Background

Overview of
Projective Geometry

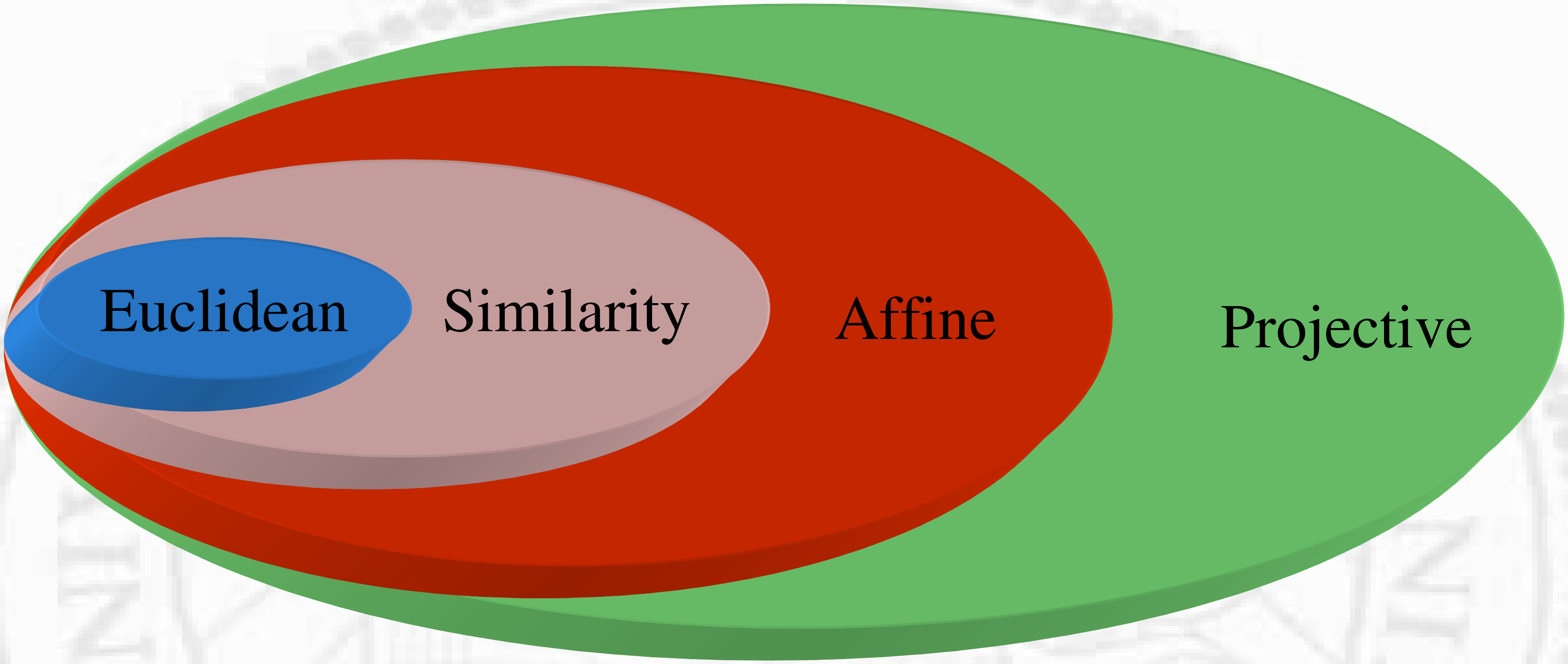
Topics Covered

- Background (Projective Geometry)
- Single View
- Double View
- Triple View
- N-view
- Robust parameter estimation

'Projective' Geometry

- Study relations between different classes of coordinate transformations
- Maintain Co-linearity (iff X'=HX)
- Not necessarily maintain
  - Angle and length (Euclidean)
  - Angle and relative length (similarity)
  - Parallelism, line and plane at infinity (affine)
  - Intersection and tangency (projective)

Class Diagram

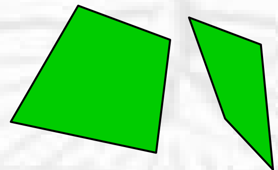


- Note that perspective camera projection is *not* in the hierarchy
- Projective geometry does not, theoretically, have to involve cameras (e.g. Euclidean)
- But many concepts are very useful in camera calibration and image analysis

Invariants

Projective
8dof

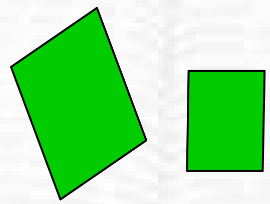
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio

Affine
6dof

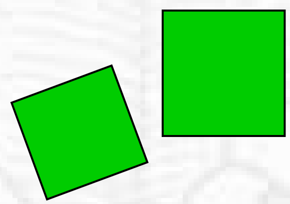
$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Parallelism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids).
The line at infinity l_∞

Similarity
4dof

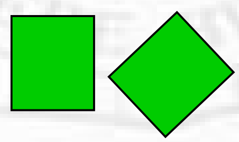
$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Ratios of lengths, angles.
The circular points I,J

Euclidean
3dof

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



lengths, areas.

Transformation Hierarchy

- Rectification
- What information (constraint) needed to be supplied to go up in the hierarchy
- Additional information limits the degrees of freedom
- Rectification is a mathematic process of limiting DOFs of H (camera or not)

	2D	3D
Projective	8	15
Affine	6	12
Similarity	4	7
Euclidean	3	6

Decomposition of projective transformations

$$\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix}$$

decomposition unique (if chosen s>0)

$$\mathbf{A} = s\mathbf{R}\mathbf{K} + \mathbf{t}\mathbf{v}^\mathsf{T}$$

$\mathbf{K}$ upper-triangular, $\det \mathbf{K} = 1$

Example:

$$\mathbf{H} = \begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 2\cos 45^\circ & -2\sin 45^\circ & 1.0 \\ 2\sin 45^\circ & 2\cos 45^\circ & 2.0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Decomposition of projective transformations

$$\mathbf{H}^{-1} = (\mathbf{H}_S\mathbf{H}_A\mathbf{H}_P)^{-1} = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{K} & 0 \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}^{-1} \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{v}^\mathsf{T} & \dfrac{1}{v} \end{bmatrix} \begin{bmatrix} \mathbf{K}^{-1} & 0 \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{s}\mathbf{R}^T & -\dfrac{1}{s}\mathbf{R}^T\mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$

$$\Rightarrow \mathbf{H} = \mathbf{H}_P\mathbf{H}_A\mathbf{H}_S$$

Decomposition of affine transformations

$$affine = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} = \mathbf{H}_S \mathbf{H}_A = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} = \begin{bmatrix} s\mathbf{RK} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

$$\mathbf{A} = s\mathbf{RK}$$

- Nonsingular matrix A has a QR decomposition
  - Q is orthogonal (rotation)
  - R is upper triangular

Important Relationship in 2D

- Point
  - $x^T l = 0$
  - $l = x_1 \times x_2$ ($l.x_1 = l.x_2 = 0$)
  - $x' = Hx$
- Line
  - $l^T x = 0$
  - $x = l_1 \times l_2$ ($x.l_1 = x.l_2 = 0$)
  - $l' = H^{-T} l$ ($l^T H^{-1} H x = 0$)

- Point conic
  - $x^T C x = 0$
  - $(H^{-1} x')^T C (H^{-1} x') = x'^T H^{-T} C H^{-1} x' = 0$
  - $C' = H^{-T} C H^{-1}$
- Line conic
  - $l^T C^* l = 0$
  - $(H^T l')^T C^* (H^T l') = l'^T H C^* H^T l' = 0$
  - $C^{*\prime} = H C^* H^T$

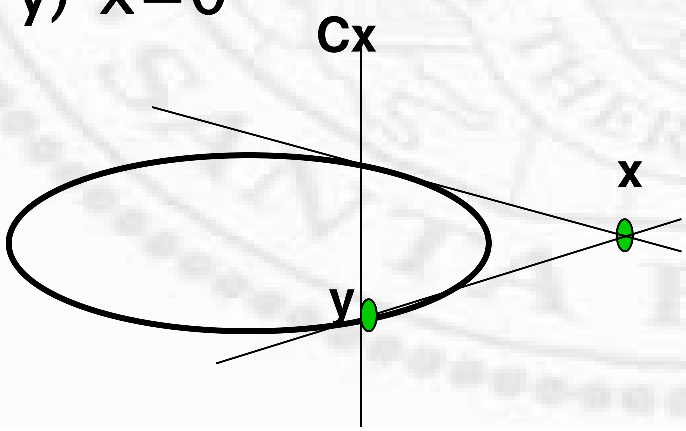
Important Relationship in 2D (cont.)

- Point at infinity
  - (rx,ry,1) r-> infinity
  - (x,y,0)
  - (x,y,0).(0,0,1) = 0
- Line at infinity
  - (a b).(x y) = r, r-> infinity
  - (0,0,1)
  - (0,0,1).(x,y,0) =0
- Degenerate point conic
  - $C=lm^T+ml^T$, ll^T
  - Rank two, rank one
  - Null space x = l x m
- Degenerate line conic
  - $C^* = xy^T + yx^T, xx^T$
  - Rank two, rank one
  - Null space l = x x y

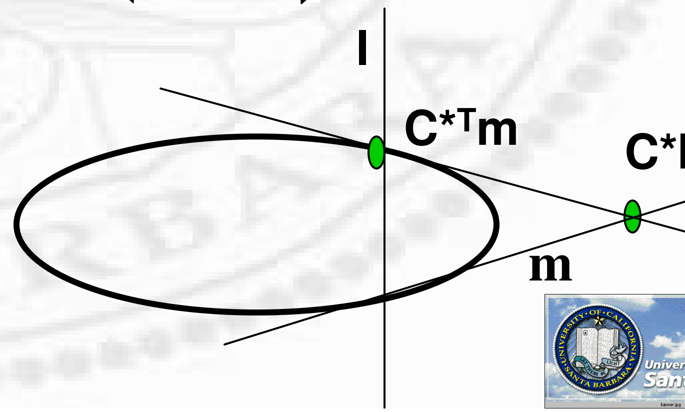
Absolute dual conic is degenerate line conics

Dual Relationship of Pole and Polar in 2D for Point and Line Conic

- Point conic $x^T C x = 0$
- Tangent $C^T x$
- Pole not on the conic x
- Polar line Cx
- Point on polar line $y^T C x = 0$
- Point also on conic
- Pole on tangent line $(C^T y)^T x = 0$



- Line conic $l^T C^* l = 0$
- Tangent point $C^{*T} l$
- Polar not tangent to conic l
- Pole $C^* l$
- Line passing through pole $m^T C^* l = 0$
- Line tangent to conic
- Polar passing through contact $(C^{*T} m)^T l = 0$



Measuring Line Direction

- Point conic
- At line of infinity
- Line of infinity encodes direction

- Line conic
- In plane

Pole-polar relation is projective invariant

Dual Relationship in 2D

- Circular points
- On line of infinity

- Dual conic
- Degenerate line conic
- $m^T C^* l = 0$, $C^* = IJ^T + JI^T$
- Measure *line* orientation

Important Relationship in 3D

- Point
  - $P^T X = 0$
  - $X' = HX$
- Plane
  - $X^T P = 0$
  - $P' = H^{-T} P$
- Point quadric
  - $X^T Q X = 0$
  - $(H^{-1} X')^T Q (H^{-1} X') = X'^T H^{-T} Q H^{-1} X' = 0$
  - $Q' = H^{-T} Q H^{-1}$
- Plane quadric
  - $P^T Q^* P = 0$
  - $(H^T P')^T Q^* (H^T P') = P'^T H Q^* H^T P' = 0$
  - $Q^{*\prime} = H Q^* H^T$

Dual Relationship of Pole and Polar in 3D for Point and Plane Quadric

- Point quadric $X^TQX=0$
- Tangent plane Q^TX
- Pole not on the quadric X
- Polar plane QX
- Point on polar plane $Y^TQX=0$
- Point also on quadric
- Pole on tangent plane $(Q^TY)^TX=0$

- Plane quadric $P^TQ^*P=0$
- Tangent point $Q^{*T}P$
- Polar plane not tangent to quadric P
- Pole Q^*P
- Plane passing through pole $S^TQ^*P=0$
- Plane tangent to quadric
- Polar passing through contact $(Q^{*T}S)^TP=0$

Measuring Direction in 3D

- Line Direction
- Point *conic* in plane of infinity
- Line conic in space
- Plane direction
- Plane *quadric* in space

A Hard to Visualize (but extremely important!) Concept

- A plane contains a line at infinity (0,0,1)
- Line at infinity contains two circular points (or a degenerate conic) $x_1^2 + x_2^2 = 0, x_3 = 0$
- 3D space contains a plane at infinity (0,0,0,1)
- Plane at infinity contains an absolute conic $x_1^2 + x_2^2 + x_3^2 = 0, x_4 = 0$

- Line at infinity lies on plane at infinity
- Two circular points lie on the absolute conic

Proof

- Line at infinity lies on plane at infinity
  - (or point at infinity lies on plane at infinity)

$$\begin{bmatrix} | & | & | & t_1 \\ \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & t_2 \\ | & | & | & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \\ 0 \end{bmatrix} = x \begin{bmatrix} | \\ \mathbf{r}_1 \\ | \\ 0 \end{bmatrix} + y \begin{bmatrix} | \\ \mathbf{r}_2 \\ | \\ 0 \end{bmatrix}$$

- Circular points lie on absolute conic

$$\begin{bmatrix} | & | & | & t_1 \\ \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & t_2 \\ | & | & | & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} | \\ \mathbf{r}_1 \\ | \\ 0 \end{bmatrix} + i \begin{bmatrix} | \\ \mathbf{r}_2 \\ | \\ 0 \end{bmatrix}$$

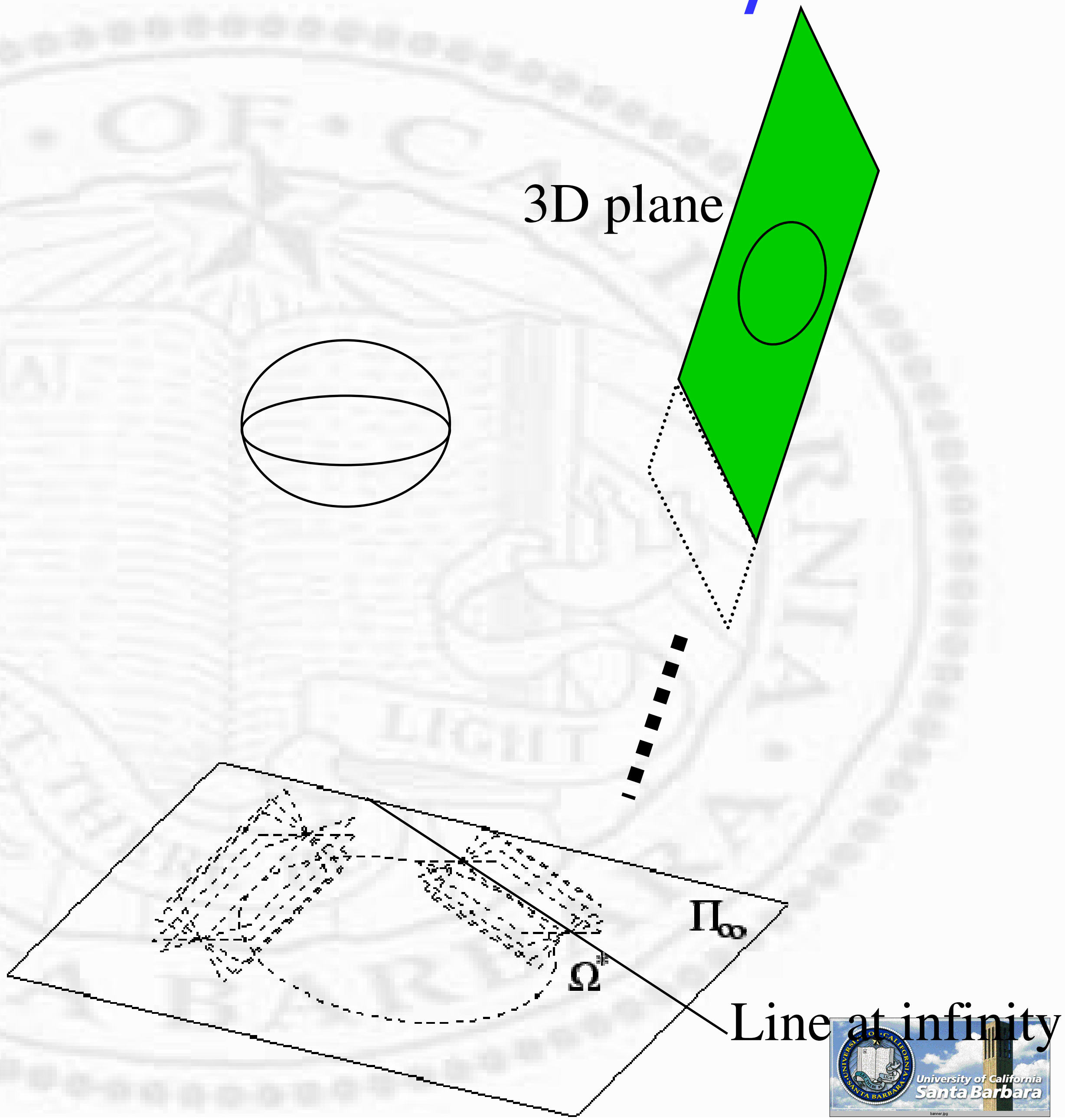
$$x_1^2 + x_2^2 + x_3^2 = (\mathbf{r}_1 + i\mathbf{r}_2) \cdot (\mathbf{r}_1 + i\mathbf{r}_2)$$

$$= |\mathbf{r}_1|^2 + 2i\mathbf{r}_1 \cdot \mathbf{r}_2 - |\mathbf{r}_2|^2 = 0$$

$$x_4 = 0$$

A Hard to Visualize Concept

- All circles intersect AC in two points (circular points)
- All spheres intersect plane at infinity in AC



Comparing 2D and 3D for Line

- Dual conic
- *Line* conic
- In line of infinity
- Pole-and-polar relation
- Line orientation

- Absolute conic
- *Point* conic
- In plane of infinity
- Pole-and-polar relation
- Line orientation

Plane of infinity encodes line direction

$$\boxed{\mathbf{l}}^T \mathbf{C}_\infty \boxed{\mathbf{l}} = 0$$

$$\mathbf{C}_\infty = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, w = 0$$

Dual Relationship in 3D

- Absolute conic (AC)
- On line of infinity
- Backproject into ADQ
- Measure *line* orientation

$$\boxed{\mathbf{l}}^T \mathbf{C}_\infty \boxed{\mathbf{l}} = 0$$

$$\mathbf{C}_\infty = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, w = 0$$

- Absolute dual quadric (ADQ)
- In space
- Project into ADC
- Measure *plane* orientation

$$\boxed{\mathbf{P}}^T \mathbf{C}^*_\infty \boxed{\mathbf{P}} = 0$$

$$\mathbf{C}^*_\infty = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix}$$

Comparing 2D and 3D

- 2 points
- In line of inf

- Line conic
- Line orientation

- Point conic
- In plane of infinity
- Line orientation
- Image of absolute conic in image

- Dual quadric
- In space
- Plane orientation
- Image of absolute dual conic in image

Rectification

- 2D
- H fixes line of infinity if and only if H is affine
- H fixes circular points if and only if H is similarity

- 3D
- H fixes plan of infinity if and only if H is affine
- H fixes AC and DAQ if and only if H is similarity

More Details (Repetition)

- Line in space
- Point in P.o.I
- Absolute conic in P.o.I

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\lim_{\lambda \to \infty} \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} \mathbf{d} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{d} \\ 0 \end{bmatrix}$$

$$\mathbf{d}^T \mathbf{C}^\infty \mathbf{d} = 0$$

- Vanishing point
- Iac

- Plane in space
- Line in P.o.I
- Absolute dual quadric in space

$$\pi = \begin{bmatrix} \mathbf{n} \\ n_4 \end{bmatrix} \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad \mathbf{d}' = \begin{bmatrix} d_1' \\ d_2' \\ d_3' \end{bmatrix}$$

$$\mathbf{l} = \mathbf{d} \times \mathbf{d}' = \mathbf{n}$$

$$\pi^T \mathbf{Q}^\infty \pi = 0$$

- Vanishing line
- iadc

Why We Even Care? (2D version)

- Because in 2D pole and polar relationship is projective invariant
  - For point conic $y^T C x = 0$
  - For line conic $m^T C^* l = 0$
  - $m'^T C^{*\prime} l' = (H^{-T} m)^T H C^* H^T (H^{-T} l) = m^T C^* l$
- This means that
  - known (orthogonal) directions in space
  - Measured them in images (no longer orthogonal, but still satisfy pole-polar relationship)
  - Constraint on H
  - In regular Cartesian frame
    - C^* is the absolute dual conic
  - In projective frame
    - $C^{*\prime}$ is the image of absolute dual conic (iadc)

Angles In 2D Case

- I, J are circular points
- Absolute dual conic is a degenerate conic (null space is line at infinity)

$$\cos\theta = \frac{d_{11}d_{21} + d_{12}d_{22}}{\sqrt{d_{11}d_{11} + d_{12}d_{12}}\sqrt{d_{21}d_{21} + d_{22}d_{22}}}$$

$$= \frac{\mathbf{d}_1^T \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \mathbf{d}_2}{\sqrt{\mathbf{d}_1^T \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \mathbf{d}_1}\sqrt{\mathbf{d}_2^T \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \mathbf{d}_2}} = \frac{\mathbf{d}_1^T C_\infty^* \mathbf{d}_2}{\sqrt{\mathbf{d}_1^T C_\infty^* \mathbf{d}_1}\sqrt{\mathbf{d}_2^T C_\infty^* \mathbf{d}_2}}$$

$$\mathbf{x}^T C_\infty^* \mathbf{x} = 0$$

$$x_1^2 + x_2^2 = 0, x_3 = 0$$

$$\mathbf{I} = (1, i, 0), \mathbf{J} = (1, -i, 0)$$

$$C_\infty^* = \mathbf{IJ}^T + \mathbf{JI}^T$$

Angles in 3D Case –Line

- Absolute (point) conic is a conic on the plane at infinity

$$\cos\theta = \frac{d_{11}d_{21}+d_{12}d_{22}+d_{13}d_{23}}{\sqrt{d_{11}d_{11}+d_{12}d_{12}+d_{13}d_{13}}\sqrt{d_{21}d_{21}+d_{22}d_{22}+d_{23}d_{23}}}$$

$$= \frac{\mathbf{l}_1^T \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \mathbf{l}_2}{\sqrt{\mathbf{l}_1^T \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \mathbf{l}_1}\sqrt{\mathbf{l}_2^T \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \mathbf{l}_2}} = \frac{\mathbf{l}_1^T \Omega_\infty \mathbf{l}_2}{\sqrt{\mathbf{l}_2^T \Omega_\infty \mathbf{l}_2}\sqrt{\mathbf{l}_2^T \Omega_\infty \mathbf{l}_2}}$$

$$\Rightarrow \mathbf{l}^T \Omega_\infty \mathbf{l} = 0$$

$$\Rightarrow x_1^2 + x_2^2 + x_3^2 = 0, x_4 = 0$$

Angles in 3D Case –Plane

- Absolute dual quadric is a degenerate quadric (null space is plane at infinity)

$$\cos\theta = \frac{d_{11}d_{21} + d_{12}d_{22} + d_{13}d_{23}}{\sqrt{d_{11}d_{11} + d_{12}d_{12} + d_{13}d_{13}}\sqrt{d_{21}d_{21} + d_{22}d_{22} + d_{23}d_{23}}}$$

$$= \frac{\boldsymbol{\pi}_1^T \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \boldsymbol{\pi}_2}{\sqrt{\boldsymbol{\pi}_1^T \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \boldsymbol{\pi}_1}\sqrt{\boldsymbol{\pi}_2^T \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \boldsymbol{\pi}_2}} = \frac{\boldsymbol{\pi}_1^T Q_\infty^* \boldsymbol{\pi}_2}{\sqrt{\boldsymbol{\pi}_2^T Q_\infty^* \boldsymbol{\pi}_2}\sqrt{\boldsymbol{\pi}_2^T Q_\infty^* \boldsymbol{\pi}_2}}$$

$$\Rightarrow \boldsymbol{\pi}^T Q_\infty^* \boldsymbol{\pi} = 0$$

$$\Rightarrow x_1^2 + x_2^2 + x_3^2 = 0, x_4 = 0$$

Why We Even Care? (3D version)

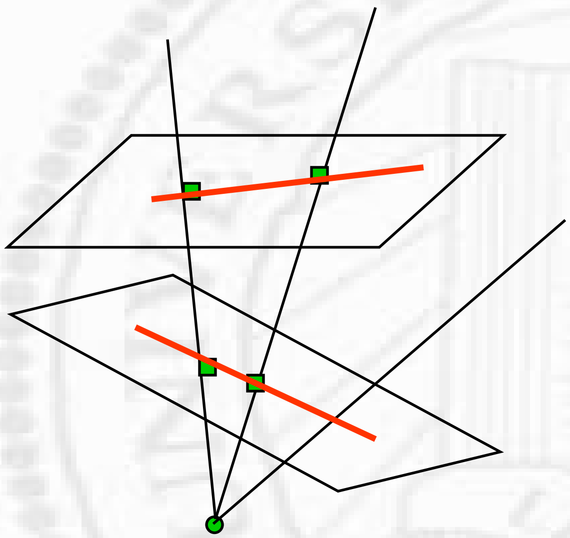
- Because in 3D pole and polar relationship is again projective invariant
  - For absolute conic $l^T \Omega m=0$ (for line)
  - For absolute dual quadric $S^T Q^* P=0$ (for plane)
  - $S'^T Q'^* P' = (H^{-T}S)^T H Q^* H^T (H^{-T}P) = S^T Q^* P$
- This means that
  - known (orthogonal) directions in space
  - Measured them in image of a3D scene
  - Constraint on H
  - In regular Cartesian frame
    - Ω , Q^* is the absolute conic and absolute dual quadric
  - In projective frame
    - ω, ω^* are iac and iadc

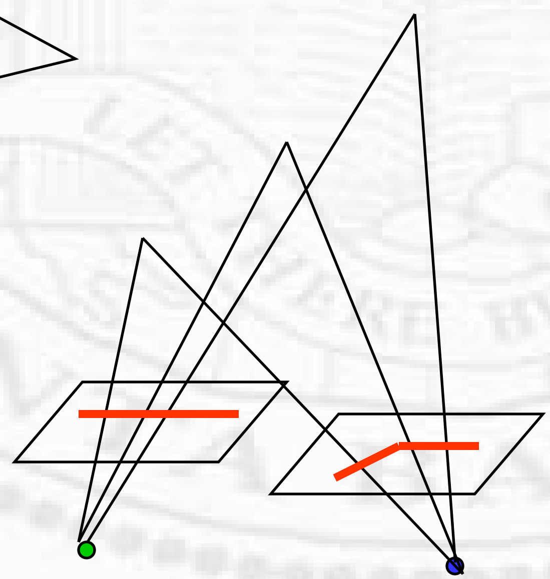
Rectification

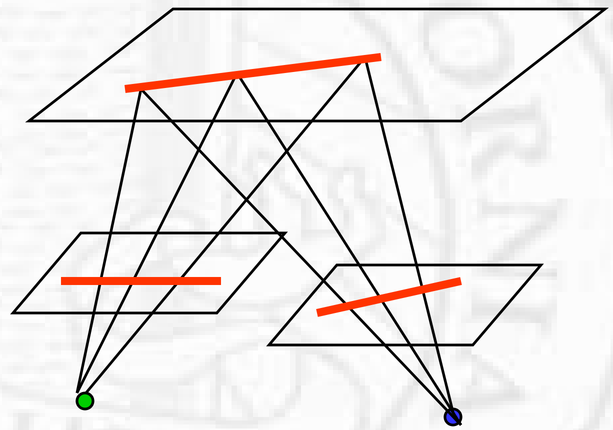
- Rectification is a mathematic process of limiting DOFs of H (camera or not)
- If H is to be decomposed further into camera matrix may not be considered (inner working of H not recovered)
- Cameras can be treated as a projective device
  - Affine camera
    - Induce affine transform
  - Perspective (pinhole) camera
    - Induce projective transform

Rectification Under Homography

- Same camera center
- Different camera centers, but planar structure







Removing projective distortion #1

Select four points in a plane with know coordinates

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

$$x'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{11}x + h_{12}y + h_{13}$$
$$y'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{21}x + h_{22}y + h_{23}$$

(linear in h_{ij})

(2 constraints/point, 8DOF $\Rightarrow$ 4 points needed)

Remark: No camera calibration necessary

Overkill, from projective to similarity are only 4 DOFs away

Removing Projective Distortion #2

- From projective to affine
  - Locate and move line of infinity
- From affine to similarity
  - Locate and move circular points

2D Rectification Hierarchy

- From perspective to affine
- H preserves line of infinity *if and only if* H is affine

$$(\Leftarrow)\mathbf{l}_\infty' = \mathbf{H}_A^{-T}\mathbf{l}_\infty = \begin{bmatrix} \mathbf{A}^{-T} & \mathbf{0} \\ -\mathbf{t}^{-T}\mathbf{A}^{-T} & 1 \end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} = \mathbf{l}_\infty$$

$$(\Rightarrow)\begin{bmatrix}x\\y\\0\end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} \rightarrow h_{31} = 0$$

$$\begin{bmatrix}x'\\y'\\0\end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} \rightarrow h_{32} = 0$$

- From affine to similarity
- H preserves absolute conic (or circular points) if and only if H is similarity

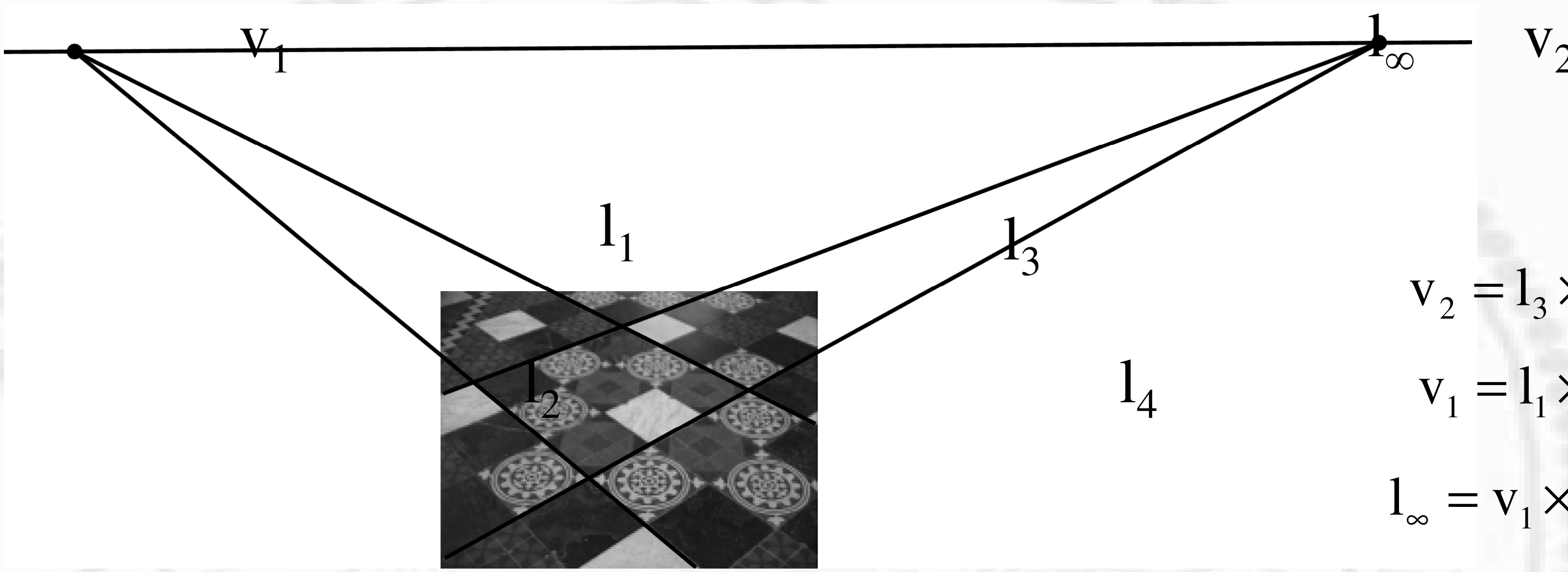
$$(\Leftarrow)\mathbf{C}^{*\prime} = \mathbf{H}_s\mathbf{C}^*\mathbf{H}_s^T = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}\begin{bmatrix}1 & & \\ & 1 & \\ & & 0\end{bmatrix}\begin{bmatrix} s\mathbf{R}^T & \mathbf{0} \\ \mathbf{t}^T & 1 \end{bmatrix}$$

$$= s^2\begin{bmatrix}1 & & \\ & 1 & \\ & & 0\end{bmatrix} = \mathbf{C}^*$$

$$(\Rightarrow)\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^T & v \end{bmatrix}\begin{bmatrix}1 & & \\ & 1 & \\ & & 0\end{bmatrix}\begin{bmatrix} \mathbf{A}^T & \mathbf{v} \\ \mathbf{t}^T & v \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathbf{A}^T & \mathbf{A}\mathbf{v} \\ \mathbf{v}^T\mathbf{A}^T & \mathbf{v}^T\mathbf{v} \end{bmatrix}$$

$$\rightarrow \mathbf{A}\mathbf{A}^T = 1, \mathbf{v} = \mathbf{0}$$

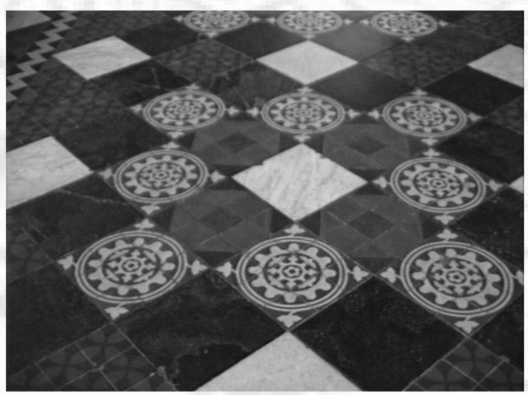
Affine rectification

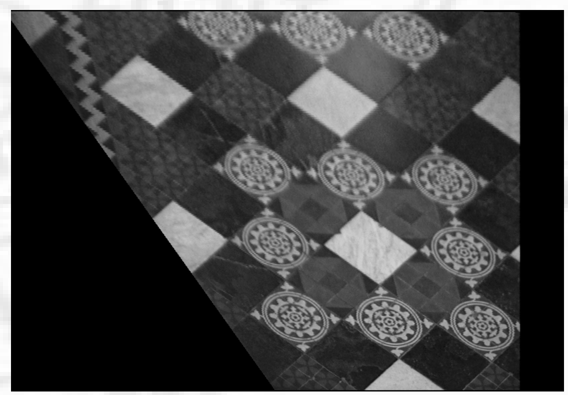


$$v_2 = l_3 \times l_4$$

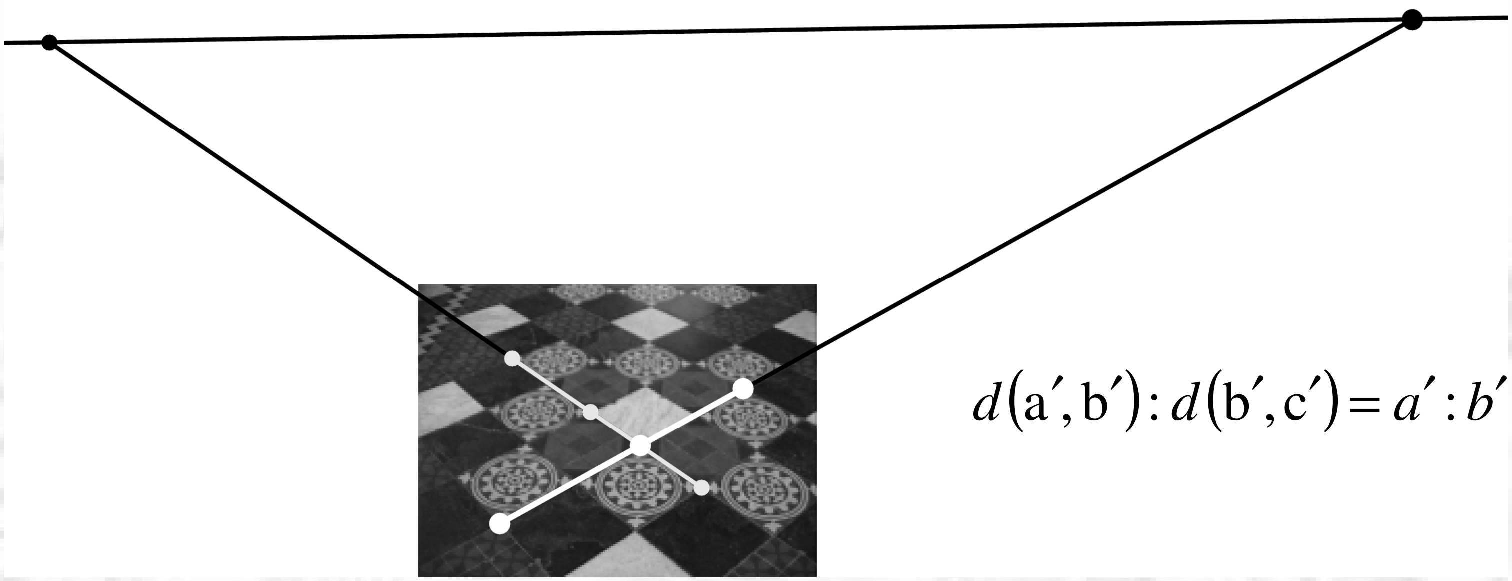
$$v_1 = l_1 \times l_2$$

$$l_\infty = v_1 \times v_2$$

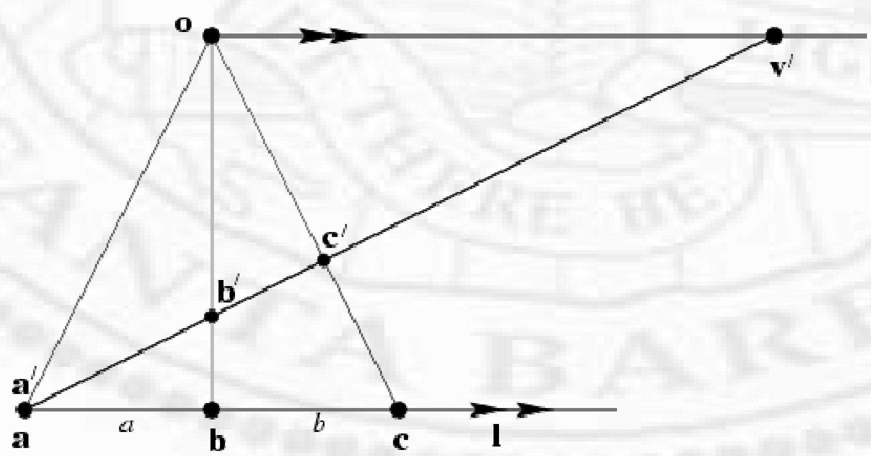




Distance ratios



$$d(\mathrm{a}',\mathrm{b}') : d(\mathrm{b}',\mathrm{c}') = a' : b'$$



$$(0,1)^\mathsf{T}, (a,1)^\mathsf{T}, (a+b,1)^\mathsf{T}$$

$$\downarrow \mathbf{H}$$

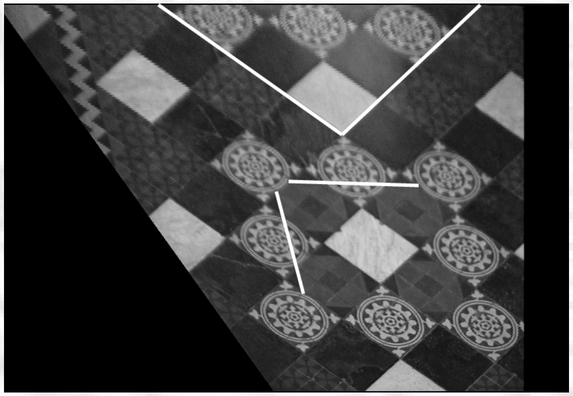
$$\mathrm{a}', \mathrm{b}', \mathrm{c}'$$

$$\mathrm{v}' = \mathbf{H}(1,0)^\mathsf{T}$$

Metric from affine

$$\begin{pmatrix} l'_1 & l'_2 & l'_3 \end{pmatrix} \begin{bmatrix} \mathbf{KK}^\mathsf{T} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} m'_1 \\ m'_2 \\ m'_3 \end{pmatrix} = 0$$

$$\left(l'_1 m'_1, l'_1 m'_2 + l'_2 m'_1, l'_2 m'_2\right)\left(k_{11}^2 + k_{12}^2, k_{11}k_{12}, k_{22}^2\right)^\mathsf{T} = 0$$



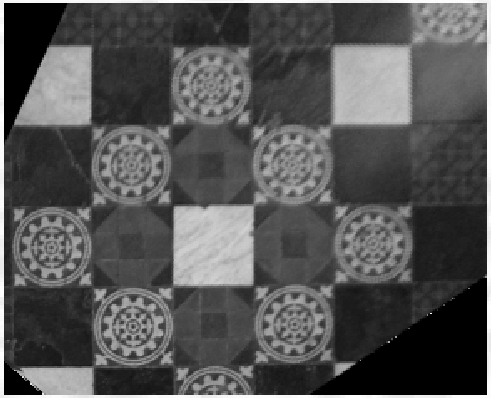


Image Formation

- Specialization: camera is used as the projective device
- Need to understand projection relationship of point, line, plane, conic, quadric, etc.
- Need to dissect the camera matrix
- Planar homography (discussed above) is a special case where 3D object is planar

Projective Relationship

- Point
- $x = PX$
- $X = P^+x$ ($P^+ = P^T(PP^T)^{-1}$, $PP^+ = I$)

$$\mathbf{P}\mathbf{P}^+ = \mathbf{I}$$

$$\Rightarrow \begin{bmatrix} \mathbf{K} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{K}^{-1} \\ \mathbf{0}^T \end{bmatrix} = \mathbf{I}$$

$$\mathbf{P}^+\mathbf{x} = \begin{bmatrix} \mathbf{K}^{-1} \\ \mathbf{0}^T \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{K}^{-1}\mathbf{x} \\ 0 \end{bmatrix}$$

$$\mathbf{d} = \mathbf{K}^{-1}\mathbf{x}$$

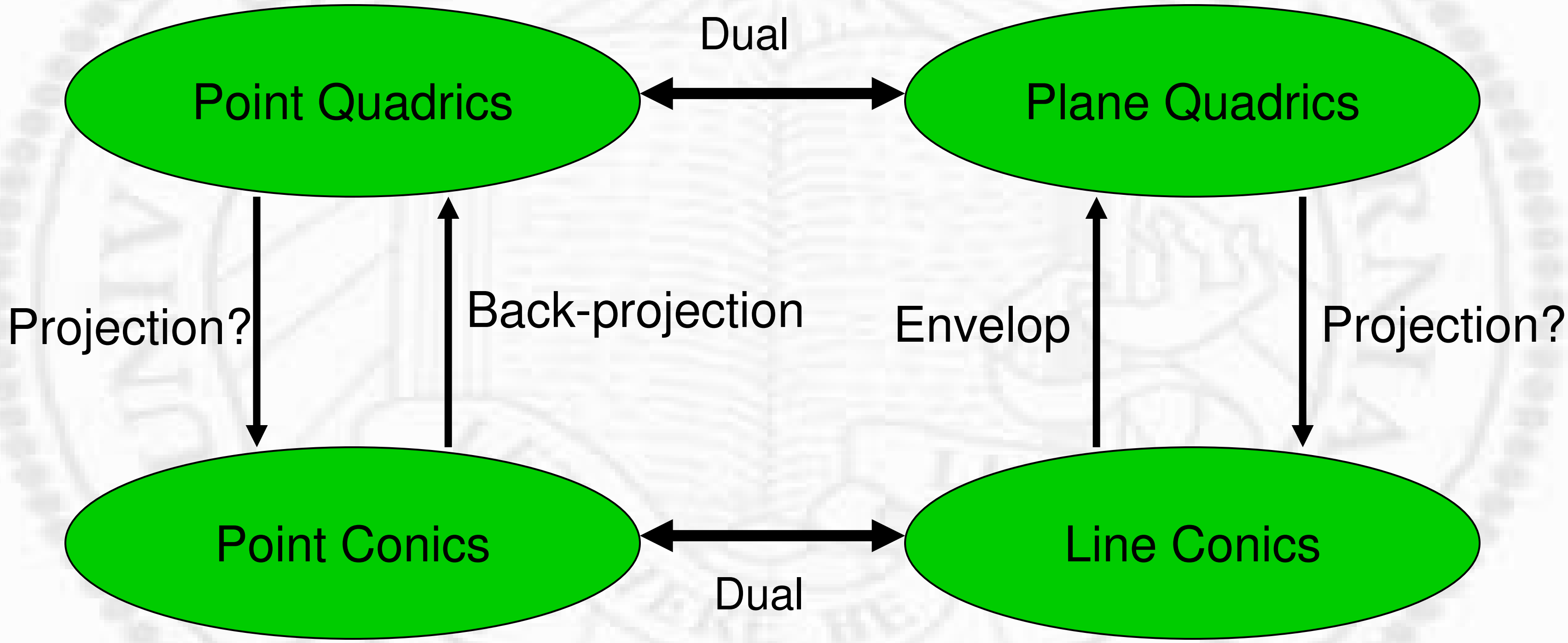
- Line
- Backprojection
- $\pi = P^T l$
  - ($x^T l = 0$, $(PX)^T l = 0$, $X^T (P^T l) = 0$)

$$\pi = \mathbf{P}^T \mathbf{l}$$

$$= \begin{bmatrix} \mathbf{K}^T \\ \mathbf{0}^T \end{bmatrix} \mathbf{l} = \begin{bmatrix} \mathbf{K}^T \mathbf{l} \\ 0 \end{bmatrix}$$

$$\mathbf{n} = \mathbf{K}^T \mathbf{l}$$

More Projective Relationships



Point Quadrics and Conics

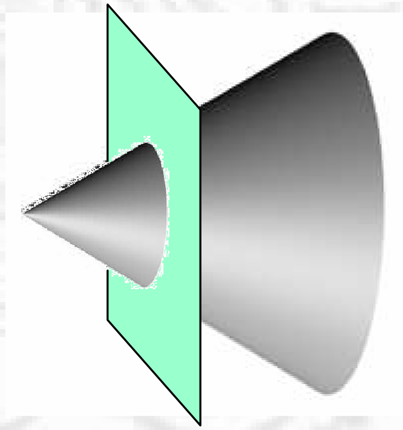
- Under the camera P a conic C back projects into a cone Q= $P^T CP$
  - Cone is a degenerate quadric
  - Camera center is the null space

$$\mathbf{x}^T \mathbf{C}\mathbf{x} = 0$$

$$\Rightarrow (\mathbf{PX})^T \mathbf{CPX} = 0$$

$$\Rightarrow \mathbf{X}^T (\mathbf{P}^T \mathbf{CP})\mathbf{X} = 0$$

$$\mathbf{P}^T \mathbf{CP}C = \mathbf{P}^T \mathbf{C}(\mathbf{P}C) = 0$$



Plane Quadrics and Line Conics

- Under the camera matrix P the outline of the quadric Q is the conic given by $C^*=PQ^*P^T$

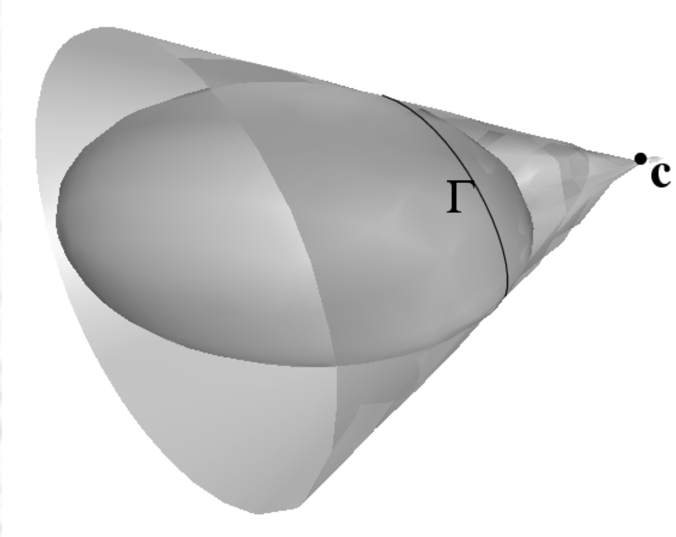
$$\mathbf{l}^T \mathbf{C}^* \mathbf{l} = 0$$

$$\Rightarrow \boldsymbol{\pi}^T \mathbf{Q}^* \boldsymbol{\pi} = 0 \qquad \boldsymbol{\pi} = \mathbf{P}^T \mathbf{l}$$

$$\Rightarrow (\mathbf{P}^T \mathbf{l})^T \mathbf{Q}^* \mathbf{P}^T \mathbf{l} = 0$$

$$\Rightarrow \mathbf{l}^T (\mathbf{P}\mathbf{Q}^* \mathbf{P}^T) \mathbf{l} = 0$$

$$\Rightarrow \mathbf{C}^* = \mathbf{P}\mathbf{Q}^* \mathbf{P}^T$$



Camera Calibration (cont.)

- Why? Because we are interested in similarity reconstruction
  - $x = PX$ implies $x = (PH^{-1})(HX)$
  - Even if we can perform reconstruction, the structure recovered is only up to HX (preserves co-linearity), which is not very useful
- How does calibration help?
  - $x = KR[I|0]X != KR[I|0]H^{-1}H X$
  - If we know K, you cannot fake R, except with another R’ (RH^{-1} is not R in general)
  - Reconstruction is similar to the original scene
    - Angle, relative length can be measured

Proof

- You can fool the camera (the same image and the same parameters) by
  - Using PH^{-1} as projection matrix (with the same K)
  - Using HX as the 3D scene
  - The matrix structure dictates similarity

$$\mathbf{x} = \mathbf{K}[\mathbf{R} \mid \mathbf{t}]\mathbf{X} = \mathbf{P}\mathbf{X}$$

$$\mathbf{H}^{-1} = \begin{bmatrix} \mathbf{R}' & \mathbf{t}' \\ \mathbf{0}^T & \lambda \end{bmatrix}$$

$$\mathbf{P}\mathbf{H}^{-1} = \mathbf{K}[\mathbf{R} \mid \mathbf{t}]\begin{bmatrix} \mathbf{R}' & \mathbf{t}' \\ \mathbf{0}^T & \lambda \end{bmatrix} = \mathbf{K}[\mathbf{R}\mathbf{R}' \mid \mathbf{R}\mathbf{t}' + \lambda\mathbf{t}]$$

Uncalibrated Camera

- The same trick won't work if you don't know K
- M has an RQ decomposition where
  - R is upper triangular
  - Q is orthogonal

$$\mathbf{x} = \mathbf{K}[\mathbf{R} \mid \mathbf{t}]\mathbf{X} = \mathbf{P}\mathbf{X}$$

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C}^T & \lambda \end{bmatrix}$$

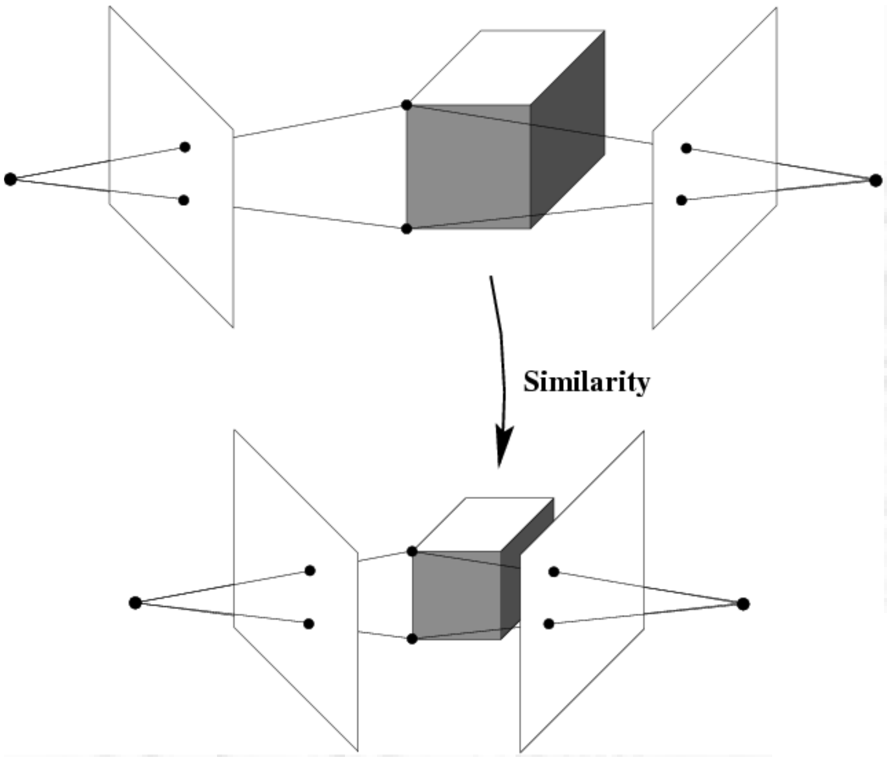
$$\mathbf{P}\mathbf{H}^{-1} = \mathbf{K}[\mathbf{R} \mid \mathbf{t}]\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C}^T & \lambda \end{bmatrix} = \mathbf{K}[\mathbf{R}\mathbf{A} + \mathbf{t}\mathbf{C}^T \mid \mathbf{R}\mathbf{B} + \lambda\mathbf{t}]$$

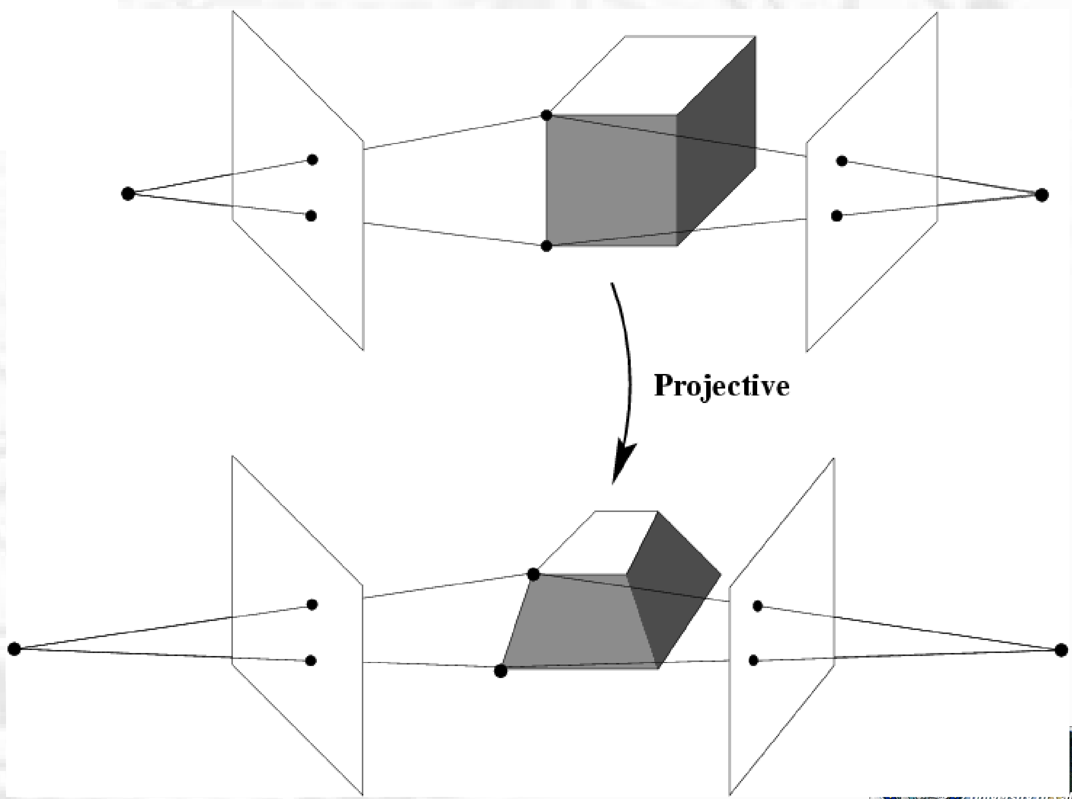
$$= \mathbf{K}'[\mathbf{R}' \mid \mathbf{t}']$$

$$\Rightarrow \mathbf{K}'\mathbf{R}' = \mathbf{K}\mathbf{R}\mathbf{A} + \mathbf{t}\mathbf{C}^T = \mathbf{M} = \mathbf{R}\mathbf{Q}$$

Camera Calibration (cont.)

- Camera calibration does not give
  - Absolute scale
  - Absolute location
  - You cannot tell if you are flying over New York City or flying over a model of New York City in Santa Barbara
  - The best you can hope for with a single camera with no object of known scale


Similarity


Projective

Camera Calibration

- There are two components
  - Intrinsic parameters: independent of placement, dependent on camera used
  - Extrinsic parameters: dependent on placement, independent of camera used
- Projective geometry constraint is on intrinsic parameters only
- Other constraints have to be brought in to determine placement

Dissecting Camera Matrix

- x=PX
- $$\mathbf{p}_{real} = \begin{bmatrix} k_u & \alpha & u_o \\ 0 & k_v & v_o \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{T} \\ 0 & 1 \end{bmatrix} \mathbf{P}_{world}$$

$$= \begin{bmatrix} k_u & \alpha & u_o \\ 0 & k_v & v_o \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{r}_1 & t_x \\ \mathbf{r}_2 & t_y \\ \mathbf{r}_3 & t_z \\ 0 & 1 \end{bmatrix} \mathbf{P}_{world}$$

$$= \mathbf{K}[\mathbf{R} \mid \mathbf{T}]\mathbf{P}_{world}$$

$$= \mathbf{KR}[\mathbf{I} \mid -\mathbf{C}]\mathbf{P}_{world} \qquad \mathbf{T} = -\mathbf{RC}$$

More Simplification

- In a camera-centered coordinate frame
  - R is identity
  - C is (0,0,0)
  - $P = KR[I|-C] = K[I|0]$
- For affine camera, the camera center is at infinity and the focal length increase to infinity
  - P becomes an affine matrix

3D Rectification Hierarchy

- From perspective to affine
- H preserves plane of infinity *if and only if* H is affine

$$(\Leftarrow)\mathbf{l}_\infty' = \mathbf{H}_A^{-T}\mathbf{l}_\infty = \begin{bmatrix} \mathbf{A}^{-T} & \mathbf{0} \\ -\mathbf{t}^{-T}\mathbf{A}^{-T} & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{l}_\infty$$

$$(\Rightarrow)\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow h_{31} = 0$$

$$\rightarrow h_{32} = h_{33} = 0$$

- From affine to similarity
- H preserves absolute conic (or absolute dual quadric) if and only if H is similarity

$$(\Leftarrow)\mathbf{C}^{*\prime} = \mathbf{H}_s\mathbf{Q}_\infty^*\mathbf{H}_s^T = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix}\begin{bmatrix} s\mathbf{R}^T & \mathbf{0} \\ \mathbf{t}^T & 1 \end{bmatrix}$$

$$= s^2\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix} = \mathbf{Q}_\infty^*$$

$$(\Rightarrow)\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^T & v \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix}\begin{bmatrix} \mathbf{A}^T & \mathbf{v} \\ \mathbf{t}^T & k \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathbf{A}^T & \mathbf{A}\mathbf{v} \\ \mathbf{v}^T\mathbf{A}^T & \mathbf{v}^T\mathbf{v} \end{bmatrix}$$

$$\rightarrow \mathbf{A}\mathbf{A}^T = 1, \mathbf{v} = \mathbf{0}$$

More Details (Repetition)

- Line in space
- Point in P.o.I
- Absolute conic in P.o.I

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\lim_{\lambda \to \infty} \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} \mathbf{d} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{d} \\ 0 \end{bmatrix}$$

$$\mathbf{d}^T \mathbf{C}^\infty \mathbf{d} = 0$$

- Plane in space
- Line in P.o.I
- Absolute dual quadric in space

$$\pi = \begin{bmatrix} \mathbf{n} \\ n_4 \end{bmatrix} \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad \mathbf{d}' = \begin{bmatrix} d_1' \\ d_2' \\ d_3' \end{bmatrix}$$

$$\mathbf{l} = \mathbf{d} \times \mathbf{d}' = \mathbf{n}$$

$$\pi^T \mathbf{Q}^\infty \pi = 0$$

- Vanishing point
- Iac

- Vanishing line
- iadc

More Details (Repetition)

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\lim_{\lambda \to \infty} \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} \mathbf{d} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{d} \\ 0 \end{bmatrix}$$

$$\mathbf{d}^T \mathbf{C}^\infty \mathbf{d} = 0$$

$$(\mathbf{K}^{-1}\mathbf{x})^T \mathbf{C}^\infty (\mathbf{K}^{-1}\mathbf{x}) = 0$$

$$\mathbf{x}^T \mathbf{K}^{-T} \mathbf{K}^{-1} \mathbf{x} = 0$$

- Vanishing point
- Iac

$$\pi = \begin{bmatrix} \mathbf{n} \\ n_4 \end{bmatrix} \quad \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad \mathbf{d}' = \begin{bmatrix} d_1' \\ d_2' \\ d_3' \end{bmatrix}$$

$$\mathbf{l} = \mathbf{d} \times \mathbf{d}' = \mathbf{n}$$

$$\pi^T \mathbf{Q}^\infty \pi = 0$$

$$\begin{bmatrix} \mathbf{l}^T \mathbf{K} & n_4 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{K}^T \mathbf{l} \\ n_4 \end{bmatrix} = \mathbf{l}^T \mathbf{K} \mathbf{K}^T \mathbf{l} = 0$$

- Vanishing line
- iadc

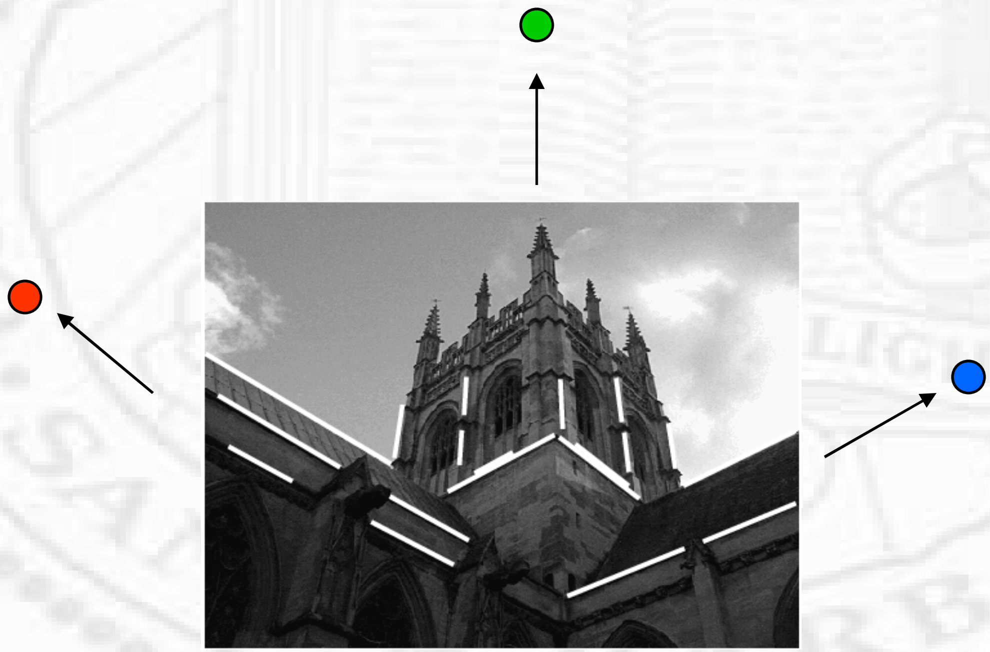
Comparison

- 2D planar homography
  - Plane to plane correspondence
- Move line of infinity
- Recover circular point
- Similarity

- 3D planar homography
  - Image plane and plane at infinity correspondence
- ?
- Recover absolute conic and dual absoluate conic
- Similarity (determining intrinsic parameters)

Stratified Reconstruction in 3D

- Not able to move plane of infinity – it is everywhere
  - Plane of infinity contains the three vanishing points



Stratified Reconstruction in 3D

- Parallel lines (of a single direction) on parallel planes have a single vanishing point (point at infinity)
- Different sets of parallel lines (of multiple directions) on parallel planes have vanishing points on a single vanishing line (line at infinity)
- Multiple sets of parallel planes (of multiple directions) have multiple vanishing lines (plane at infinity)

Measuring Directions Using camera

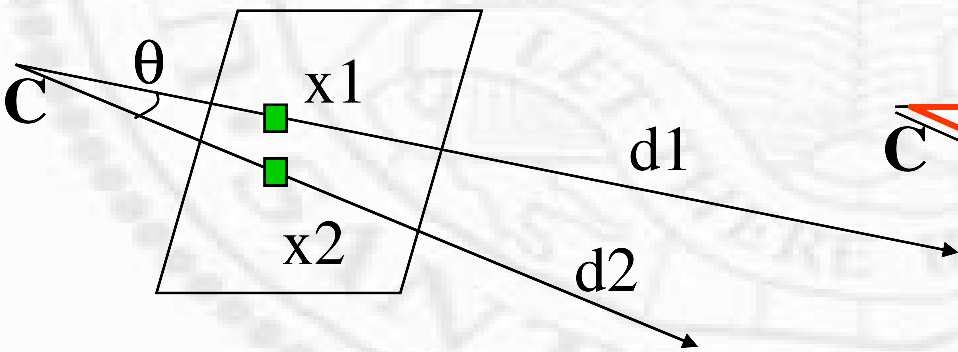
- For image points

$$\mathbf{x} = \mathbf{P}\left(\begin{bmatrix}\mathbf{C}\\1\end{bmatrix} + \lambda\begin{bmatrix}\mathbf{d}\\0\end{bmatrix}\right)$$

$$= \mathbf{K}[\mathbf{I}\,|\,0]\left(\begin{bmatrix}\mathbf{0}\\1\end{bmatrix} + \lambda\begin{bmatrix}\mathbf{d}\\0\end{bmatrix}\right)$$

$$= \mathbf{K}\mathbf{d} \rightarrow$$

$$\boxed{\mathbf{d} = \mathbf{K}^{-1}\mathbf{x}}$$



- For image lines

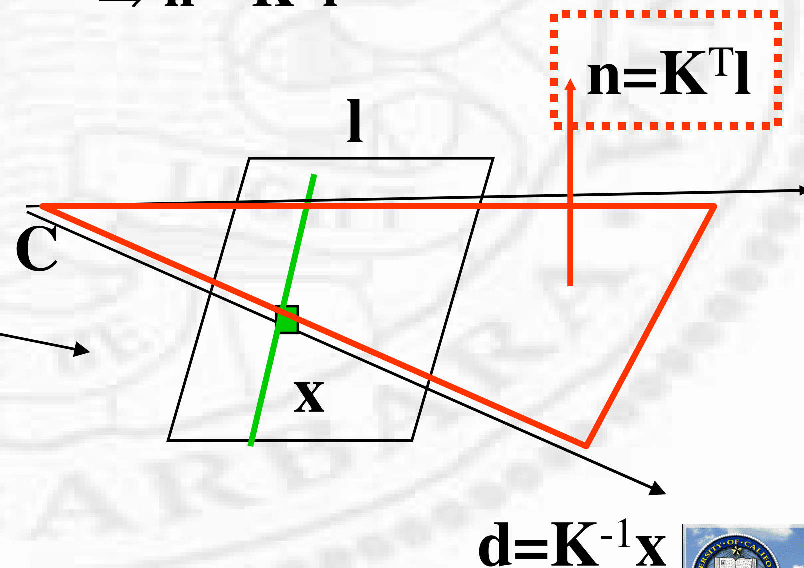
$$\mathbf{x}^T\mathbf{l} = 0, \mathbf{d}^T\mathbf{n} = 0$$

$$\Rightarrow (\mathbf{K}^{-1}\mathbf{x})^T\mathbf{n} = 0$$

$$\Rightarrow \mathbf{x}^T(\mathbf{K}^{-T}\mathbf{n}) = 0$$

$$\Rightarrow \mathbf{K}^{-T}\mathbf{n} = \mathbf{l}$$

$$\Rightarrow \mathbf{n} = \mathbf{K}^T\mathbf{l}$$



An Important Homography Relation

- Images of plane at infinity and image plane form a planar homography

$$\mathbf{x} = P\mathbf{X}_{\infty} = \mathbf{KR}[\mathbf{I} \mid -\mathbf{C}]\begin{pmatrix} \mathbf{d} \\ 0 \end{pmatrix} = \mathbf{KRd} = \mathbf{Hd}$$

Projection of AC and ADQ

- Image of absolute conic (iac)

$$\omega \rightarrow \mathbf{H}^{-T}\mathbf{C}_{\infty}\mathbf{H}^{-1}$$

$$\omega = (\mathbf{KR})^{-T}\mathbf{I}(\mathbf{KR})^{-1} = (\mathbf{KK}^{T})^{-1}$$

- Image of absolute dual quadric (iadc)

$$\begin{aligned}\omega^{*} &= \mathbf{PQ}_{\infty}^{*}\mathbf{P}^{T} \\ &= \mathbf{KR}[\mathbf{I} \mid -\mathbf{C}]\begin{bmatrix}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{I} \\ -\mathbf{C}\end{bmatrix}\mathbf{R}^{T}\mathbf{K}^{T} \\ &= \mathbf{KRR}^{T}\mathbf{K}^{T} = \mathbf{KK}^{T}\end{aligned}$$

If iac or idac can be recovered, camera can be calibrated

But How?

- The pole-polar relation

$$\mathbf{x} = \mathbf{K}[\mathbf{I} \mid 0]\begin{pmatrix} \lambda\mathbf{d} \\ 1 \end{pmatrix} = \mathbf{K}\mathbf{d} \rightarrow \mathbf{d} = \mathbf{K}^{-1}\mathbf{x}$$

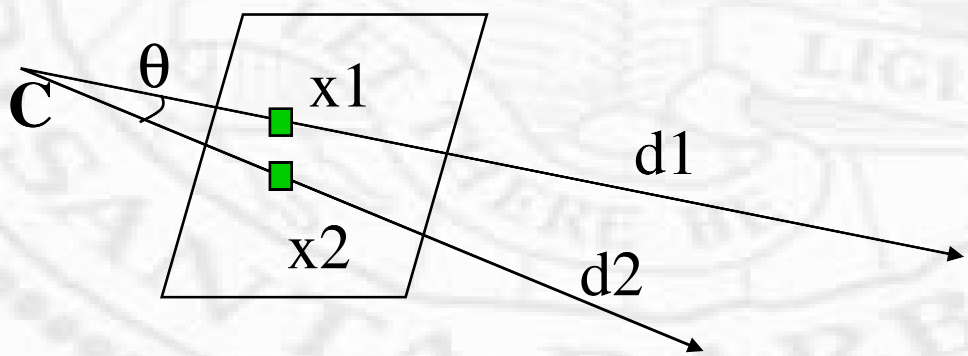
$$\cos\theta = \frac{\mathbf{d}_1^T\mathbf{d}_2}{\sqrt{\mathbf{d}_1^T\mathbf{d}_1}\sqrt{\mathbf{d}_2^T\mathbf{d}_2}} = \frac{(\mathbf{K}^{-1}\mathbf{x}_1)^T(\mathbf{K}^{-1}\mathbf{x}_2)}{\sqrt{(\mathbf{K}^{-1}\mathbf{x}_1)^T(\mathbf{K}^{-1}\mathbf{x}_1)}\sqrt{(\mathbf{K}^{-1}\mathbf{x}_2)^T(\mathbf{K}^{-1}\mathbf{x}_2)}}$$

$$= \frac{\mathbf{x}_1^T(\mathbf{K}\mathbf{K}^T)^{-1}\mathbf{x}_2}{\sqrt{\mathbf{x}_1^T(\mathbf{K}\mathbf{K}^T)^{-1}\mathbf{x}_1}\sqrt{\mathbf{x}_2^T(\mathbf{K}\mathbf{K}^T)^{-1}\mathbf{x}_2}}$$

- Similar to planar homography case, where conic of circular points determines pole-polar relationship in a plane (special case)

General Procedure

- Measure 2D image points (x1, x2)
- Relate that to 3D direction (d1, d2)
- Impose constraints on θ (e.g., some directions are orthogonal)
- But how do we know two directions are perpendicular?



Yet Another Dual Relation

- Vanishing point
- Images of lines
- Invariant measure using AC

- Vanishing line
- Images of planes
- Invariant measure using IAC

$$\mathbf{d}_1^T \mathbf{C}_\infty \mathbf{d}_2 = 0$$

$$\boldsymbol{\omega} \to \mathbf{H}^{-T} \mathbf{C}_\infty \mathbf{H}^{-1}, \boldsymbol{\omega} = (\mathbf{KR})^{-T} \mathbf{I} (\mathbf{KR})^{-1} = (\mathbf{KK}^T)^{-1}$$

$$\mathbf{d} \to \text{vanishing point}$$

$$\boldsymbol{\pi}_1{}^T \mathbf{Q}_\infty^* \boldsymbol{\pi}_2 = 0$$

$$\boldsymbol{\omega}^* \to \mathbf{P}\mathbf{Q}_\infty^* \mathbf{P}^T, \omega^* = \mathbf{KR}[\mathbf{I} \mid -\mathbf{C}] \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ -\mathbf{C} \end{bmatrix} \mathbf{R}^T \mathbf{K}^T$$

$$= \mathbf{KRR}^T \mathbf{K}^T = \mathbf{KK}^T$$

$$\boldsymbol{\pi} \to \textit{vanishing line}$$

Proof for Vanishing Points

$$\mathbf{d}_1^T \mathbf{C}_\infty \mathbf{d}_2 = 0, \mathbf{x} = \mathbf{H}\mathbf{d}$$

$$\boldsymbol{\omega} \to \mathbf{H}^{-T} \mathbf{C}_\infty \mathbf{H}^{-1}, \boldsymbol{\omega} = (\mathbf{K}\mathbf{R})^{-T} \mathbf{I} (\mathbf{K}\mathbf{R})^{-1} = (\mathbf{K}\mathbf{K}^T)^{-1}$$

$$\Rightarrow \mathbf{d}_1^T \mathbf{H}^T \mathbf{H}^{-T} \mathbf{C}_\infty \mathbf{H}^{-1} \mathbf{H}\mathbf{d}_2 = 0$$

$$\Rightarrow (\mathbf{H}\mathbf{d}_1)^T (\mathbf{H}^{-T} \mathbf{C}_\infty \mathbf{H}^{-1})(\mathbf{H}\mathbf{d}_2) = 0$$

$$\Rightarrow \mathbf{x}_1 \boldsymbol{\omega} \mathbf{x}_2 = 0$$

Proof for Vanishing Lines

$$\boldsymbol{\pi}_1{}^T \mathbf{Q}^*_\infty \boldsymbol{\pi}_2 = 0, \mathbf{n} = \mathbf{K}^{\mathrm{T}} \mathbf{l}$$

$$\boldsymbol{\omega}^* \to \mathbf{P}\mathbf{Q}^*_\infty \mathbf{P}^T, \omega^* = \mathbf{KR}[\mathbf{I} \mid -\mathbf{C}] \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ -\mathbf{C} \end{bmatrix} \mathbf{R}^T \mathbf{K}^T$$

$$= \mathbf{KRR}^T \mathbf{K}^T = \mathbf{KK}^T$$

$$\Rightarrow \begin{bmatrix} \mathbf{n}_1^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{n}_2 \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} \mathbf{l}_1 \mathbf{K} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{K}^{\mathrm{T}} \mathbf{l}_2 \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow \mathbf{l}_1 \mathbf{KK}^T \mathbf{l}_2 = 0$$

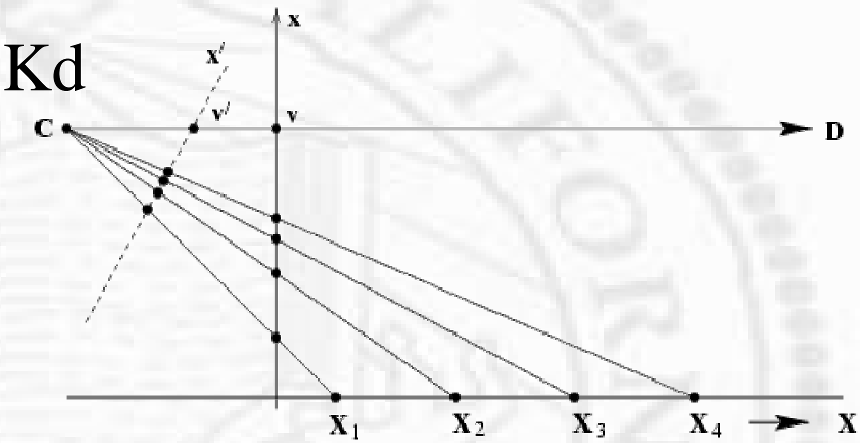
$$\Rightarrow \mathbf{l}_1 \boldsymbol{\omega}^* \mathbf{l}_2 = 0$$

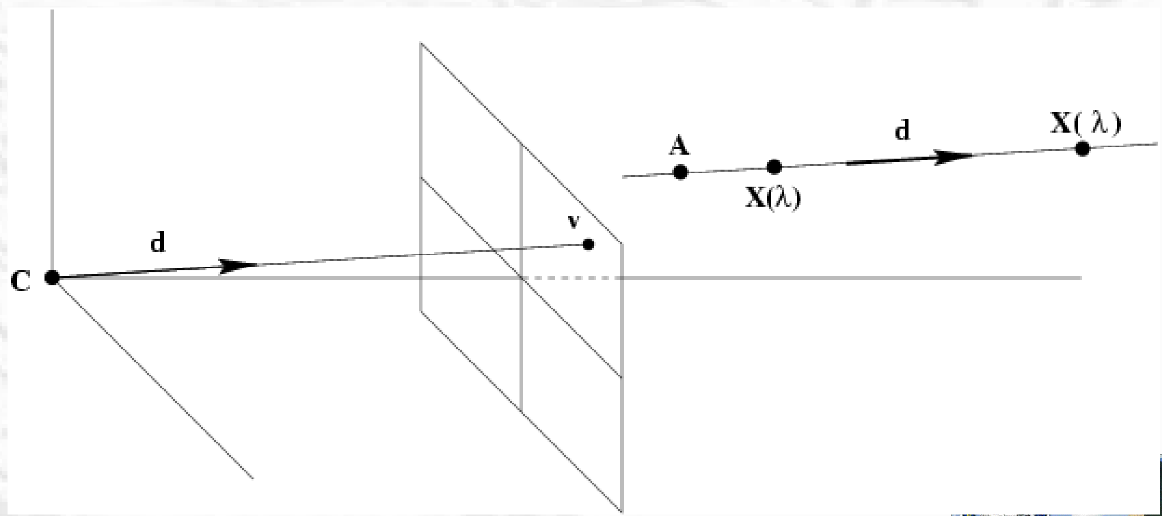
Vanishing points

$$x(\lambda) = PX(\lambda) = PA + \lambda PD = a + \lambda Kd$$

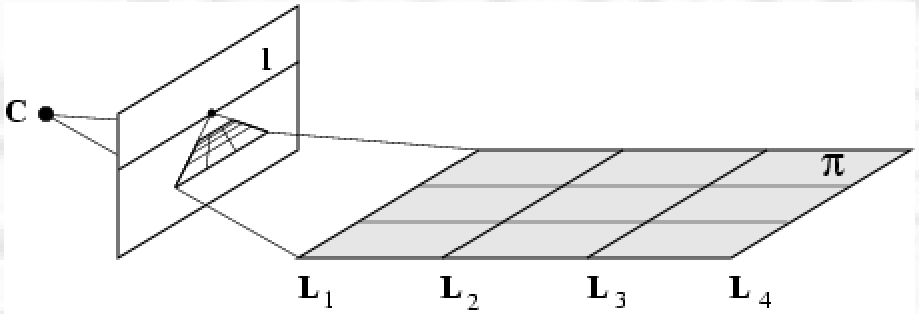
$$v = \lim_{\lambda \to \infty} x(\lambda) = \lim_{\lambda \to \infty} (a + \lambda Kd) = Kd$$

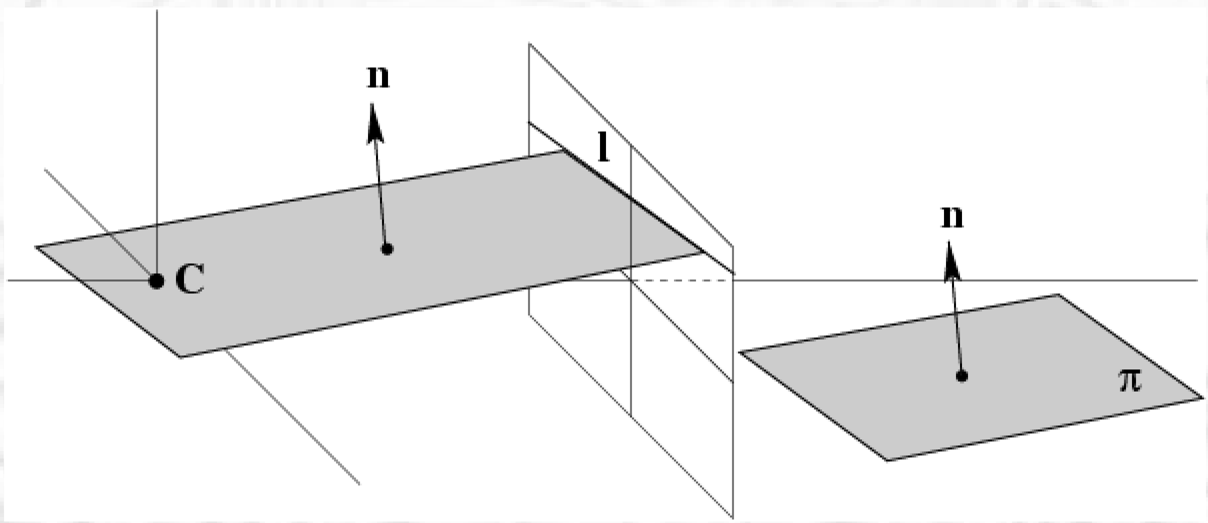
$$v = PX_{\infty} = Kd$$





Vanishing lines





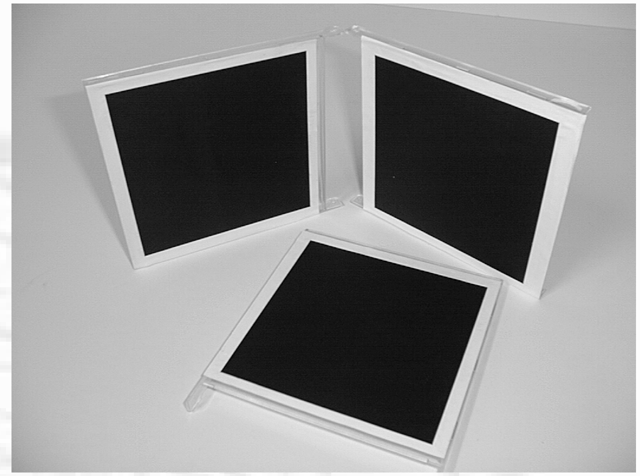
Vanishing Points and Vanishing Lines

- If you can locate vanishing points and lines in images, then you can know their 3D directions (in camera's coordinate system)
- Using domain knowledge you can impose constraints on K

A Hybrid Relation

- For a line and a plane in space
- We have a point (x) and a line (l) in image
- l=wx, x=w*l if they are orthogonal
- Proof left as an exercise

Using Homography



- compute H for each square
- (corners ✁ (0,0),(1,0),(0,1),(1,1))
- compute the imaged circular points $H(1,\pm i,0)^T$
- fit a conic to 6 circular points
- compute K from w through Cholesky factorization

Using Other Relationships

- Image plane
- Two vanishing points
- 1 vanishing point + 1 vanishing line
- Plane

- 3D space
- Two orthogonal lines
- Orthogonal line and plane
- Plane