Complex numbers are essential in quantum theory

(Scott Aaronson's blog).

<u>complex number</u>. A complex number $\lambda \in \mathbb{C}$ is defined by $\lambda = a+bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$

<u>complex conjugate</u>: For a complex number $\lambda = a+bi$, its complex conjugate is $\lambda^* = a-bi$.

For a vector v (e.g. a column vector $\underset{\text{ket}}{|v\rangle}$ in quantum computing), its complex conjugate $v^\dagger$ is a row vector $\underset{\text{bra}}{\langle v|}$, where each entry is a complex conjugate of that in $|v\rangle$.

For a complex matrix A, its conjugate transpose $A^\dagger$ (dagger) is obtained by transposing A and apply complex conjugate on ~~each~~ entry of ~~the~~ matrix.

Examples: $|v\rangle = \begin{bmatrix} 1 \\ e^{i\theta} \end{bmatrix} \rightarrow \langle v| = [1, e^{-i\theta}]$

$$A = \begin{bmatrix} 3 & i+1 \\ -i & 2 \end{bmatrix} \rightarrow A^\dagger = \begin{bmatrix} 3 & i \\ -i+1 & 2 \end{bmatrix}$$

Hilbert space: complex vector space + inner product

↳ A quantum state for n qubits can be described as a vector in a 2^n dimensional Hilbert space.

✓ Let us start with a simple example.

the state of a qubit can be a superposition of "0" and "1".

✓ Formally these "0" and "1" represents two basis states in $\mathbb{C}^2$ space

d=2 complex vector space

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \leftarrow \text{Z basis}$$

$$|v\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{C}^2$$

✓ We can choose either basis states.

e.g.

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftarrow X \text{ basis}$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$|v\rangle = \alpha'|+\rangle + \beta'|-\rangle$$

✓ Note: superposition is regarding to some basis states

· $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$ is a superposition state of $|0\rangle$ and $|1\rangle$

but not a superposition of $|+\rangle$ and $|-\rangle$

✓ For two vector $|v\rangle = \begin{bmatrix} \alpha_v \\ \beta_v \end{bmatrix}$, $|w\rangle = \begin{bmatrix} \alpha_w \\ \beta_w \end{bmatrix}$ in $\mathbb{C}^2$,

their inner product is

$$\langle w|v\rangle = [\alpha_w^*, \beta_w^*] \cdot \begin{bmatrix} \alpha_v \\ \beta_v \end{bmatrix} = \alpha_w^* \alpha_v + \beta_w^* \beta_v$$

This definition also generalize to $\mathbb{C}^n$.

✓ orthogonal: $|w\rangle$ and $|v\rangle$ are orthogonal if $\langle w|v\rangle = 0$

$\langle 0|1\rangle = (1, 0)\cdot\begin{pmatrix}0\\1\end{pmatrix} = 0$

$\langle +|-\rangle = \frac{1}{2}(1, 1)\begin{pmatrix}1\\-1\end{pmatrix} = 0$

✓ norm of a vector $\||v\rangle\| = \sqrt{\langle v|v\rangle} \geq 0$

✓ A unit vector is a vector $|v\rangle$ such that $\||v\rangle\| = 1$

✓ qubit state vector is unit vector.

$|v\rangle = \alpha|0\rangle + \beta|1\rangle \Rightarrow$ if we measure the qubit in $|0\rangle, |1\rangle$ basis

then $|0\rangle \rightarrow \alpha^2$ probability

$|1\rangle \rightarrow \beta^2$ probability

$\Rightarrow \alpha^2 + \beta^2 = 1$

$\||v\rangle\| = \sqrt{\alpha^2+\beta^2} = 1$

It still holds if we measure in a different basis, e.g. $|+\rangle, |-\rangle$.

$|v\rangle = \alpha|0\rangle + \beta|1\rangle = \alpha\frac{|+\rangle + |-\rangle}{\sqrt{2}} + \beta\frac{|+\rangle - |-\rangle}{\sqrt{2}}$

$= \frac{\alpha+\beta}{\sqrt{2}}|+\rangle + \frac{\alpha-\beta}{\sqrt{2}}|-\rangle$

$\Rightarrow |+\rangle \rightarrow \frac{(\alpha+\beta)^2}{2}$ probability

$|-\rangle \rightarrow \frac{(\alpha-\beta)^2}{2}$ probability

$\Rightarrow \frac{(\alpha+\beta)^2}{2} + \frac{(\alpha-\beta)^2}{2} = \alpha^2+\beta^2 = 1$

✓ Bloch sphere of a qubit see slides P

N-qubit state.

<u>composition principle</u>: the state basis of N-qubit state is the tensor products of all individual qubit state basis.

<u>Tensor product</u>: $\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} \otimes \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \alpha_0 \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \\ \alpha_1 \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{bmatrix}$

√ Take a 2-qubit state as an example.

- A set of basis states can be obtained from the $|0\rangle, |1\rangle$ states of single-qubit state

$$|0\rangle \otimes |0\rangle = |00\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|0\rangle \otimes |1\rangle = |01\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|1\rangle \otimes |0\rangle = |10\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$|1\rangle \otimes |1\rangle = |11\rangle = \cdots = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Any 2-qubit state can be written as

$$|\varphi_{AB}\rangle = \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix}$$

when measuring in the above basis, the probability of getting
$$\begin{cases} |00\rangle: & |\alpha_{00}|^2 \\ |01\rangle: & |\alpha_{01}|^2 \\ |10\rangle: & |\alpha_{10}|^2 \\ |11\rangle: & |\alpha_{11}|^2 \end{cases}$$

- If we only ~~measure~~ measure qubit A in $|0\rangle, |1\rangle$ basis, the probability of getting 0 is $|\alpha_{00}|^2 + |\alpha_{01}|^2$ because both $|00\rangle$ and $|01\rangle$ are compatible with $|0\rangle$ for qubit A.

✓ product states VS. entangled states

- The joint state of two seperated, unentangled quantum system A and B, is the tensor product of $|\varphi_A\rangle$ and $|\varphi_B\rangle$

$$|\varphi_{AB}\rangle = |\varphi_A\rangle \otimes |\varphi_B\rangle$$
$$= \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} \otimes \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{bmatrix} \begin{array}{l} \rightarrow |00\rangle \\ \rightarrow |01\rangle \\ \rightarrow |10\rangle \\ \rightarrow |11\rangle \end{array}$$

Quick check: $|\varphi_{AB}\rangle$ is a valid quantum state

$$|\alpha_0\beta_0|^2 + |\alpha_0\beta_1|^2 + |\alpha_1\beta_0|^2 + |\alpha_1\beta_1|^2 = (|\alpha_0|^2 + |\alpha_1|^2)(|\beta_0|^2 + |\beta_1|^2) = 1$$

As a product state, there is some constraints of the coeffecients!

- States that can not be written as product state are entangled states.

- Example ① $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle = \underbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}_{|\varphi_A\rangle} \underbrace{|0\rangle}_{|\varphi_B\rangle}$

② $\boxed{\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \neq |\varphi_A\rangle \otimes |\varphi_B\rangle}$

In fact, most states are entangled.

Bell state, EPR pair

Einstein, Podolsky, Rosen

✓ Generalize to n-qubit state in 2^n-dimensional Hilbert space

$$|\varphi\rangle = \sum_{j \in \{0,1\}^n} \alpha_j |j\rangle,$$ where $\alpha_j \in \mathbb{C}$ satisfying the normalization condition $\sum_j |\alpha_j|^2 = 1$

Hermitian Matrix: For a complex matrix, if $A^{+} = A$, then A is a hermitian matrix.

Property
A basic fact for a Hermitian matrix is that the eigenvalues are real and eigenvectors of distinct eigenvalues are orthogonal.

simple proof:
① $AX = \lambda X \Rightarrow X^{+}AX = \lambda \Rightarrow \lambda^{+} = (X^{+}AX)^{+} = \lambda$
$\Rightarrow \lambda$ is real.

② suppose X_1, X_2 are eigenvectors for two different eigenvalues λ_1, λ_2

$$\left.\begin{aligned} (AX_1)^{+}X_2 &= X_1^{+}AX_2 = \lambda_2 X_1^{+}X_2 \\ (AX_1)^{+}X_2 &= (X_2^{+}AX_1)^{+} = (\lambda_1 X_2^{+}X_1)^{+} = \lambda_1 X_1^{+}X_2 \end{aligned}\right\} \Rightarrow \lambda_1 = \lambda_2 \text{ or } X_1^{+}X_2 = 0$$

Hermitian matrices $\Longleftrightarrow$ observables (measurements) in quantum mechanics, as all observables shall be real.

Unitary matrix: For a complex, invertible matrix, if $A^{+} = A^{-1}$ (i.e., $A^{+}A = AA^{+} = I$), then A is a unitary matrix

property
✓ norm preserving $\| A|\psi\rangle \|^2 = \langle\psi|A^{+}A|\psi\rangle = \| |\psi\rangle \|^2$
$\Rightarrow$ A valid quantum state will be mapped to another valid quantum state.

✓ Invertible ("Reversible") $U^{-1} = U^{+}$

Unitary matrix $\Longleftrightarrow$ quantum gates.

Normal matrix: Matrix A is normal $\Leftrightarrow$ $A^\dagger A = A A^\dagger$

☆ Unitary and Hermitian are special types of normal matrix:

$$\begin{cases} \text{Unitary} & A^\dagger A = A A^\dagger = I \\ \text{Hermitian} & A = A^\dagger \Rightarrow A A^\dagger = A^\dagger A = A^2 \end{cases}$$

Spectral decomposition: A matrix A is normal iff there exists a diagonal matrix Λ and a unitary matrix U such that

P72 proof

$$A = U \Lambda U^\dagger,$$ where the diagonal entries of Λ are the eigenvalues of A, and the columns of U are the corresponding eigenvectors of A.

Another common format for spectral decomposition is:

$$A = \sum_j \lambda_j |j\rangle\langle j|$$

where the column vectors $|j\rangle$ are orthonormal eigenvectors of A and λ_j are the corresponding eigenvalues.

Applications of the theorem (operator function)

$$A^3 = \left(\sum_j \lambda_j |j\rangle\langle j|\right)^3 = \sum_j \lambda_j^3 |j\rangle\langle j| \ldots$$

$$f(A) = \sum_j f(\lambda_j) |j\rangle\langle j|$$

$|0\rangle \to \begin{bmatrix}1\\0\end{bmatrix}$ $|+\rangle \to \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

$|1\rangle \to \begin{bmatrix}0\\1\end{bmatrix}$ $|-\rangle \to \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |1\rangle)$

$X \begin{bmatrix}0&1\\1&0\end{bmatrix}$

$\boxed{\begin{aligned} X|0\rangle &\to |1\rangle \\ X|1\rangle &\to |0\rangle \end{aligned}}$

Conclusion { Hermitian, Unitary

$+ |0\rangle + |1\rangle + |+\rangle\ |-\rangle$)

proof.

$X\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$

$X\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$

$\boxed{\begin{aligned} X|+\rangle &= X\frac{(|0\rangle + |1\rangle)}{2} = |+\rangle \\ X|-\rangle &= X\frac{|0\rangle - |1\rangle}{2} = -|-\rangle \end{aligned}}$

$X^+ = X \to X \cdot X = \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix} = I$

$Z \begin{bmatrix}1&0\\0&-1\end{bmatrix}$

$\boxed{\begin{aligned} Z|0\rangle &= |0\rangle \\ Z|1\rangle &= -|1\rangle \end{aligned}}$

$Z\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$

$Z\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = -\begin{bmatrix}0\\1\end{bmatrix}$

$\boxed{\begin{aligned} Z|+\rangle &= Z\left(\frac{|0\rangle + |1\rangle}{2}\right) = |-\rangle \\ Z|-\rangle &= Z\left(\frac{|0\rangle - |1\rangle}{2}\right) = |+\rangle \end{aligned}}$

$Z^+ = Z = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$ $Z^2 = \begin{bmatrix}1&0\\0&1\end{bmatrix} = I$

unitary $A^+ \cdot A = I$

Hermitian $A^+ = A$

$\checkmark\ H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}(X+Z)$

$H^{+} = H$

$H \cdot H = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I$

$= I$

$H|0\rangle = \frac{1}{\sqrt{2}}(X+Z)|0\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} = |+\rangle$

$H|1\rangle = \frac{1}{\sqrt{2}}(X+Z)|1\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}} = |-\rangle$

$|H_+\rangle = \cos\frac{\pi}{8}|0\rangle + \sin(\frac{\pi}{8})|1\rangle$

$H\rangle_- = -\sin\frac{\pi}{8}|0\rangle + \cos\frac{\pi}{8}|1\rangle$

$H|+\rangle = H\frac{|0\rangle+|1\rangle}{\sqrt{2}} = \frac{|+\rangle+|-\rangle}{\sqrt{2}} = |0\rangle$

$H|-\rangle = H\frac{|0\rangle-|1\rangle}{\sqrt{2}} = \frac{|+\rangle-|-\rangle}{\sqrt{2}} = |1\rangle$

Question: $\cancel{*}|0\rangle, |1\rangle$ eigenstates of X

$|+\rangle\ |-\rangle$ of Z

An operaton of the summation of X ~~+~~ and Z allow us to change these two state basis sets. Is this a coincedance?

$AX = \lambda X \rightarrow (A \cancel{\Rightarrow} \lambda I)X = 0 \cdots$

$\checkmark\ Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$\underset{X}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}\underset{\cancel{Z}}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -iY$

$Y =$ ~~$XZ = iZ$~~

$Y^{+} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = Y$

$Y^{+} \cdot Y = Y^2 =$

$X^2 = Y^2 = Z^2 = I$

Anticommute

$XY = -Y\cancel{X} =$

$\cancel{X} \cdot Z = -ZX = iY$

$YZ = -YZ$

$\underset{X}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}\underset{Y}{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = iZ$

$\underset{Y}{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}}\underset{Z}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = iX$

$\underset{Z}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}\underset{X}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$= iY$

$Z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

$e^{i\theta Z}$ $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1\,0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\langle 1| = (1, 0)$ $|1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$e^{-i\theta Z} = \sum_{n=0}^{\infty} \frac{(-i\theta Z)^n}{n!} = \sum_{n=0,2,4}^{\infty} \frac{(-i\theta)^n}{n!} \cdot I + \sum_{n=1,3,5}^{\infty} \frac{(-i\theta)^n}{n!} \cdot Z$$

$$= \cos\theta \cdot I - i\sin\theta \cdot Z$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5!} - \dots \text{ for all } x$$

$X^2 = Y^2 = Z^2 = I$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4} - \dots \text{ for all } x$$

$$e^{-i\theta X} = \cos\theta \cdot I - i\sin\theta \cdot X$$
$$e^{-i\theta Y} = \cos\theta \cdot I - i\sin\theta\, Y$$
$$e^{-i\theta Z} = \cos\theta \cdot I - i\sin\theta\, Z$$

$\Rightarrow$ $\boxed{\text{diagonale matrix}}$ $\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$ $\rightarrow$ next page

$\rightarrow$ for whatever matrix $A^2 = I$

Exercise 1

$R_{\hat{n}}(\theta) \equiv \exp(-i\theta\, \hat{n}\cdot\hat{\sigma}) = \cos\theta - i\sin\theta\,(n_x \cdot X + n_y Y \cdot n_z \cdot Z)$

Easy to prove: $(\hat{n}\cdot\hat{\sigma})^2 = I$ $\Rightarrow$ anticommutate + unite vector.

$$\Rightarrow (n_x \cdot X + n_y Y + n_z Z)^2$$
$$= n_x^2 \cdot X^2 + n_y^2 \cdot Y^2 + n_z^2 \cdot Z^2 + 0 \cdots$$
$$= I$$

Excersise 2

Any arbitrary single qubit gate

$U = \underbrace{\exp(i\alpha)}_{\text{global phases}} \underbrace{R_{\hat{n}}(\theta)}_{\uparrow\text{rotation}}$

$$e^{-i\theta Z} \doteq \begin{pmatrix} e^{-i\theta} & \\ & e^{i\theta} \end{pmatrix} \qquad e^{0} = 1 \quad e^{i\frac{\pi}{2}} = i \quad e^{i\pi} = -1 \quad e^{i\frac{3}{2}\pi} = -i$$

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} e^{0} & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix} = e^{\frac{\pi}{4}i} \begin{pmatrix} e^{-\frac{\pi}{4}i} & \\ & e^{\frac{\pi}{4}i} \end{pmatrix} = \frac{\pi}{4} \text{ gate}$$

phase gate

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = e^{i\frac{\pi}{8}} \begin{pmatrix} e^{-i\frac{\pi}{8}} & \\ & e^{i\frac{\pi}{8}} \end{pmatrix} . \quad \frac{\pi}{8} \text{ gate.}$$