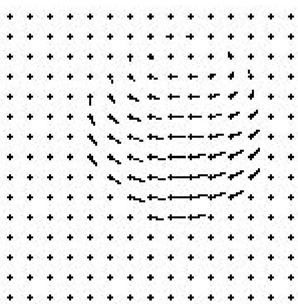
Figure 8.6 The motion field of a pilot looking to the right in level flight. The f
of expansion here is off at infinity to the left of the figure; equivalently, the focu
contraction is off at infinity to the right of the figure. (From [Gibson 1950] w
permission. Copyright © 1977, 1950 by Houghton Mifflin Company.)





Example





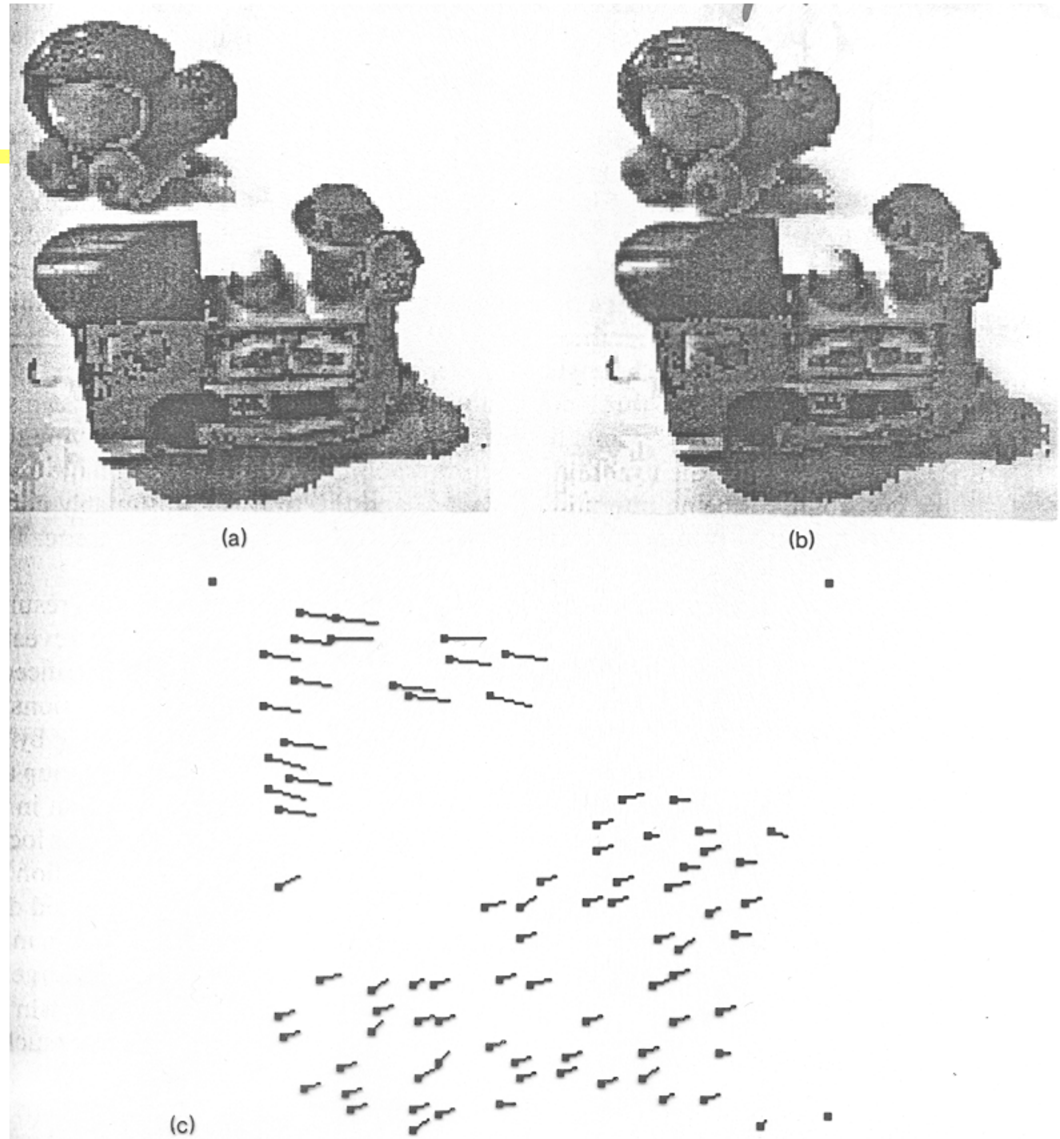
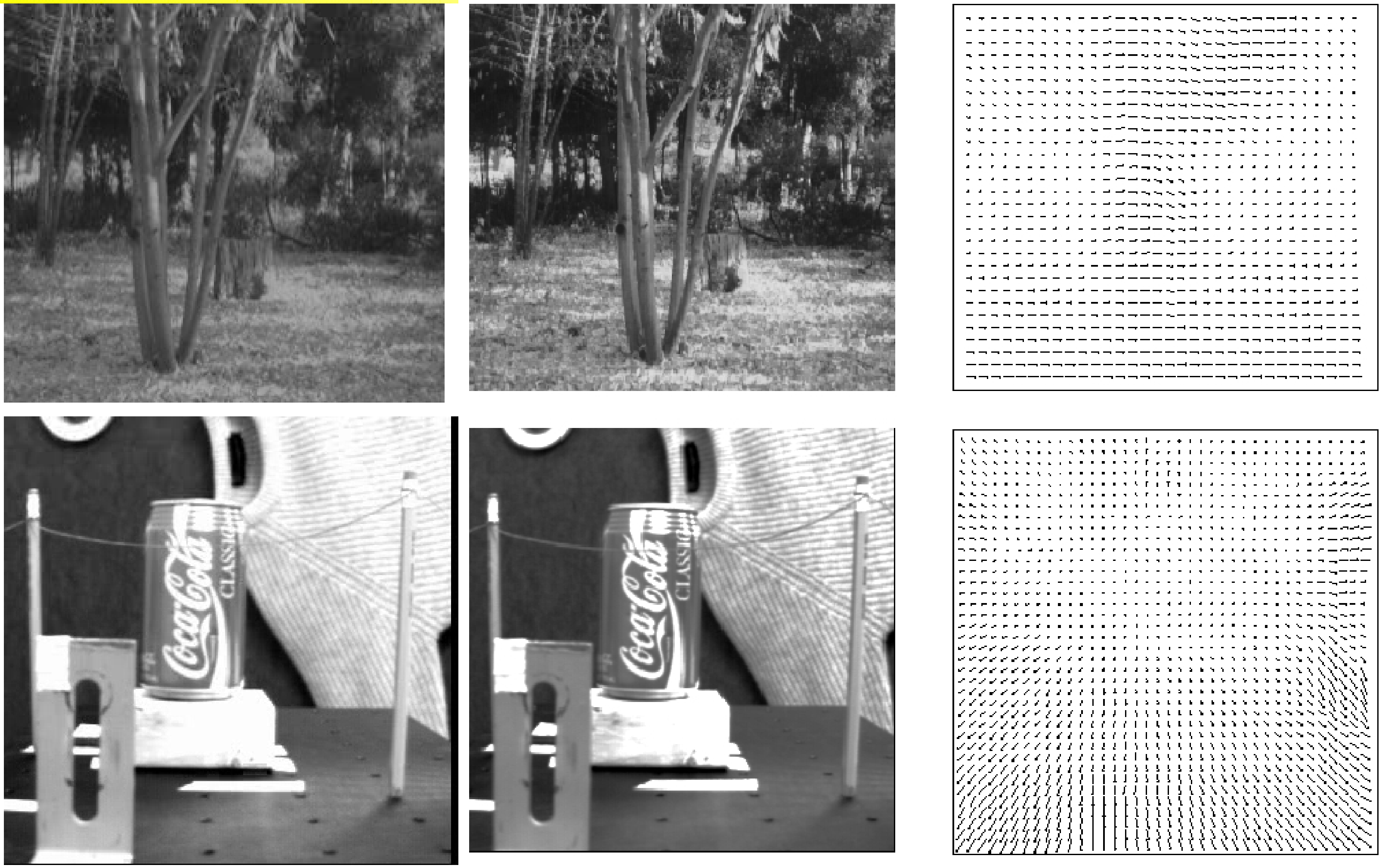
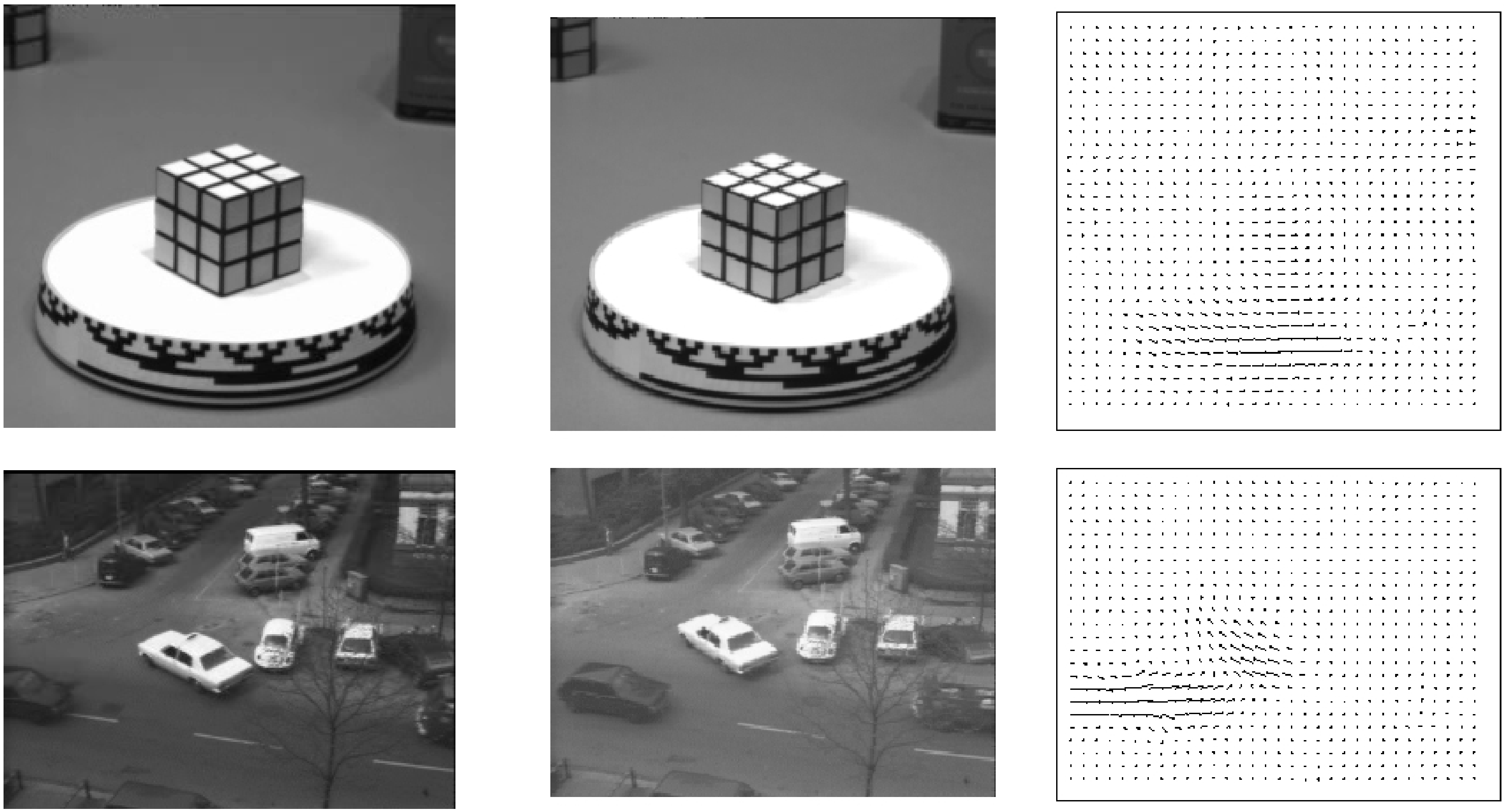


Fig. 7.7 Optical flow from feature point analyses. (a) An image. (b) Later image. (c) Optical flow found by relaxation.


Coca-Cola
CLASSIC

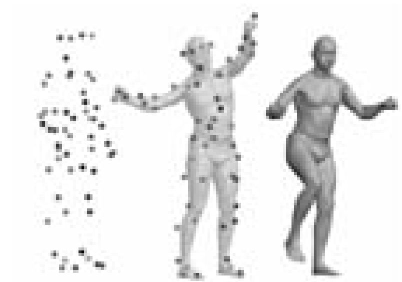


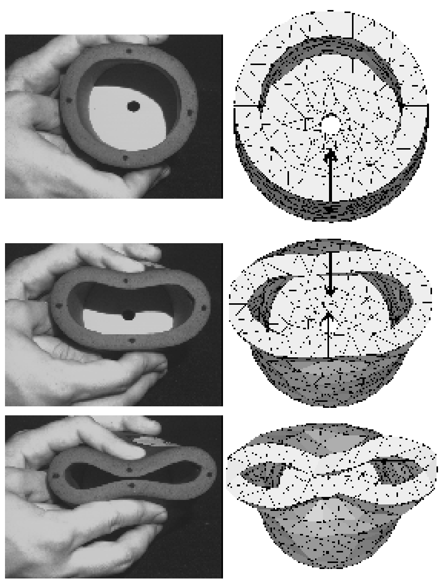
Caveats

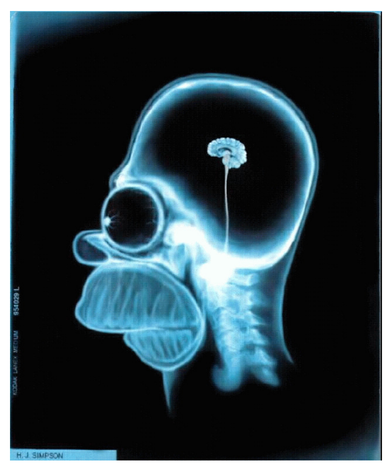
- Motion analysis a very important and popular area in computer vision
- A large body of literature exits with maybe hundreds of different formulations (At CVPR, you will find at least 2 or 3 sessions on motion)
- Many of them can be very mathematical
- Apparent motion != True motion

Rigid vs. nonrigid motion

- Camera motion is 6 DOF rigid motion
- Object motion may be rigid or nonrigid
  - Rigid: coffee mugs, silverware, baseballs, jets, ...
  - Nonrigid: humans, face, medical imagery, beach balls, scissors, grass, ...
    - Includes ***articulated*** motion

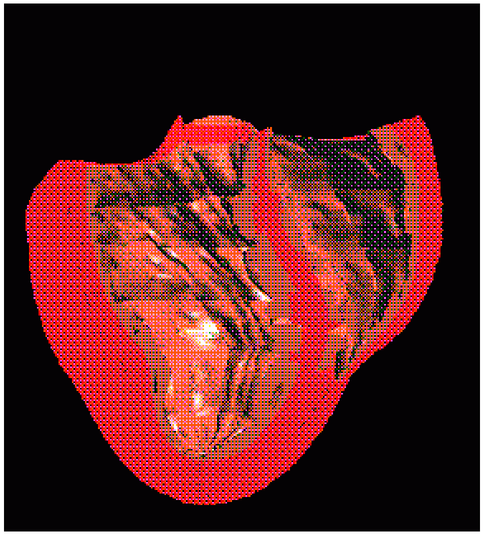






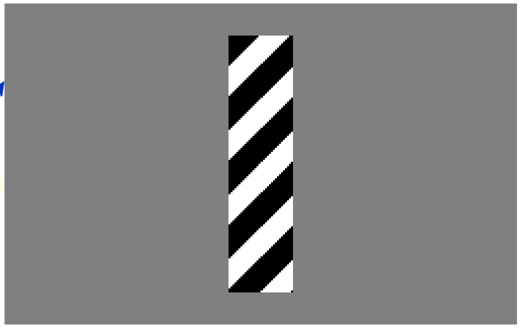
Nonrigid motion

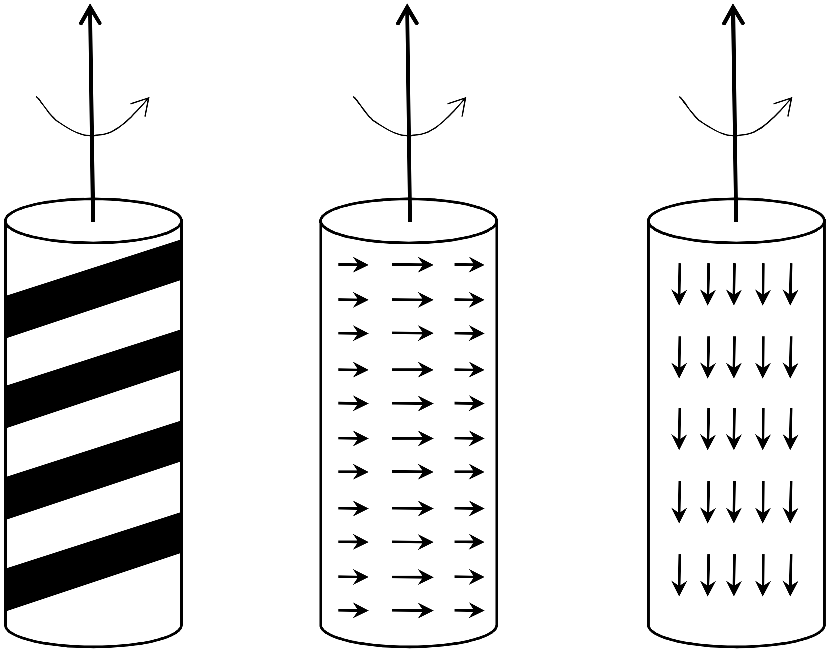
- Nonrigid motion is complicated and difficult, especially with little prior knowledge on what is being viewed
  - Typical problem: What are the parameters of the known nonrigid model of the object being viewed?



We'll just focus on rigid motion

The barber's pole illus





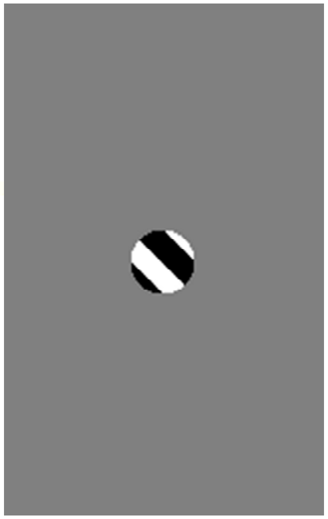
Motion field

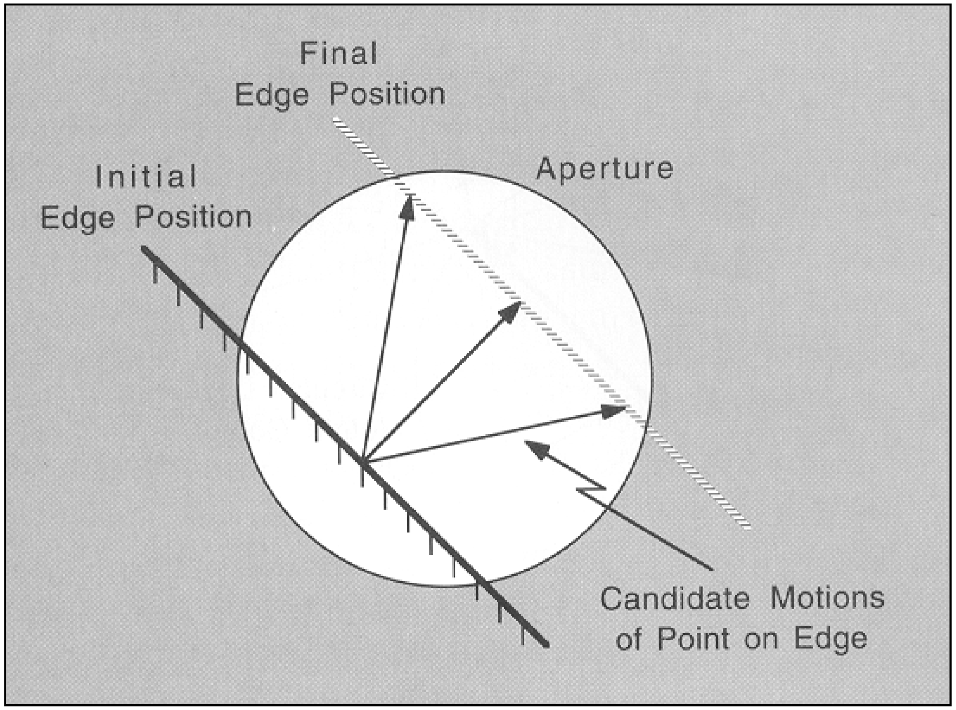
Optical flow

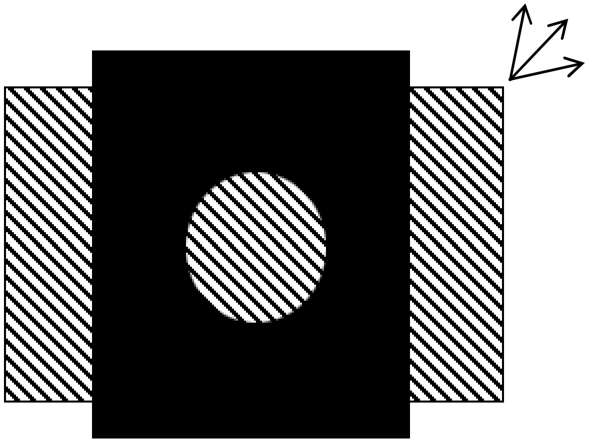
Web example

The aperture problem

- In local processing, we can only measure motion perpendicular to the image gradient







First steps

- Motion processing starts with estimating optical flow from frame to frame, either *densely* or *sparsely*
- The typical approaches are:
  - Dense correspondence:
    - Differential methods, local area/correlation based
    - This could be hierarchical (coarse-to-fine approach)
  - Sparse correspondence
    - Matching methods, feature based
- Asumption: Points/features can be matched in nearby images

Brightness constancy equation

Total derivative $\longrightarrow$ $$\frac{dI}{dt} = \frac{dI(x(t),\, y(t),\, t)}{dt} = 0$$ For a given scene point

$$\frac{dI(x(t),\, y(t),\, t)}{dt} = \frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t}\frac{dt}{dt} = 0$$

$$\left(\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y}\right)^T \left(\frac{dx}{dt}, \frac{dy}{dt}\right) + \frac{\partial I}{\partial t} = 0$$

$$\boxed{\nabla I \cdot \mathbf{v} + I_t = 0}$$

∇I Image gradient

v Optical flow

I_t Time difference

Brightness constancy equation (method #2)

$$I(x, y, t) = I(x + \delta x, y + \delta y, t + \delta t)$$

For a given scene point

$$\boxed{I(x + \delta x, y + \delta y, t + \delta t)} - I(x, y, t) = 0$$

$$\Big\downarrow \approx$$

$$I(x, y, z) + \frac{\partial I(x, y, t)}{\partial x} dx + \frac{\partial I(x, y, t)}{\partial y} dy + \frac{\partial I(x, y, t)}{\partial t} dt$$

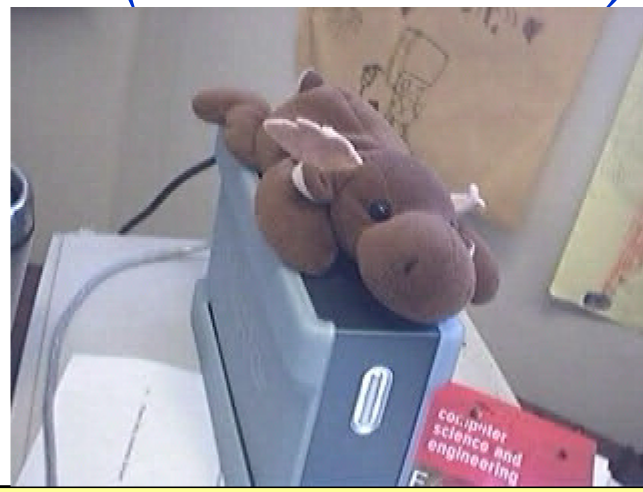
by Taylor expansion

$$\frac{\partial I(x, y, t)}{\partial x} dx + \frac{\partial I(x, y, t)}{\partial y} dy + \frac{\partial I(x, y, t)}{\partial t} dt = 0$$

$$\frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

$$\boxed{\nabla I \cdot v + I_t = 0}$$

Brightness constancy equation (method #2)



$$I(x, y, t) = I(x + \delta x, y + \delta y, t + \delta t)$$

Image at time t

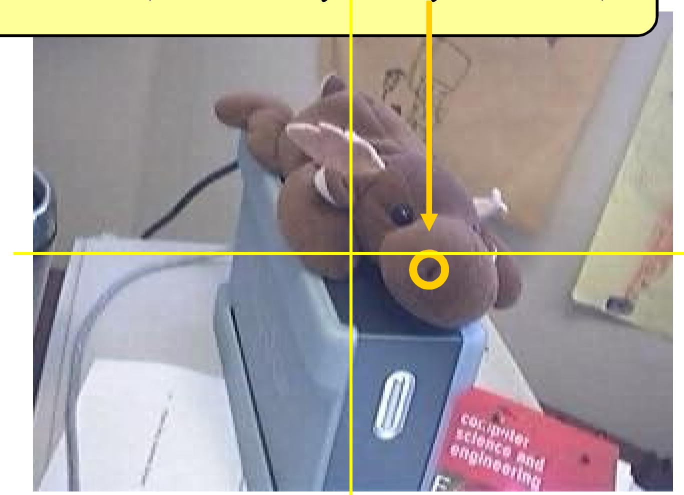
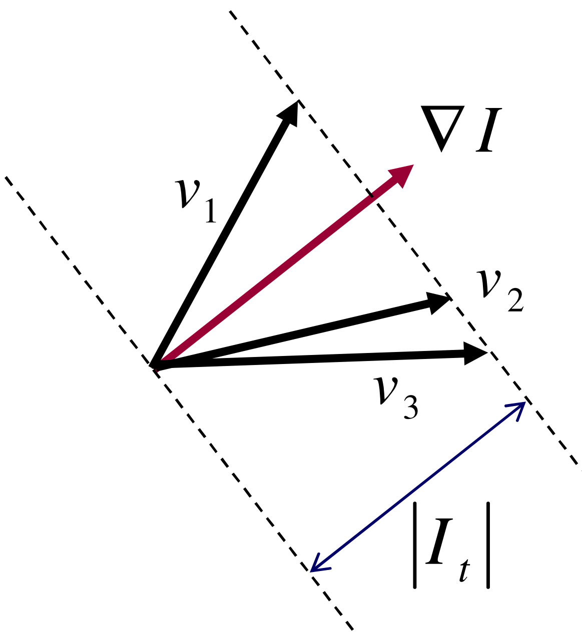


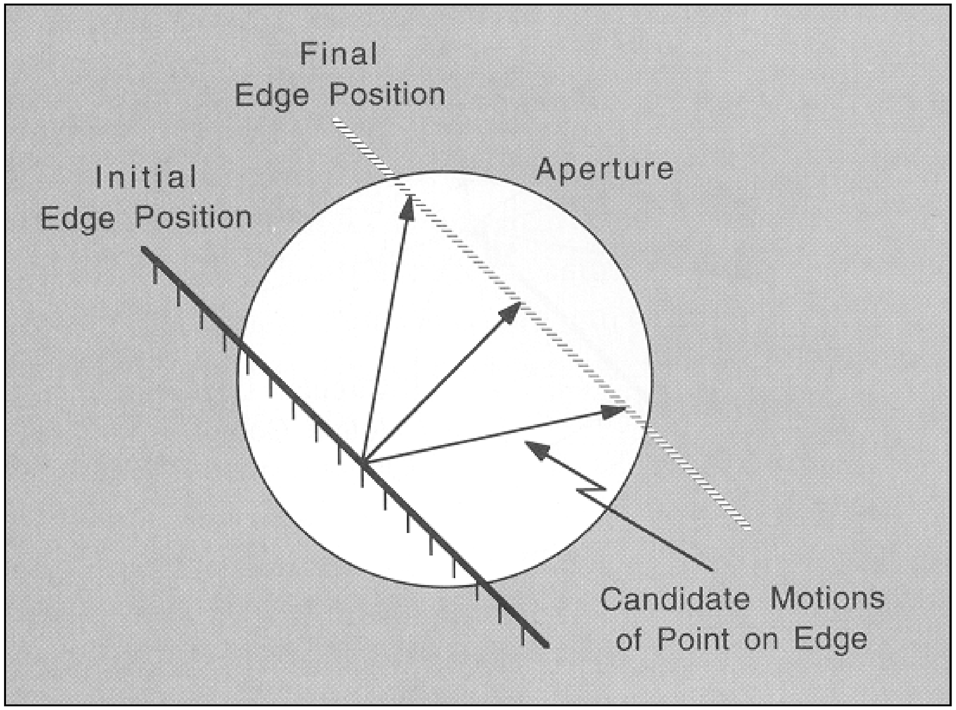
Image at time $t+\delta t$

Back to the aperture problem

$$\nabla I \cdot v + I_t = 0$$

$$\nabla I \cdot v = -I_t$$

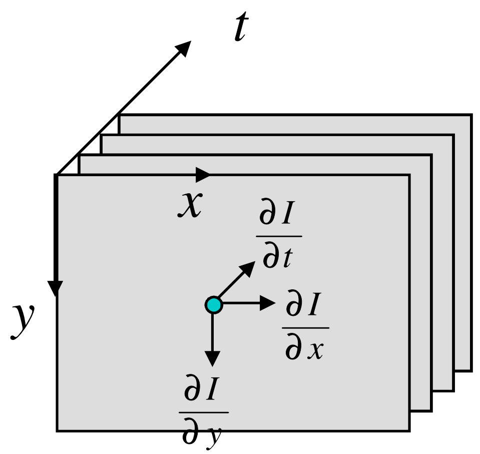


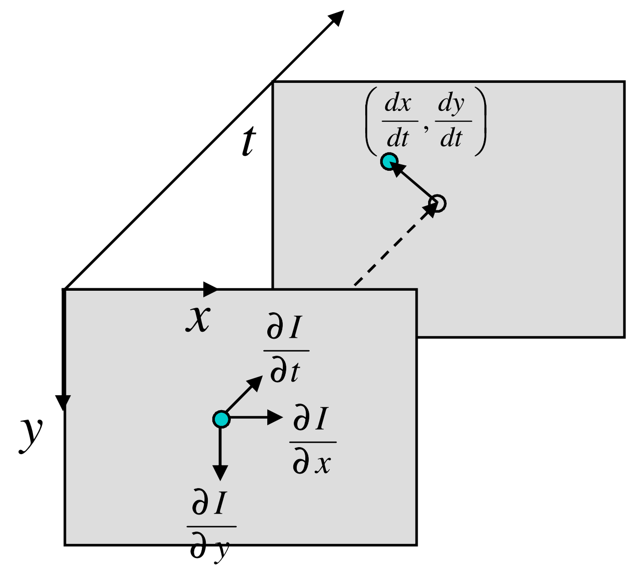


Many vectors ***v*** satisfy this

Only the normal direction is constrained

On images...



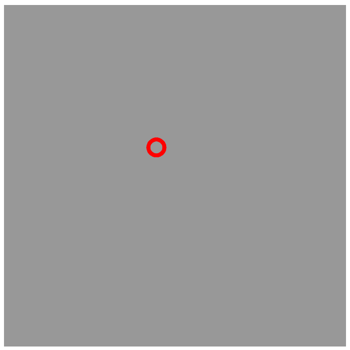


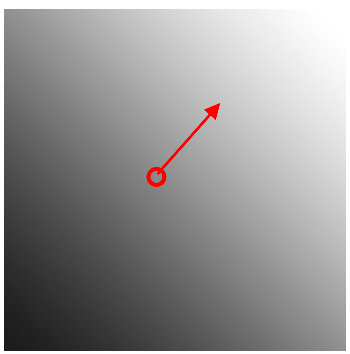
$$\nabla I \cdot v + I_t = 0$$

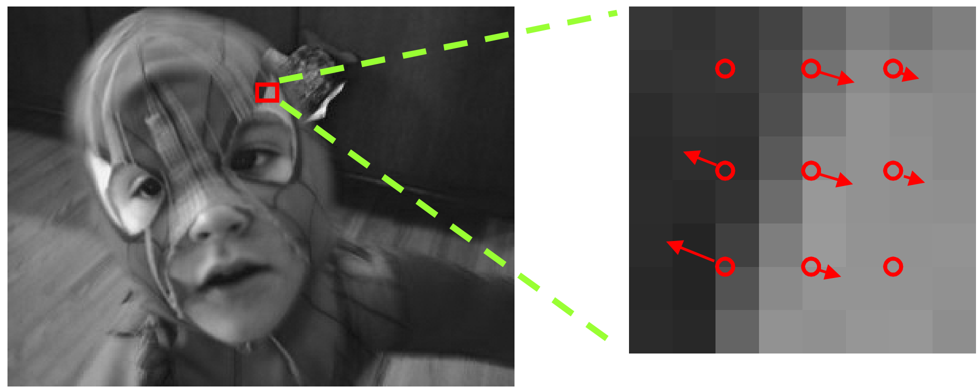
This equation defines and constrains the optical flow ***v*** **(*x*, *y*)**

What is the image gradient?

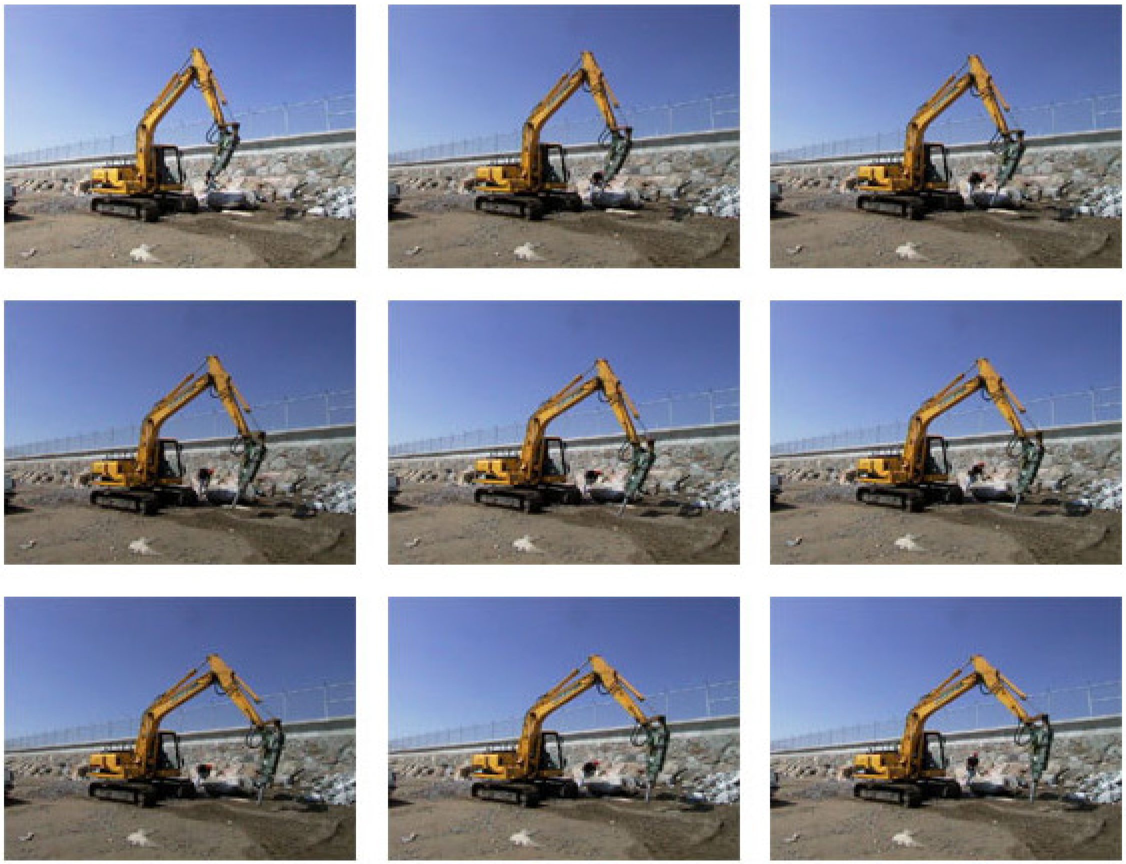
Image gradient – the first derivative (slope) of the intensity variation in (x, y)







What is the temporal gradient?



t_1 t_2

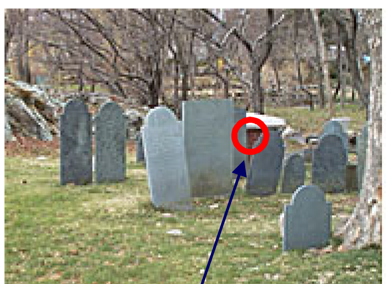
$$\frac{\partial I}{\partial t}$$

Brightness constancy of a point

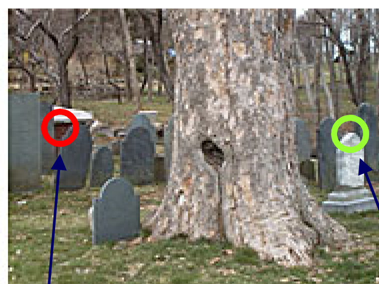
Scene

Image sequence

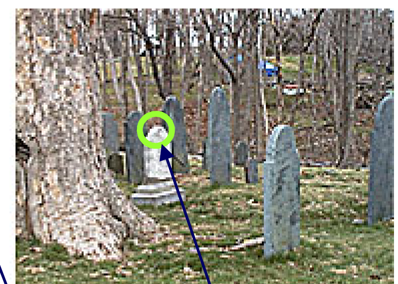
I_1



I_2



I_3

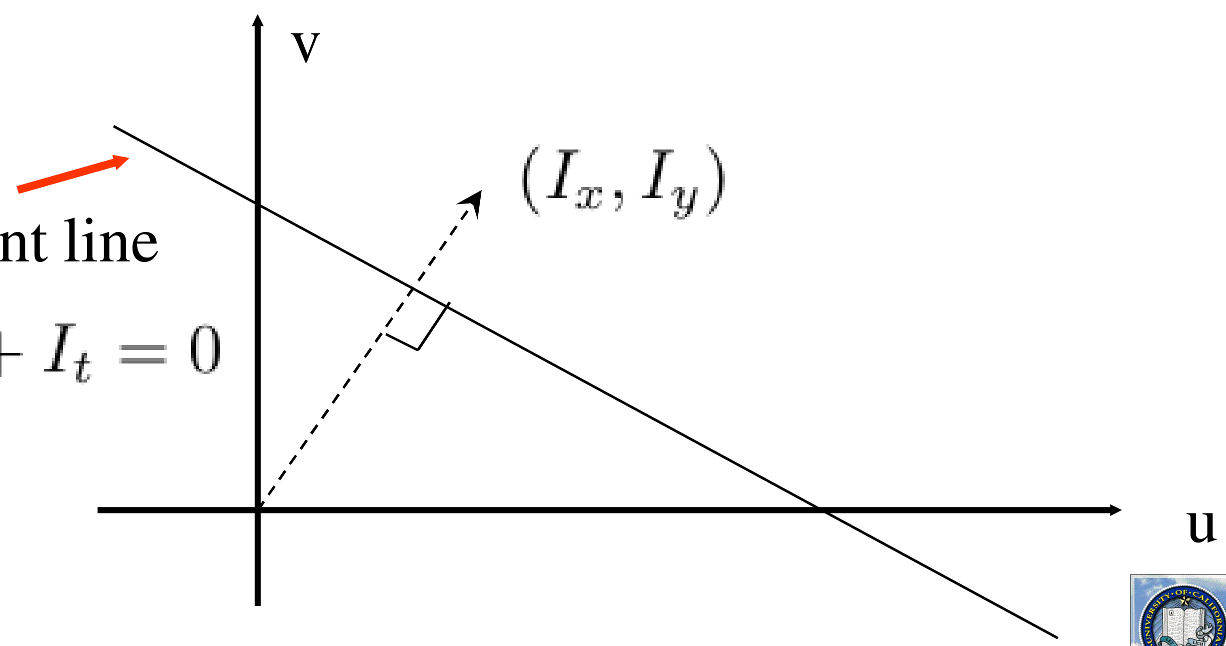


$$I(x(t_1),y(t_1),t_1) = I(x(t_2),y(t_2),t_2)$$

$$I(x(t_2),y(t_2),t_2) = I(x(t_3),y(t_3),t_3)$$

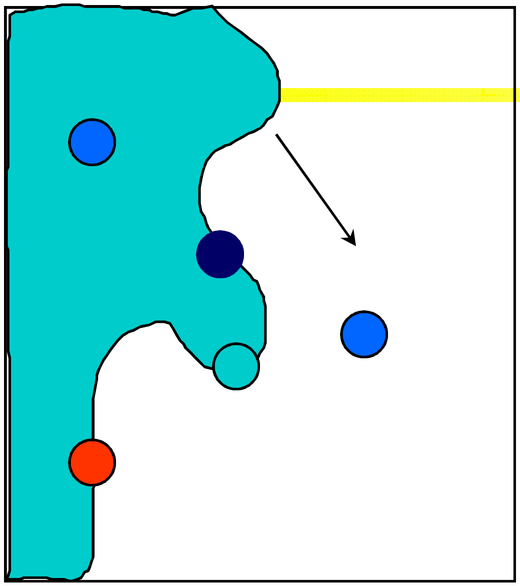
Difficulty

- One equation with two unknowns
- Aperture problem
  - spatial derivatives use only a few adjacent pixels (limited aperture and visibility)
  - many combinations of (u,v) will satisfy the equation



Constraint line

$$I_x u + I_y v + I_t = 0$$



- intensity gradient is zero
  no constraints on (u,v) $\quad (0,0)\cdot(u,v)=0$
  interpolated from other places

- intensity gradient is nonzero
  but is *constant*
  one constraints on (u,v)
  only the component along the gradient
  are recoverable

$$\left(\frac{\partial I}{\partial x},\frac{\partial I}{\partial y}\right)\cdot(u,v)=-\frac{\partial I}{\partial t}$$

- intensity gradient is nonzero
  and *changing*
  multiple constraints on (u,v)
  motion recoverable

$$\left(\frac{\partial I}{\partial x_1},\frac{\partial I}{\partial y_1}\right)\cdot(u,v)=-\frac{\partial I}{\partial t}_{(x_1,y_1)}$$
$$\left(\frac{\partial I}{\partial x_2},\frac{\partial I}{\partial y_2}\right)\cdot(u,v)=-\frac{\partial I}{\partial t}_{(x_2,y_2)}$$

Patch Translation [Lucas-Kanade]

Assume a single velocity for all pixels within an image patch

$$E(u,v) = \sum_{x,y\in\Omega} \left(I_x(x,y)u + I_y(x,y)v + I_t \right)^2$$

Minimizing

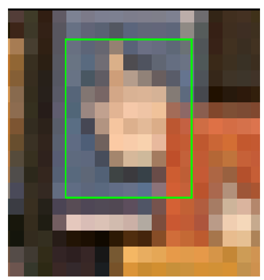
$$\begin{bmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_x I_y & \sum I_y^2 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} \sum I_x I_t \\ \sum I_y I_t \end{pmatrix}$$

$$\left(\sum \nabla I \nabla I^T \right) \vec{U} = -\sum \nabla I I_t$$

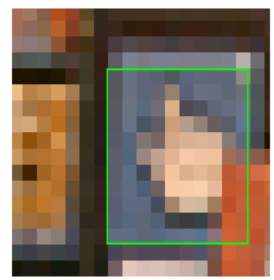
LHS: sum of the 2x2 outer product of the gradient vector

Image motion

How do we determine correspondences?



I



J

Assume all change between frames is due to **motion:**

$$J(x, y) \approx I(x + u(x, y), y + v(x, y))$$

The Aperture Problem

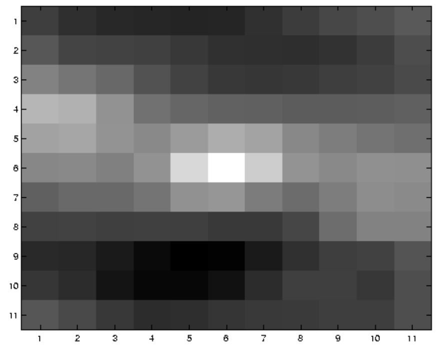
Let $M = \sum (\nabla I)(\nabla I)^T$ and $b = \begin{bmatrix} -\sum I_x I_t \\ -\sum I_y I_t \end{bmatrix}$

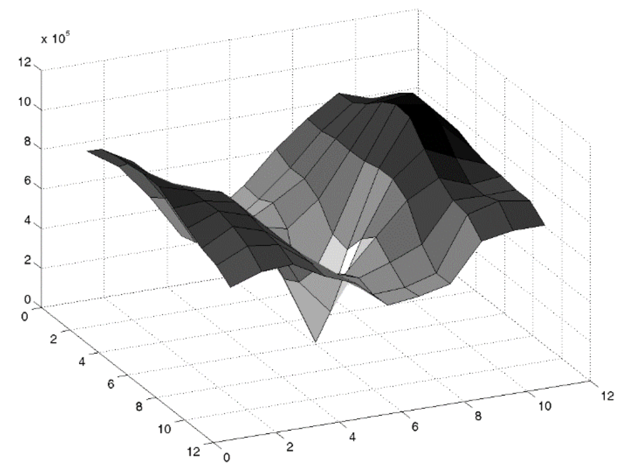
- Algorithm: At each pixel compute U by solving $MU = b$

- M is singular if all gradient vectors point in the same direction
  - e.g., along an edge
  - of course, trivially singular if the summation is over a single pixel or there is no texture
  - i.e., only *normal flow* is available (aperture problem)

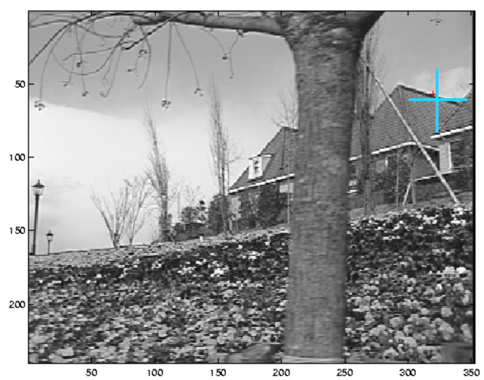
- Corners and textured areas are OK

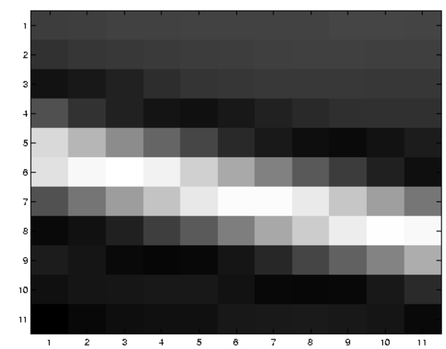
SSD Surface – Textured area

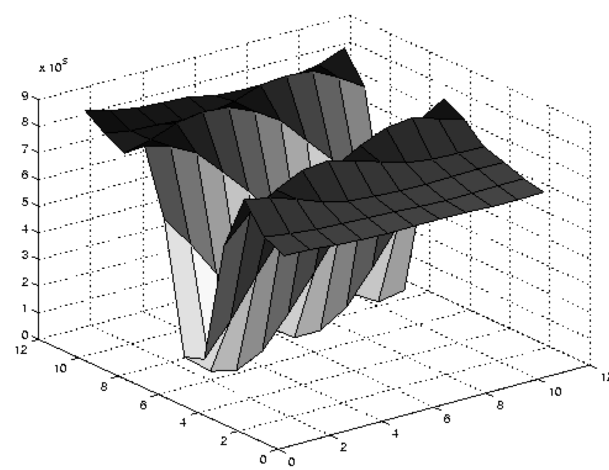




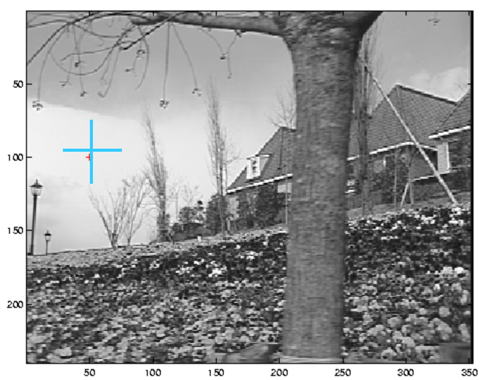
SSD Surface -- Edge

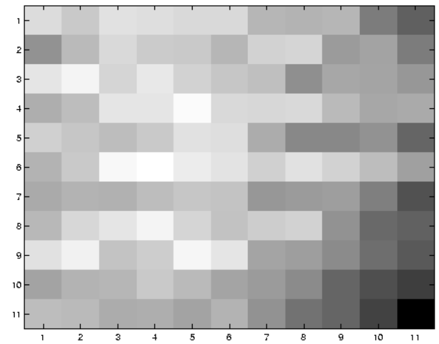


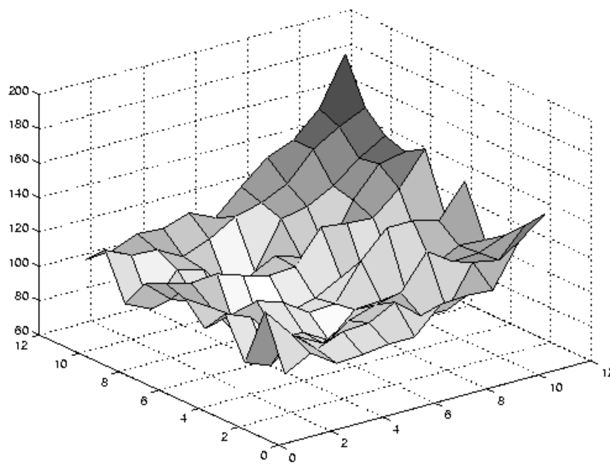




SSD – homogeneous area







Limits of the gradient method

Fails when intensity structure in window is poor

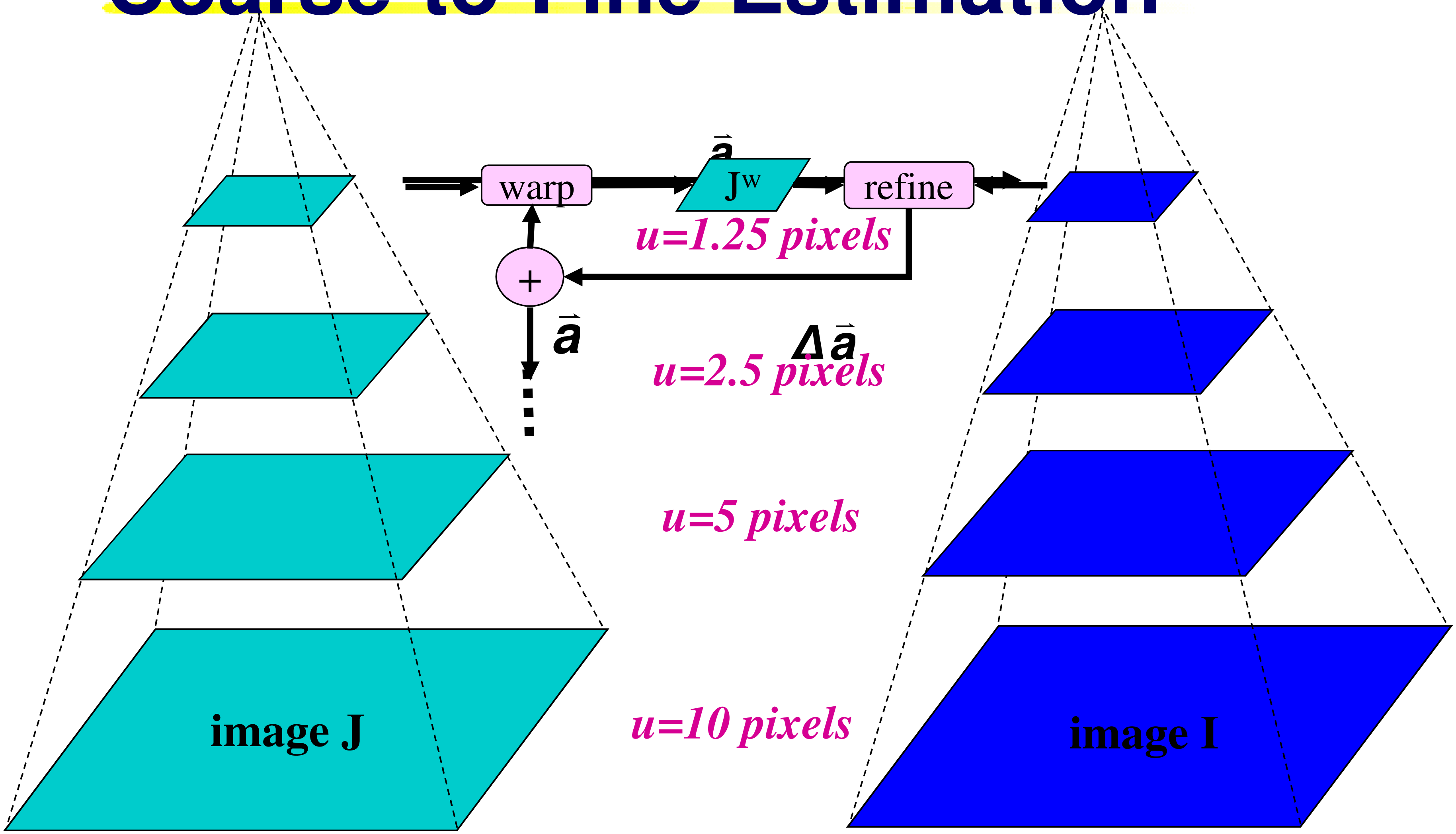
Fails when the displacement is large (typical operating range is motion of 1 pixel)

Linearization of brightness is suitable only for small displacements

- Also, brightness is not strictly constant in images

actually less problematic than it appears, since we can pre-filter images to make them look similar

Coarse-to-Fine Estimation



Pyramid of image J

Pyramid of image I

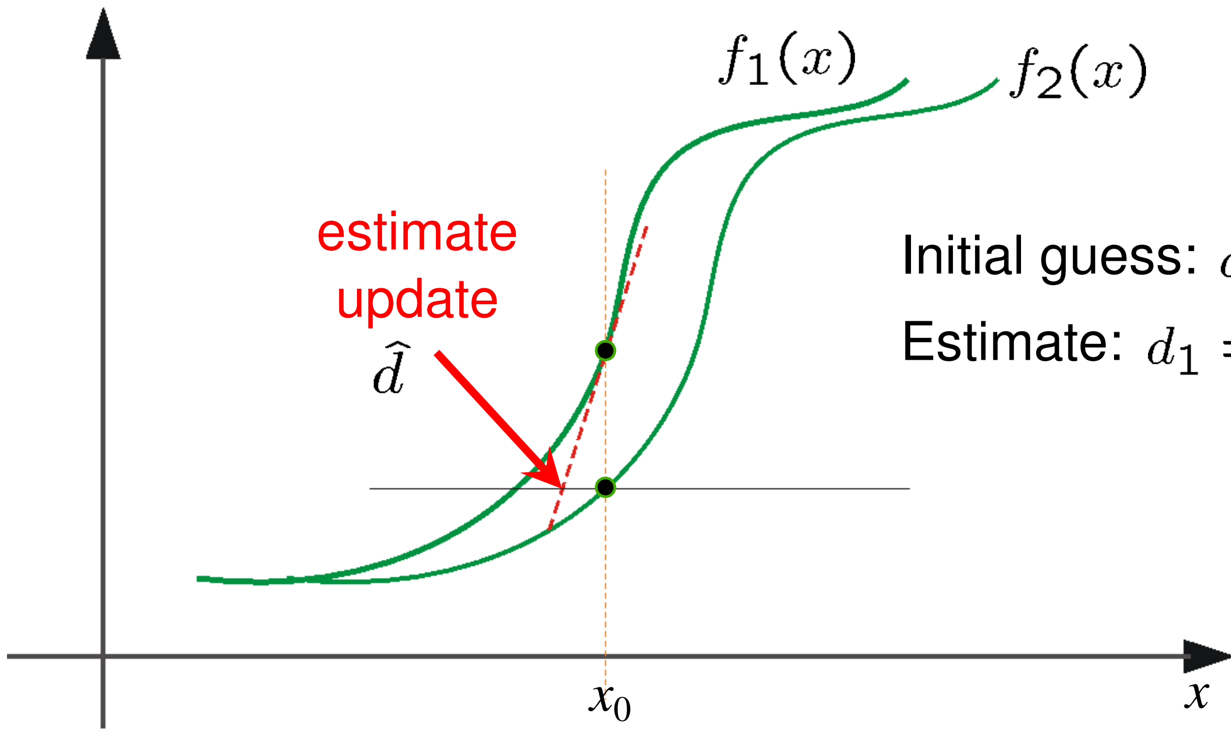
Szelisk

Iterative Refinement

- Estimate velocity at each pixel using one iteration of Lucas and Kanade estimation
- Warp one image toward the other using the estimated flow field

  (easier said than done)
- Refine estimate by repeating the process

Optical Flow: Iterative Estimation

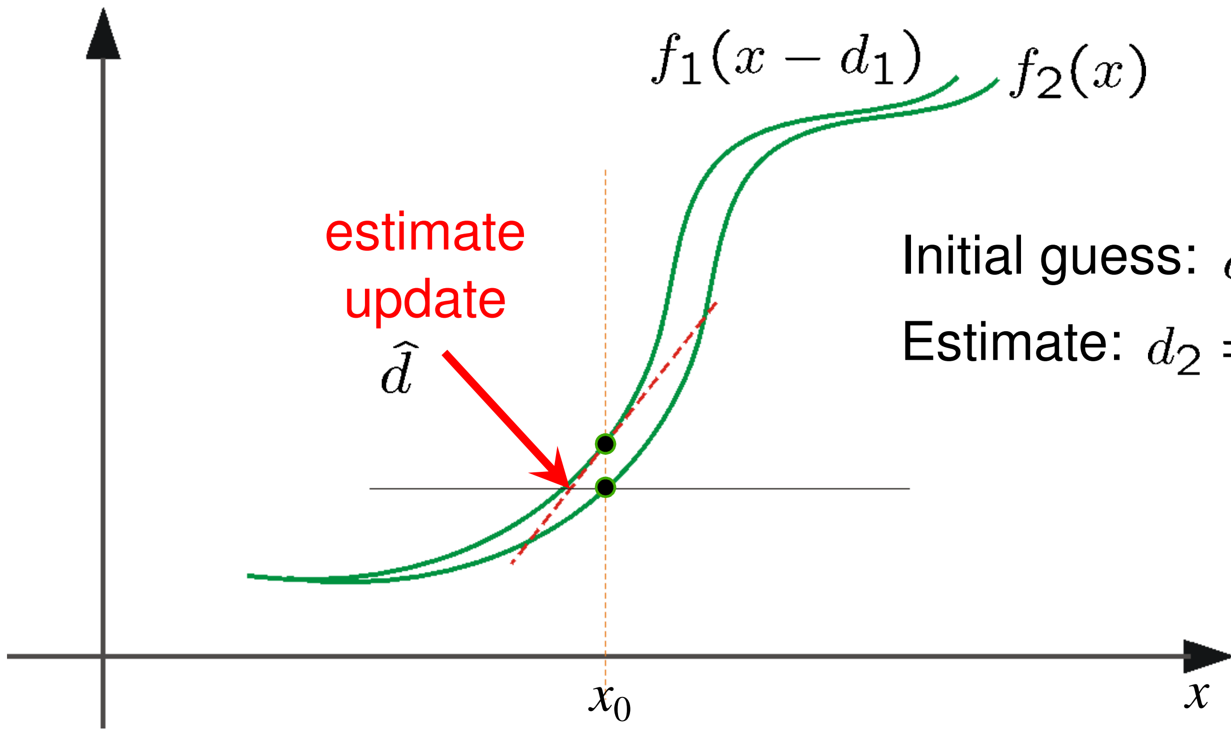


Initial guess: $d_0 = 0$

Estimate: $d_1 = d_0 + \hat{d}$

(using d for *displacement* here instead of u)

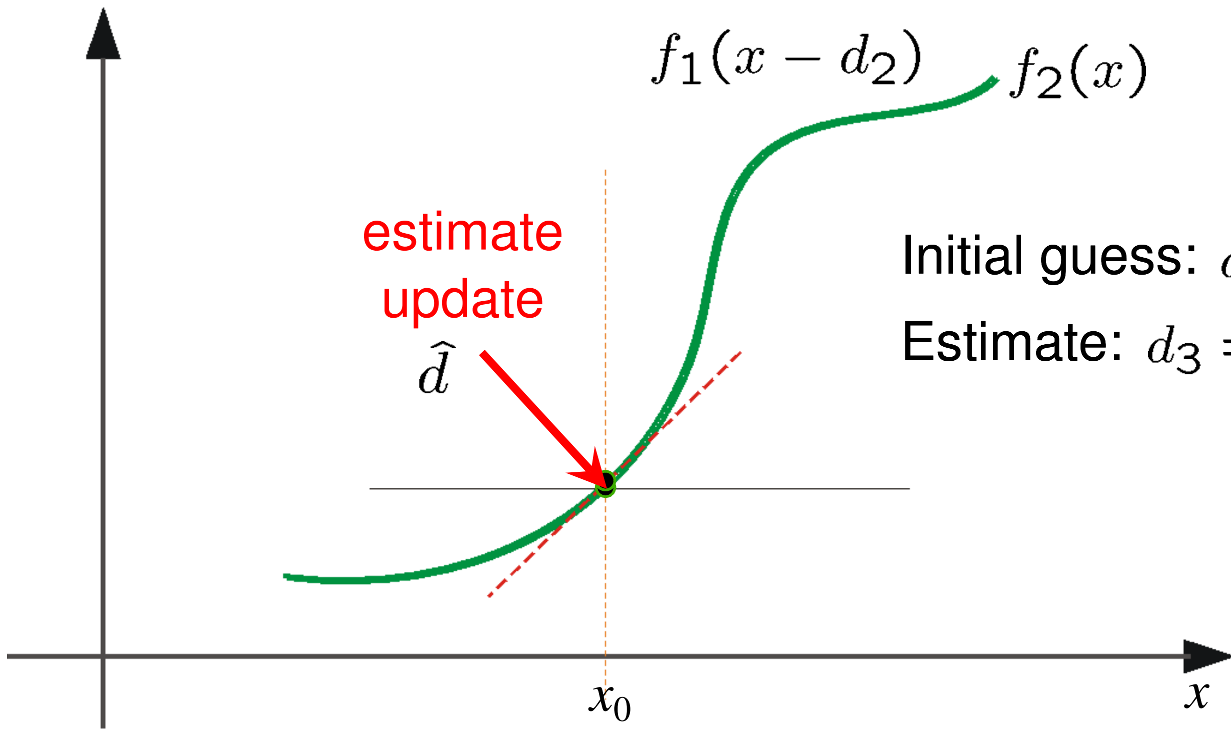
Optical Flow: Iterative Estimation



Initial guess: d_1

Estimate: $d_2 = d_1 + \hat{d}$

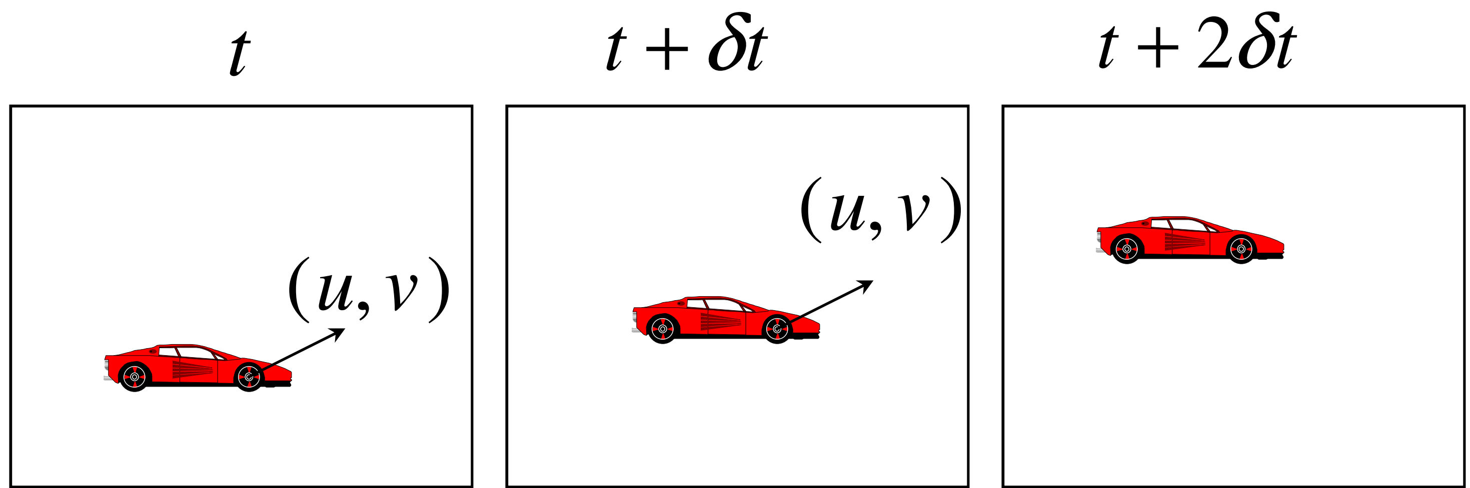
Optical Flow: Iterative Estimation



Initial guess: d_2

Estimate: $d_3 = d_2 + \hat{d}$

Temporal coherency



$$\left(\frac{\partial I}{\partial x},\frac{\partial I}{\partial y}\right)\cdot(u,v) = -\frac{\partial I}{\partial t} \qquad \left(\frac{\partial I}{\partial x},\frac{\partial I}{\partial y}\right)\cdot(u,v) = -\frac{\partial I}{\partial t}$$

- Caveat:
  - (*u,v*) must stay the same across several frames
  - scenes highly textured
  - *(u,v)* at the same location actually refers to different object points

Spatial coherency

- neighboring pixels should have “similar” flow vector
- Q: What do you mean by “similar”
- A1: identical
- A2: change slowly

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \cong 0$$

$$(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial v}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 \text{ small}$$

Mathematical formulation

- Based on Lagrange Multiplier
- Incorporate smoothness as an additional constraint
- Can be thought of as a weighting of two terms:
  - optical flow constraint
  - smoothness constraint

$$(\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y}) \cdot (u, v) = -\frac{\partial I}{\partial t}$$

$$(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial v}{\partial x})^2 + (\frac{\partial v}{\partial y})^2$$

- Optimize over all image plane:

$$E=\iint\left(\frac{\partial I}{\partial x}u+\frac{\partial I}{\partial y}v+\frac{\partial I}{\partial t}\right)^2+\lambda\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]dxdy$$

- Discretize the governing equation, at (*i*,*j*):

$$\frac{\partial u}{\partial x}=u_{i+1,j}-u_{i,j} \qquad \frac{\partial u}{\partial y}=u_{i,j+1}-u_{i,j}$$

$$\frac{\partial v}{\partial x}=v_{i+1,j}-v_{i,j} \qquad \frac{\partial v}{\partial y}=v_{i,j+1}-v_{i,j}$$

- Discretized expression:

$$E=\sum_i\sum_j\left(\left.\frac{\partial I}{\partial x}\right|_{i,j}u_{i,j}+\left.\frac{\partial I}{\partial y}\right|_{i,j}v_{i,j}+\left.\frac{\partial I}{\partial t}\right|_{i,j}\right)^2$$

$$+\lambda\left[(u_{i+1,j}-u_{i,j})^2+(u_{i,j+1}-u_{i,j})^2+(v_{i+1,j}-v_{i,j})^2+(v_{i,j+1}-v_{i,j})^2\right]$$

- At a pixel location (k,l):

$$\frac{\partial E}{\partial u_{k,l}} = 2\left(\frac{\partial I}{\partial x}_{k,l} u_{k,l} + \frac{\partial I}{\partial y}_{k,l} v_{k,l} + \frac{\partial I}{\partial t}_{k,l}\right)\frac{\partial I}{\partial x}_{k,l}$$
$$- 2\lambda[(u_{k-1,l} - u_{k,l}) + (u_{k,l-1} - u_{k,l}) + (u_{k+1,l} - u_{k,l}) + (u_{k,l+1} - u_{k,l})] = 0$$

$$\frac{\partial E}{\partial v_{k,l}} = 2\left(\frac{\partial I}{\partial x}_{k,l} u_{k,l} + \frac{\partial I}{\partial y}_{k,l} v_{k,l} + \frac{\partial I}{\partial t}_{k,l}\right)\frac{\partial I}{\partial y}_{k,l}$$
$$- 2\lambda[(v_{k-1,l} - v_{k,l}) + (v_{k,l-1} - v_{k,l}) + (v_{k+1,l} - v_{k,l}) + (v_{k,l+1} - v_{k,l})] = 0$$

$$\frac{\partial E}{\partial \lambda} = \cdots$$

- Putting it all together:

$$\left(\frac{\partial I}{\partial x}\bigg|_{k,l}\right)^2 u_{k,l} + \frac{\partial I}{\partial x}\bigg|_{k,l}\frac{\partial I}{\partial y}\bigg|_{k,l} v_{k,l} + \frac{\partial I}{\partial x}\bigg|_{k,l}\frac{\partial I}{\partial t}\bigg|_{k,l} - 4\lambda(\bar{u} - u_{k,l}) = 0$$

$$\frac{\partial I}{\partial x}\bigg|_{k,l}\frac{\partial I}{\partial y}\bigg|_{k,l} u_{k,l} + \left(\frac{\partial I}{\partial y}\bigg|_{k,l}\right)^2 v_{k,l} + \frac{\partial I}{\partial y}\bigg|_{k,l}\frac{\partial I}{\partial t}\bigg|_{k,l} - 4\lambda(\bar{v} - v_{k,l}) = 0$$

$$\bar{u} = (u_{k-1,l} + u_{k,l-1} + u_{k+1,l} + u_{k,l+1})/4$$

$$\bar{v} = (v_{k-1,l} + v_{k,l-1} + v_{k+1,l} + v_{k,l+1})/4$$

- Or:

$$[4\lambda + (\frac{\partial I}{\partial x}_{k,l})^2]u_{k,l} + \frac{\partial I}{\partial x}_{k,l}\frac{\partial I}{\partial y}_{k,l} v_{k,l} = 4\lambda\bar{u} - \frac{\partial I}{\partial x}_{k,l}\frac{\partial I}{\partial t}_{k,l}$$

$$\frac{\partial I}{\partial x}_{k,l}\frac{\partial I}{\partial y}_{k,l} u_{k,l} + [4\lambda + (\frac{\partial I}{\partial y}_{k,l})^2]v_{k,l} = 4\lambda\bar{v} - \frac{\partial I}{\partial y}_{k,l}\frac{\partial I}{\partial t}_{k,l}$$

$$u_{k,l} = \bar{u} - \frac{\frac{\partial I}{\partial x}_{k,l}\bar{u} + \frac{\partial I}{\partial y}_{k,l}\bar{v} + \frac{\partial I}{\partial t}_{k,l}}{4\lambda + (\frac{\partial I}{\partial x}_{k,l})^2 + (\frac{\partial I}{\partial y}_{k,l})^2}\frac{\partial I}{\partial x}_{k,l}$$

$$v_{k,l} = \bar{v} - \frac{\frac{\partial I}{\partial x}_{k,l}\bar{u} + \frac{\partial I}{\partial y}_{k,l}\bar{v} + \frac{\partial I}{\partial t}_{k,l}}{4\lambda + (\frac{\partial I}{\partial x}_{k,l})^2 + (\frac{\partial I}{\partial y}_{k,l})^2}\frac{\partial I}{\partial y}_{k,l}$$

$$u_{k,l} = \bar{u} - \frac{\left.\frac{\partial I}{\partial x}\right|_{k,l}\bar{u} + \left.\frac{\partial I}{\partial y}\right|_{k,l}\bar{v} + \left.\frac{\partial I}{\partial t}\right|_{k,l}}{4\lambda + \left(\left.\frac{\partial I}{\partial x}\right|_{k,l}\right)^2 + \left(\left.\frac{\partial I}{\partial y}\right|_{k,l}\right)^2} \left.\frac{\partial I}{\partial x}\right|_{k,l}$$

$$v_{k,l} = \bar{v} - \frac{\left.\frac{\partial I}{\partial x}\right|_{k,l}\bar{u} + \left.\frac{\partial I}{\partial y}\right|_{k,l}\bar{v} + \left.\frac{\partial I}{\partial t}\right|_{k,l}}{4\lambda + \left(\left.\frac{\partial I}{\partial x}\right|_{k,l}\right)^2 + \left(\left.\frac{\partial I}{\partial y}\right|_{k,l}\right)^2} \left.\frac{\partial I}{\partial y}\right|_{k,l}$$

- estimate based on smoothness
- how much does the smooth estimate violate optical flow constraint
- how much does the optical flow constraint matters
- direction for correction

Algorithms

1. Compute $\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y}, \frac{\partial I}{\partial t}$ from a pair of input images
2. Choose a weighting factor λ
3. Compute $(\bar{u}, \bar{v})$
4. At each pixel location (k,l), do

$$u_{k,l}^{(n+1)} = \bar{u}^{(n)} - \frac{\left.\frac{\partial I}{\partial x}\right|_{k,l} \bar{u}^{(n)} + \left.\frac{\partial I}{\partial y}\right|_{k,l} \bar{v}^{(n)} + \left.\frac{\partial I}{\partial t}\right|_{k,l}}{4\lambda + \left(\left.\frac{\partial I}{\partial x}\right|_{k,l}\right)^2 + \left(\left.\frac{\partial I}{\partial y}\right|_{k,l}\right)^2} \left.\frac{\partial I}{\partial x}\right|_{k,l}$$

$$v_{k,l}^{(n+1)} = \bar{v}^{(n)} - \frac{\left.\frac{\partial I}{\partial x}\right|_{k,l} \bar{u}^{(n)} + \left.\frac{\partial I}{\partial y}\right|_{k,l} \bar{v}^{(n)} + \left.\frac{\partial I}{\partial t}\right|_{k,l}}{4\lambda + \left(\left.\frac{\partial I}{\partial x}\right|_{k,l}\right)^2 + \left(\left.\frac{\partial I}{\partial y}\right|_{k,l}\right)^2} \left.\frac{\partial I}{\partial y}\right|_{k,l}$$

5. Iterate steps 3 and 4 until no change or count exceeds

Results

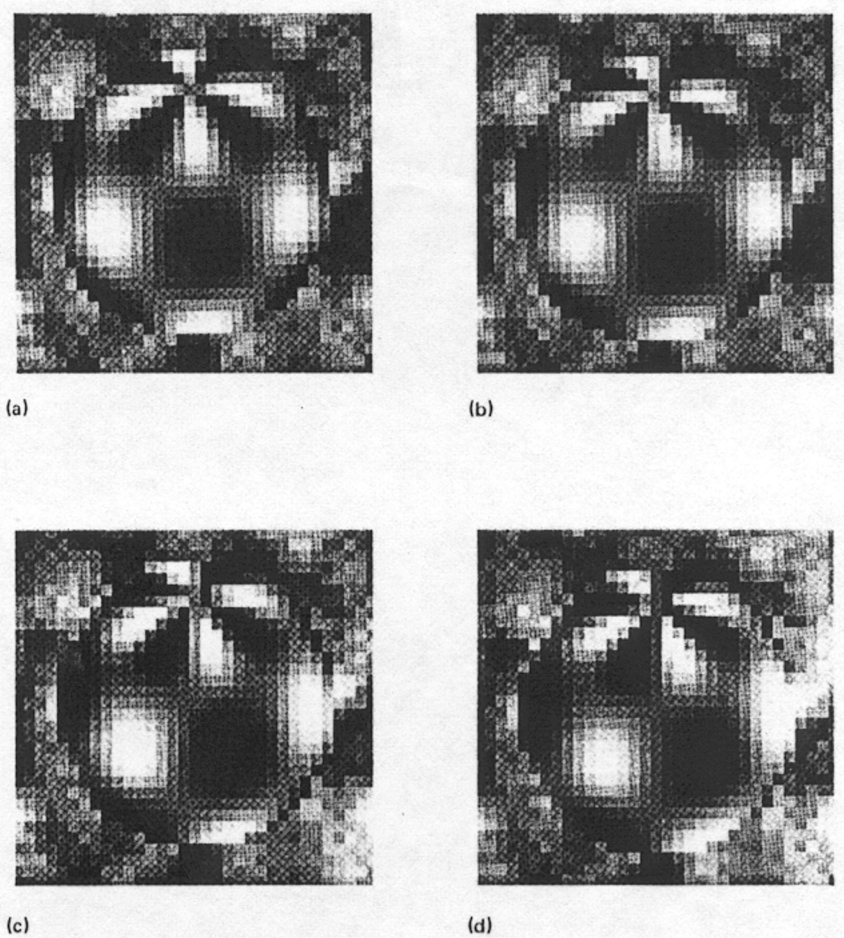
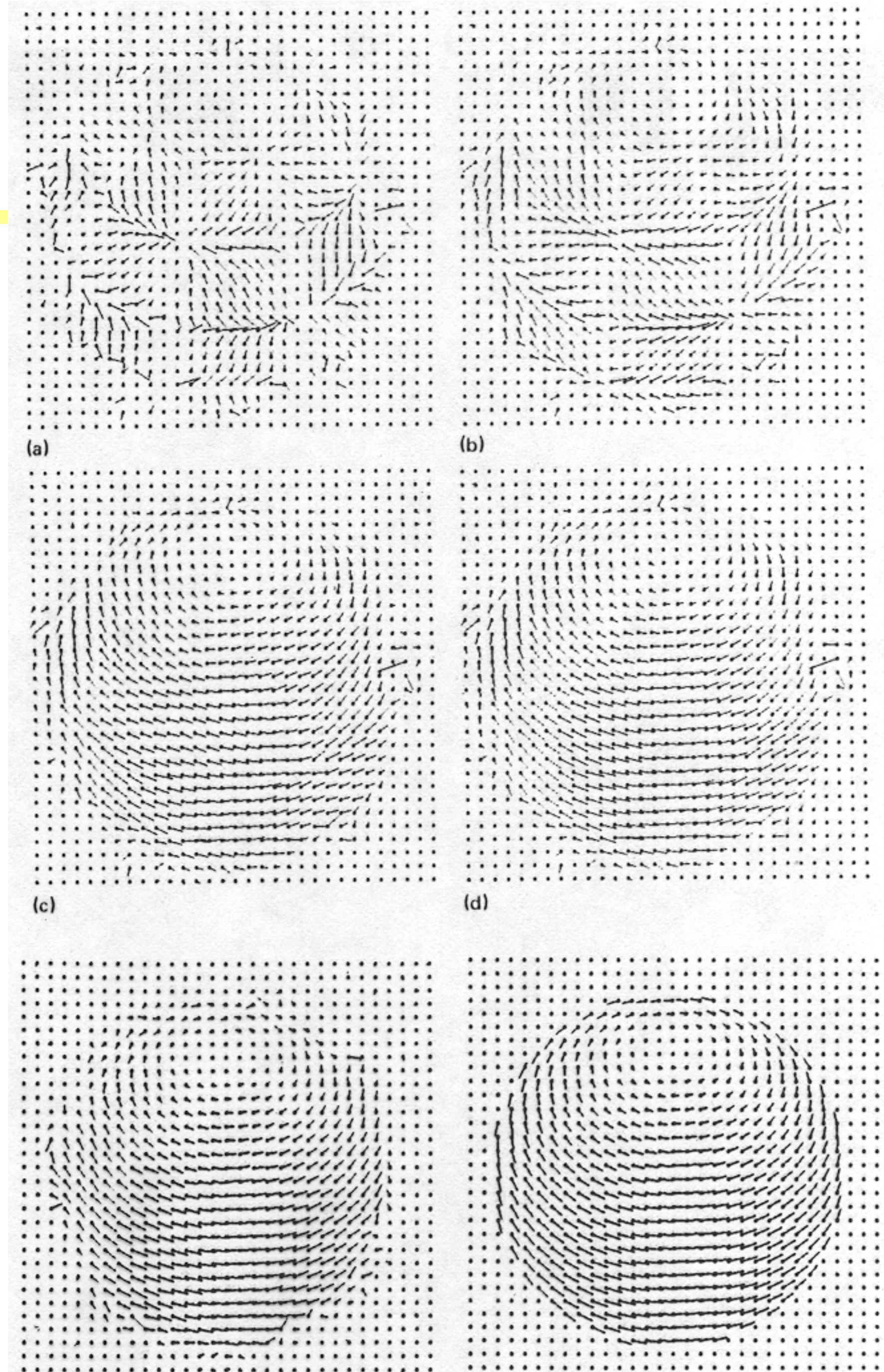


Figure 12-8. Four frames of a synthetic image sequence showing a sphere slowly rotating in front of a randomly patterned background.

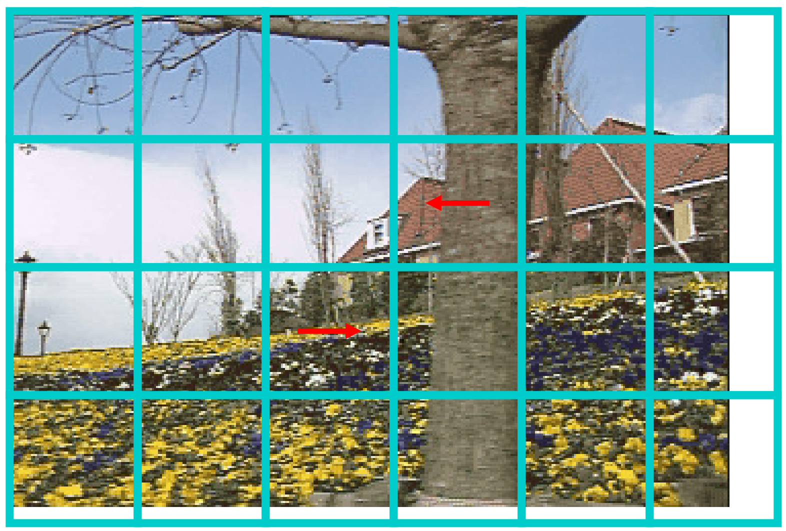


Motion representations

- How can we describe this scene?

Block-based motion prediction

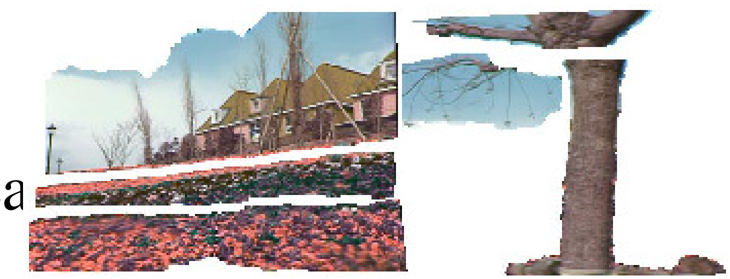
- Break image up into square blocks
- Estimate translation for each block
- Use this to predict next frame, code difference (MPEG-2)



Layered motion

- Break image sequence up into “layers”:

-  ·  =



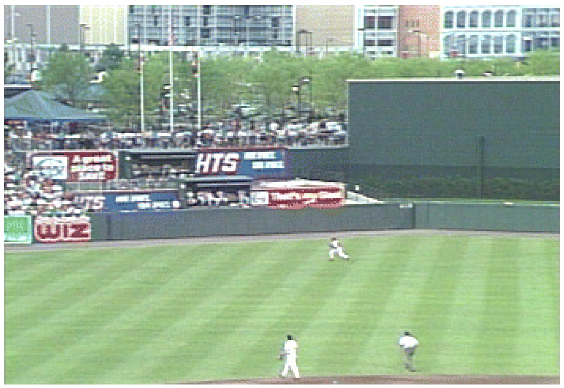
- Describe ea

Szelisk

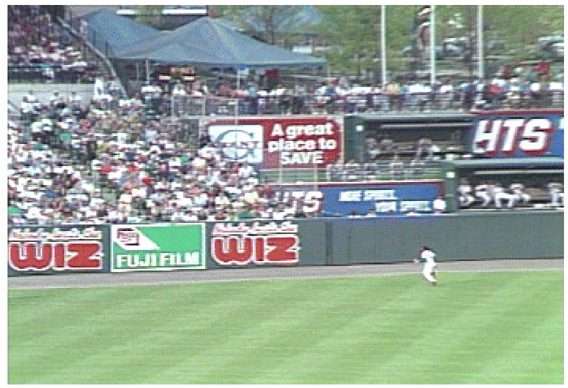
Layered motion

- Advantages:
  - can represent occlusions / disocclusions
  - each layer’s motion can be smooth
  - video segmentation for semantic processing
- Difficulties:
  - how do we determine the correct number?
  - how do we assign pixels?
  - how do we model the motion?

Layers for video summarization



Frame 0



Frame 50

Frame 80

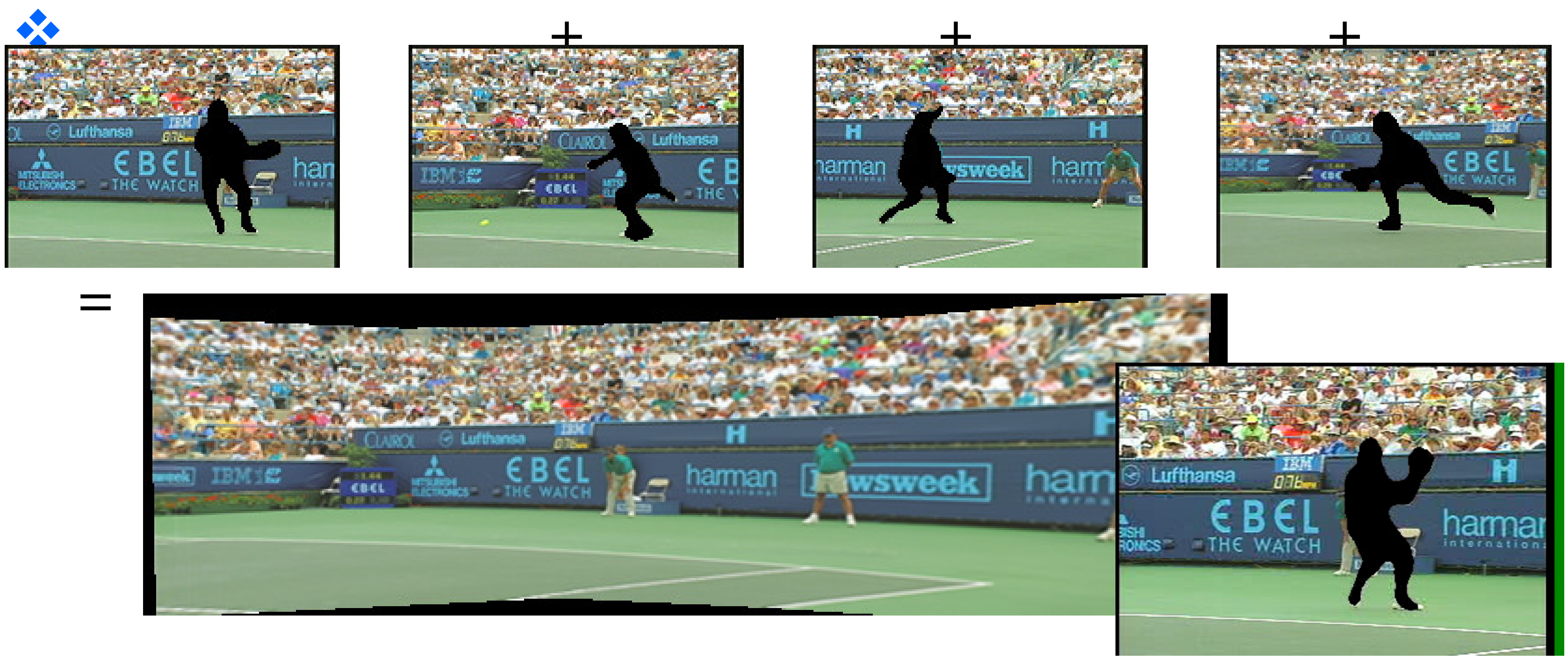
Background scene (players removed)

Complete synopsis of the video

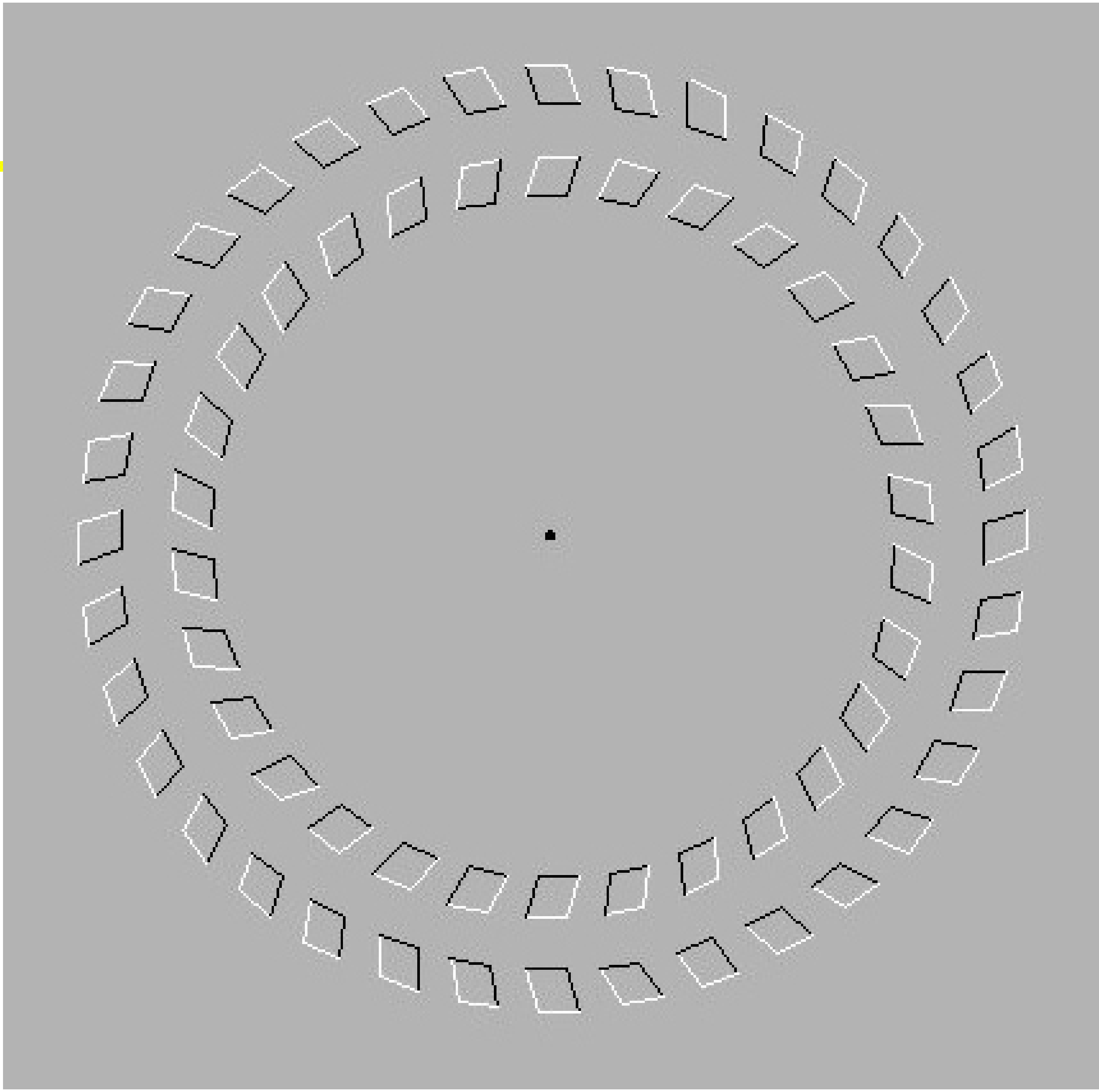
Szelisk

Background modeling (MPEG-4)

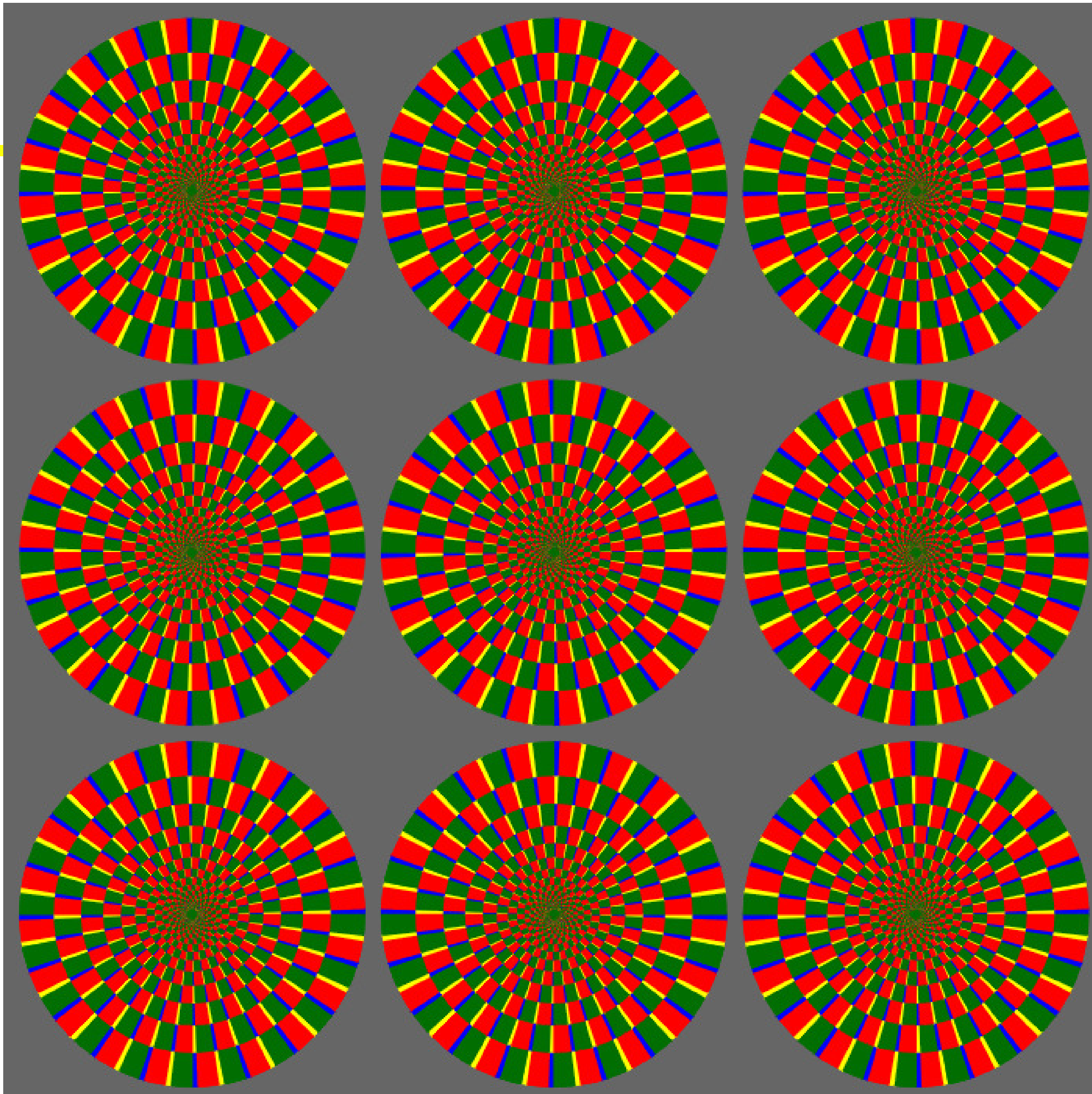
- Convert masked images into a background sprite for layered video coding

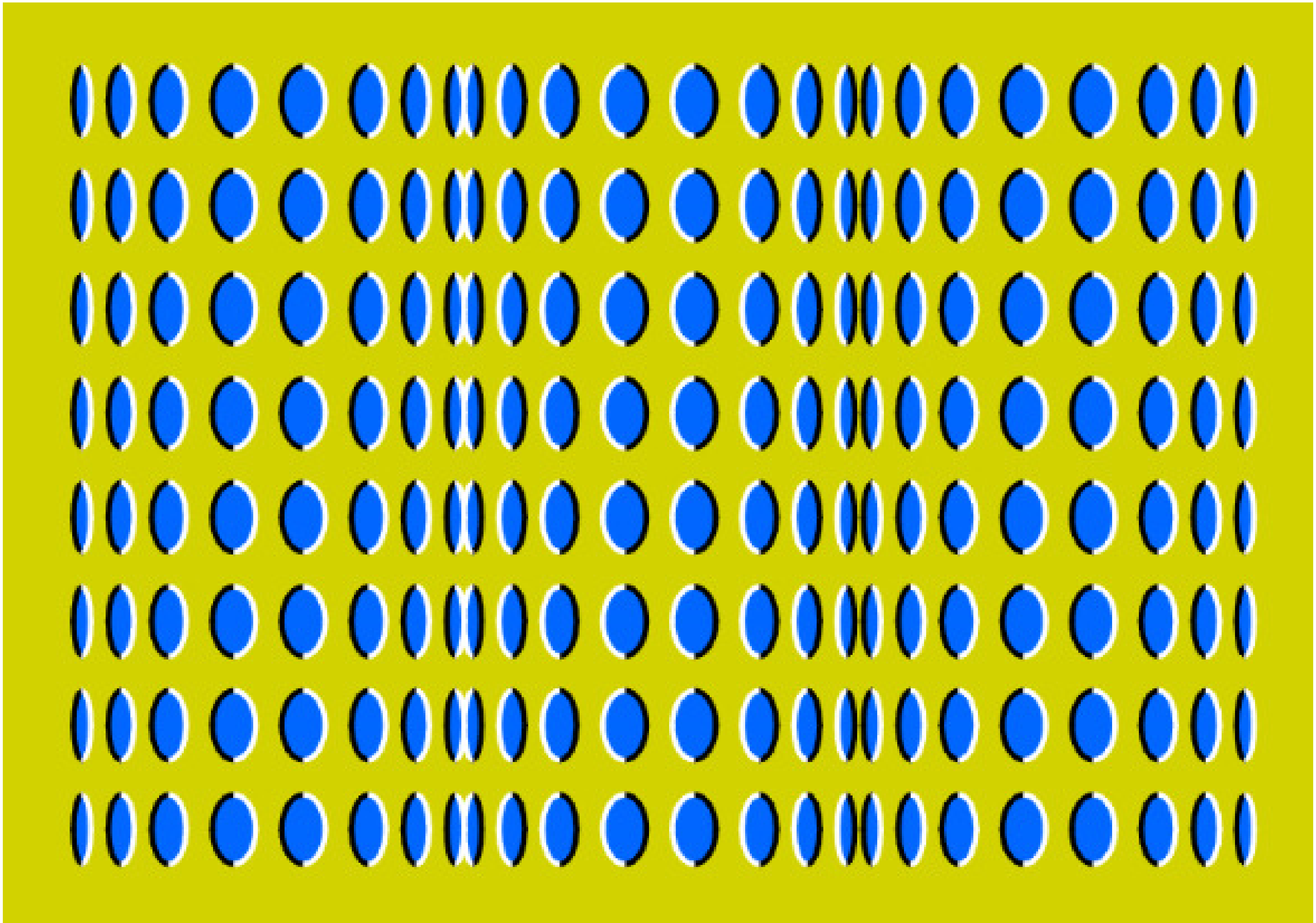


Szelisk

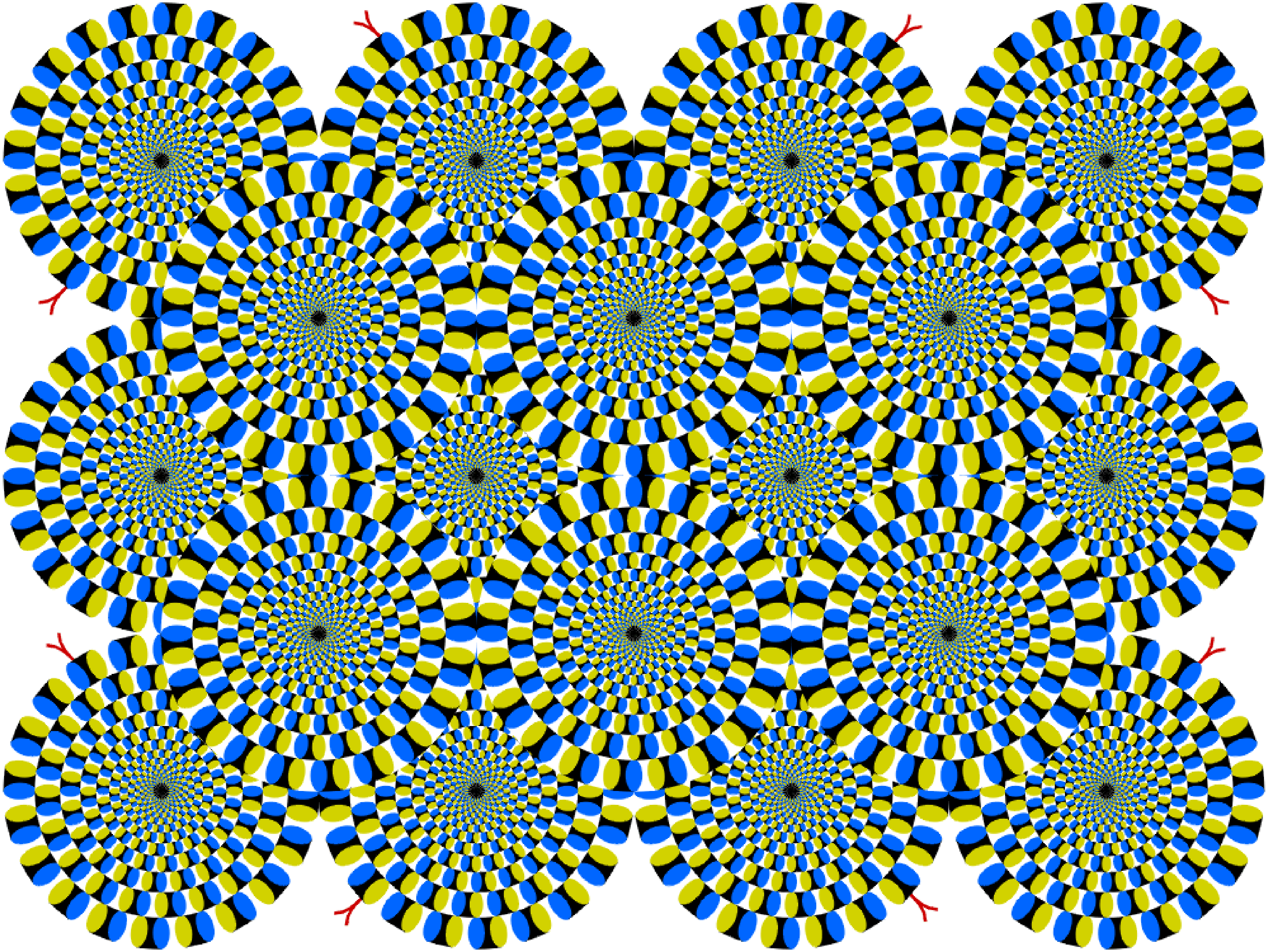


of California
Barbara





University of California
Santa Barbara



Optical flow summary

- Optical flow techniques:
  - Techniques that estimate the motion field from the image brightness constancy equation
- Optical flow:
  - Is best estimated (least noisy) at image points with high spatial image gradients. (Why?)
  - Works best for Lambertian surfaces (Why?)
  - Works best for very high frame rates (Why?)
- From optical flow, we can compute shape/structure/depth, motion parameters, segmentation, etc.
  - But if you primarily want to track an object, other methods may be preferred

Tracking

- Tracking is the process of updating an object's position (and orientation, and articulation?) over time through a video sequence
  - Estimate the object ***pose*** at each time point
    - "Pose" – position and orientation

- Applications
  - Surveillance
  - Targeting
  - Motion-based recognition (e.g., gesture recognition, computation of egomotion)
  - Motion analysis (golf swing, gait, character animation)
  - ……..

Tracking vs. optical flow

- In tracking, we are generally acknowledging that some sparse features are the points to track
  - Corners, lines, regions, patterns, contours....
- Rather than computing the full motion field from optical flow, let's keep track of the time-varying position of these sparse features
  - Then compute {egomotion, object pose, etc.} from this
- This typically involves a loop of prediction, measurement, and correction
  - Often with presumed models of motion dynamics and measurement noise

Tracking vs. Matching

- Tracking requires videos
- Small displacement is assumed
- Simple features
- Use image constraint (similar to optical flow constraint)

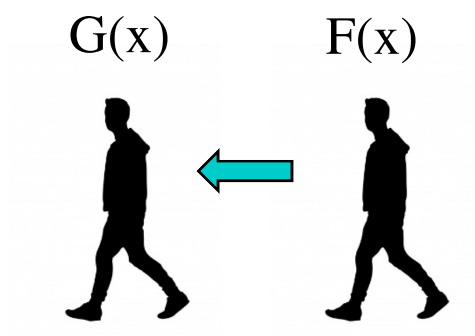
- Matching can be done on discrete frames
- Displacement can be large (>10 pixels)
- Often more elaborate features
- Independent detection in each frame and then match

Examples LKT tracker

$$F(x+h) \approx F(x) + hF'(x)$$

$$E = \sum_x [F(x+h) - G(x)]^2$$

$$\begin{aligned} 0 &= \frac{\partial E}{\partial h} \\ &\approx \frac{\partial}{\partial h} \sum_x [F(x) + hF'(x) - G(x)]^2 , \\ &= \sum_x 2F'(x)\,[F(x) + hF'(x) - G(x)] \\ \Rightarrow h &\approx \frac{\sum_x F'(x)[G(x) - F(x)]}{\sum_x F'(x)^2} \end{aligned}$$



KLT tracker

- An iterative update algorithm
- The estimate is more accurate if F is indeed linear
- Penalize pixels with large 2[nd] derivatives (w(x))

$$h \approx \frac{G(x) - F(x)}{F'(x)}$$

$$F''(x) \approx \frac{G'(x) - F'(x)}{h} \Rightarrow w(x) = \frac{1}{|G'(x) - F'(x)|}$$

$$\begin{cases} h_0 = 0 \\ h_{k+1} = h_k + \dfrac{\sum_x w(x) F'(x + h_k) \left[G(x) - F(x + h_k) \right]}{\sum_x w(x) F'(x + h_k)^2} \end{cases}$$