Two-qubit gates

why? Entanglement:

product state + single-qubit gates → product state

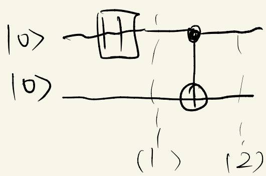
what kind of two-qubit gate? Almost any, as we can use one + single-qubit gates to generate others.

CNOT gate
CX

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$

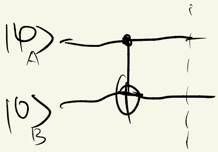
$|00\rangle \to |00\rangle$
$|01\rangle \to |01\rangle$
$|10\rangle \to |11\rangle$
$|11\rangle \to |10\rangle$

On z-basis states,
CNOT $|z_1, z_2\rangle$
$= |z_1, z_2 \oplus z_1\rangle$



(1) $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$

(2) $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ ↑ Bell state

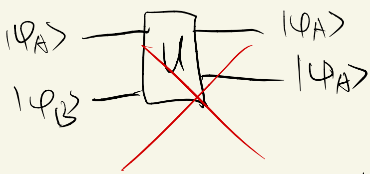


$|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$

$\Rightarrow |\varphi_{AB}\rangle = \alpha|00\rangle + \beta|11\rangle$

Quantum Fanout

Quantum Non-cloning theorem: it's impossible to create an independent and identical copy of an arbitrary unknown quantum state.



Proof: Take two particular state $|\varphi_1\rangle$ and $|\varphi_2\rangle$

P532

$$U(|\varphi_1\rangle \otimes |\varphi_B\rangle) = |\varphi_1\rangle \otimes |\varphi_1\rangle$$

$$U(|\varphi_2\rangle \otimes |\varphi_B\rangle) = |\varphi_2\rangle \otimes |\varphi_2\rangle$$

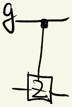
Take the inner product $\Rightarrow \langle\varphi_1|\varphi_2\rangle = (\langle\varphi_1|\varphi_2\rangle)^2$

$\Rightarrow \langle\varphi_1|\varphi_2\rangle = 0$ or 1

$\varphi_1\rangle, \varphi_2\rangle$ are either orthogonal or the same.

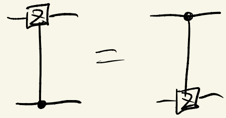
$\Rightarrow$ not arbitrary.

/Other 2-qubit gates and their relationship to CNOT.

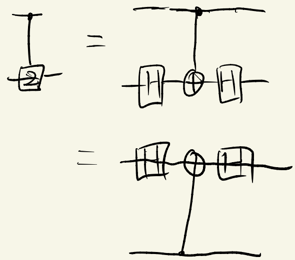
CZg  $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$

$|00\rangle \to |00\rangle$
$|01\rangle \to |01\rangle$
$|10\rangle \to |10\rangle$
$|11\rangle \to |11\rangle$

symmetric effect on two qubits



Can you also prove



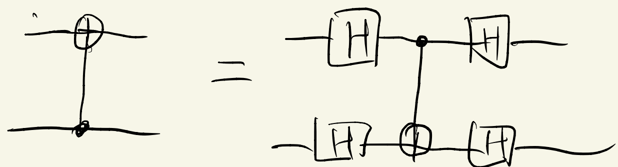
?

Hint: $Z = HXH$

Two general proving strategies
1. matrix format
2. check results on basis states + linear operations

Exercise: given a set of basis states $|\lambda_j\rangle$ in the hilbert space if $U|\lambda_j\rangle = V|\lambda_j\rangle$ for all $|\lambda_j\rangle$, then $U = V$.

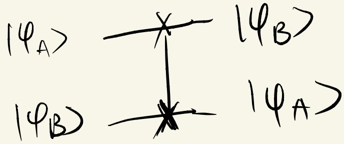
From the above circuit identities, we can also get:



why is this potentially benefitial? Hardware implementation.

We generally hope to simplify the hardware implementation while not sacrificing too much performance.

swap gate



Q: Given the desired functionality, can you draw its matrix format and prove it is a unitary matrix?

A: Think of basis states

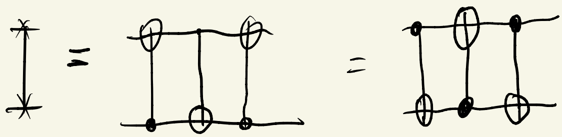
$|00\rangle \rightarrow |00\rangle$
$|01\rangle \rightarrow |10\rangle$
$|10\rangle \rightarrow |01\rangle$
$|11\rangle \rightarrow |11\rangle$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$|00\rangle$ $[1\,0\,0\,0]^{\dagger}$
$|01\rangle$ $[0\,1\,0\,0]^{\dagger}$
$|10\rangle$ $[0\,0\,1\,0]^{\dagger}$
$|11\rangle$ $[0\,0\,0\,0]^{\dagger}$

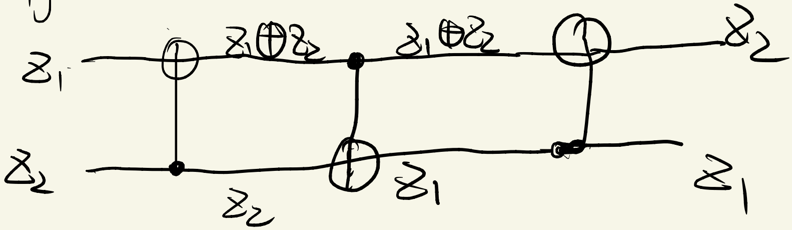
You can check whether it is a unitary..

Q: can you prove the following?



A: ① matrix format

② Again, with the 8 basis states $|z_1 z_2\rangle$



Classical analogy: XOR swap algorithm, which uses three XOR operations to swap the value of two binary values without using extra temporary memory

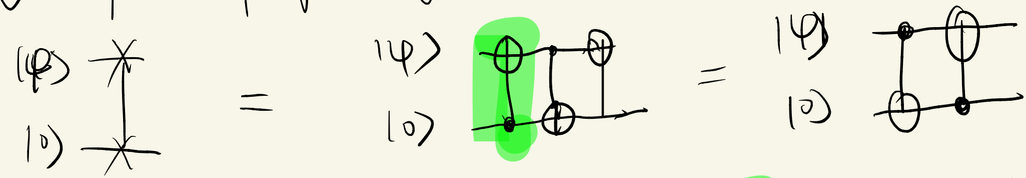
e.g.

$$
\begin{array}{ccccc}
x & & y & & \\
1010 & \oplus & 0011 & = & 1001 \to x \\
1001 & \oplus & 0011 & = & 1010 \to y \\
1001 & \oplus & 1010 & = & 0011 \to x
\end{array}
$$

Q: Can we get CNOT gate from swap gate along with arbitrary single-qubit gates?

A: no. Swap gate does not create entanglement, so do arbitrary single-qubit gate.

✓ Input-specific gate transformation



Key insights: partial equilence under special inputs

very common optimization techniques for quantum program optimization

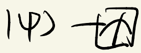
More excercises: See slides.

Measurement :

Z measurement = measure in Z-basis $\neq$ Z gate

$Z = \underbrace{|0\rangle\langle 0|}_{P_0} - \underbrace{|1\rangle\langle 1|}_{P_1} = P_0 - P_1$ (both P_0 and P_1 are projective matrix)

projective matrix $P^2 = P$



$\begin{cases} |0\rangle & \text{probability} = p_0 = \langle\psi|P_0|\psi\rangle = \langle 0|\psi\rangle^2 \\ |1\rangle & \text{probability} = p_1 = \langle\psi|P_1|\psi\rangle = \langle 1|\psi\rangle^2 \end{cases}$

What is the expectation value of Z measurement ?

$$\langle Z\rangle = (+1)\cdot p_0 + (-1)\cdot p_1 = \langle\psi|P_0 - P_1|\psi\rangle = \langle\psi|Z|\psi\rangle$$

what if we did the measurement, but we don't know the result?

Density matrix : Ensemble of pure quantum states.

$$\rho \equiv \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

For the example above, $\rho = p_0|0\rangle\langle 0| + p_1|1\rangle\langle 1|$

Unitary operator on ρ

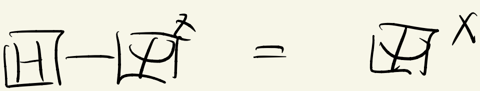
$$\rightarrow \sum_i p_i U|\psi_i\rangle\langle\psi_i|U^\dagger = U\rho U^\dagger$$

X measurement = measure in X-basis $\neq$ X gate

$X = \underbrace{|+\rangle\langle +|}_{P_0} - \underbrace{|-\rangle\langle -|}_{P_1} = P_0 - P_1$

Q: Can we implement X measurement with Z measurement as it make our hardware design easier?

A: Yes.



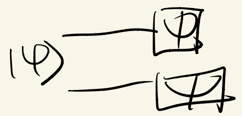
Proof ? Hint : use basis states for testing
$|+\rangle, |-\rangle$

Two-qubit composite measurement

✓ measurement according to a set of basis states in the composite Hilbert space.

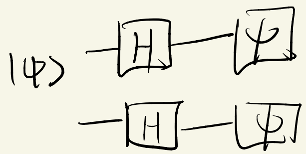
ZZ measurement: 4 eigenvectors $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$

$P_1 = |00\rangle\langle 00|$ $\quad P_2 = |01\rangle\langle 01|$ $\quad P_3 = |10\rangle\langle 10|$ $\quad P_4 = |11\rangle\langle 11|$



$$|\psi\rangle = \alpha_1|00\rangle + \alpha_2|01\rangle + \alpha_3|10\rangle + \alpha_4|11\rangle \quad \left\{ \begin{array}{ll} p_1 = \alpha_1^2 & |00\rangle \\ p_2 = \alpha_2^2 & |01\rangle \\ p_3 = \alpha_3^2 & |10\rangle \\ p_4 = \alpha_4^2 & |11\rangle \end{array} \right.$$

XX measurement 4 eigenvectors $|++\rangle$, $|+-\rangle$, $|-+\rangle$, $|--\rangle$



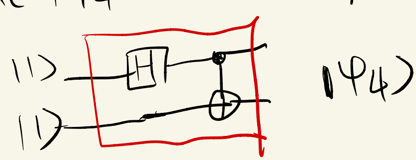
Bell state measurement

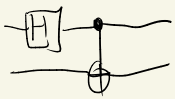
4 orthonormal 2-qubit states

$$\left\{ \begin{array}{l} |\psi_1\rangle = \frac{1}{\sqrt{2}} |00\rangle + |11\rangle \\ |\psi_2\rangle = \frac{1}{\sqrt{2}} |00\rangle - |11\rangle \\ |\psi_3\rangle = \frac{1}{\sqrt{2}} |01\rangle + |10\rangle \\ |\psi_4\rangle = \frac{1}{\sqrt{2}} |01\rangle - |10\rangle \end{array} \right.$$

Q: Can you use a circuit with H, X, Z, CNOT to create these states from ZZ states?

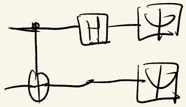
A: Take $|\psi_4\rangle$ as an example





maps $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ to
$|\varphi_1\rangle, |\varphi_2\rangle, |\varphi_3\rangle, |\varphi_4\rangle$ respectively.

$\Longrightarrow$



gives the measurement according to the bell state basis.

$P_1 = |\varphi_1\rangle\langle\varphi_1|$, $P_2 = |\varphi_2\rangle\langle\varphi_2|$, ...

Give a input $|\psi\rangle$ to the measurement circuit.

$p_1 = \langle\varphi_1|\psi\rangle^2$ · collapse to $|\varphi_1\rangle$ state

...

Partial measurement

How to measure entangled qubit?

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

cannot be written as $|\psi_A\rangle \otimes |\psi_B\rangle$

① what if B measure his qubit first in Z basis?

Observable? $I \otimes \sigma_z = (+1)\underbrace{I \otimes |0\rangle\langle 0|}_{P_1} + (-1)\underbrace{I \otimes |1\rangle\langle 1|}_{P_2}$

$$\text{Prob}(|0\rangle_B) = \langle \psi_{AB}|P_0|\psi_{AB}\rangle$$

$$= \langle \psi_{AB}| \frac{1}{\sqrt{2}}|00\rangle = \frac{1}{2} \quad \text{(By linearly)}$$

$$|\psi_{AB}\rangle = \frac{P_1 |\psi_{AB}\rangle}{\sqrt{\text{Prob}(|0\rangle_B)}} = |00\rangle$$

$$|\psi_A\rangle = |0\rangle$$

$$\text{Prob}(|1\rangle_B) = \langle \psi_{AB}|P_1|\psi_{AB}\rangle$$

$$= \langle \psi_{AB}| \frac{1}{\sqrt{2}}|11\rangle = \frac{1}{2}$$

$$|\psi_{AB}\rangle = \frac{P_2 |\psi_{AB}\rangle}{\sqrt{\text{Prob}(|1\rangle_B)}} = |11\rangle$$

$$|\psi_A\rangle = |1\rangle$$

② what if we measure B in X basis?

Your answer here.

A equivalent, but easier calculation for partial measurement

For a composite system in state $|\varphi_{AB}\rangle = \sum_{i,j} \alpha_{ij} |i,j\rangle$, we only measure part B in basis set of $|j\rangle$. what would be the result?

$$|\psi_{AB}\rangle = \sum_{ij} \alpha_{ij} |ij\rangle = \sum_{j} \left(\sum_{i} \alpha_{ij} |i\rangle \right) \otimes |j\rangle$$

Let $\beta_j = \sum_{i} a_{ij}^2$, then we can re-write the sum as

$$|\psi_{AB}\rangle = \sum_{j} \beta_j \left(\frac{1}{\beta_j} \sum_{i} a_{ij} |i\rangle \right) |j\rangle$$

Now, if we measure B in $|j\rangle$ basis set, we will see outcome $|j\rangle$ with probability $\beta_j = \sum_{j} a_{ij}^2$

the state for A after seeing $|j\rangle$ is $\frac{1}{\beta_i} \sum_{i} a_{ij} |i\rangle$

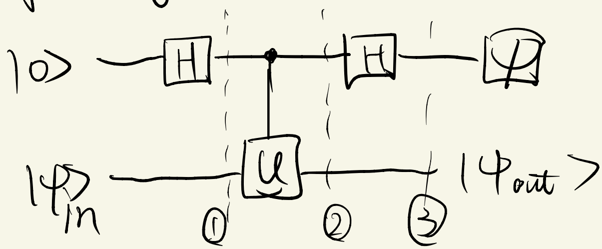
Other things about measurement.

1. measuring an operator

suppose we have an operator U with eigenvalues of ± 1. so the U is both Hermitian and unitary, so it can be regarded as both an observable and gates.

Goal: we want to measure U, and leaving the post measurement state is the corresponding eigenvector

show the following circuit satisfy the needs:



Proof: we can write $|\varphi_{in}\rangle = \alpha|\varphi_{+1}\rangle + \beta|\varphi_{-1}\rangle$

where $U|\varphi_{+1}\rangle = |\varphi_{+1}\rangle$

$U|\varphi_{-1}\rangle = -|\varphi_{-1}\rangle$

① $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)(\alpha|\varphi_{+1}\rangle + \beta|\varphi_{-1}\rangle)$

② $\frac{1}{\sqrt{2}}|0\rangle(\alpha|\varphi_{+}\rangle + \beta|\varphi_{+}\rangle)$

$+\frac{1}{\sqrt{2}}|1\rangle(\alpha|\varphi_{+}\rangle - \beta|\varphi_{-1}\rangle)$

$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\alpha|\varphi_{+}\rangle + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\beta|\varphi_{-1}\rangle$

③ $|0\rangle \cdot \alpha|\varphi_{+}\rangle + |1\rangle\beta|\varphi_{-1}\rangle$

$= \alpha|0\varphi_{+}\rangle + \beta|1\varphi_{-}\rangle$

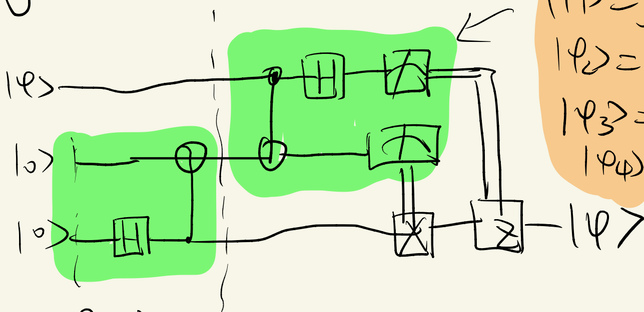
Quantum Teleportation

Now we have finish all the basics for quantum computation (e.g., quatum state, 2-q and 1-q gates, measurement), we can have a in-depth analysis of one of the most important quantum circuit/program, quantum teleportation.

✓ Teleportation enable us with quantum communication without directly transmitting a qubit that directly encodes the information.

It also lets us to perform universal fault-tolerent computation. and measurement based quantum computing.

See slide:



$|\varphi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$|\varphi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

$|\varphi_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

$|\varphi_4\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |10\rangle)$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\psi_{ABC}\rangle = \alpha|0\rangle + \beta|1\rangle \left(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right)$$

$$= \frac{\alpha}{\sqrt{2}}|00\rangle|0\rangle + \frac{\alpha}{\sqrt{2}}|01\rangle|1\rangle + \frac{\beta}{\sqrt{2}}|10\rangle|0\rangle + \frac{\beta}{\sqrt{2}}|11\rangle|1\rangle$$

$$= \alpha(|\varphi_1\rangle + |\varphi_2\rangle)|0\rangle + \alpha(|\varphi_3\rangle + |\varphi_4\rangle)|1\rangle$$

$$+ \beta(|\varphi_3\rangle - |\varphi_4\rangle)|0\rangle + \beta(|\varphi_1\rangle - |\varphi_2\rangle)|1\rangle$$

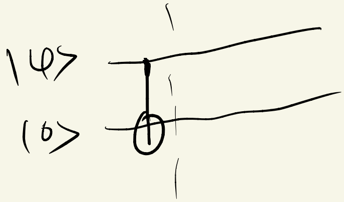
$$= |\varphi_1\rangle(\alpha|0\rangle + \beta|1\rangle) + |\varphi_2\rangle(\alpha|0\rangle - \beta|1\rangle)$$

$$|\varphi_3\rangle(\alpha|1\rangle + \beta|0\rangle) + |\varphi_4\rangle(\alpha|1\rangle - \beta|0\rangle)$$

$$= \cdots$$

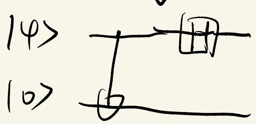
Can we use just one CNOT (for entanglement) to push the state $|\psi\rangle$ to the ancilla qubit?

Hint: we may use single-qubit gate and measurement in the transformation process.



$\alpha|00\rangle + \beta|11\rangle$

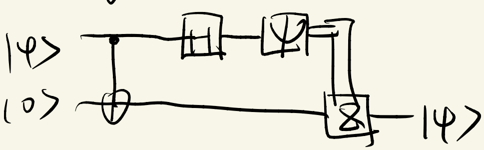
$\Downarrow$



$\frac{\alpha}{\sqrt{2}}(|0\rangle+|1\rangle)|0\rangle + \frac{\beta}{\sqrt{2}}(|0\rangle-|1\rangle)|1\rangle)$

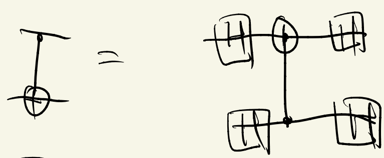
$= \frac{|0\rangle}{\sqrt{2}}(\alpha|0\rangle+\beta|1\rangle) + \frac{|1\rangle}{\sqrt{2}}(\alpha|0\rangle-\beta|1\rangle$

$\Downarrow$

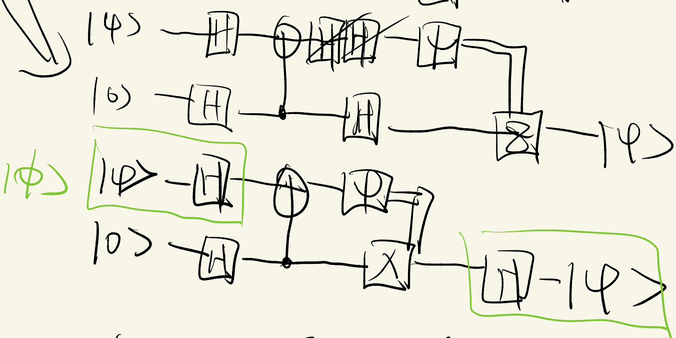


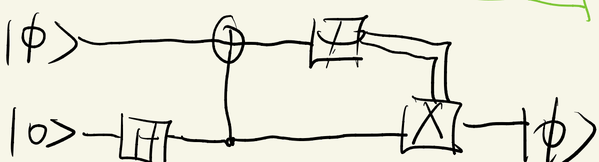
"Z-swap circuit"

Recall



$H - Z = Z - H$





"X-swap circuit"

Multi-qubit gate

√ A very special type of gate

$Z_1 Z_2 Z_3$ can you tell me its matrix formate without doing the tensor product? of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Diagonal matrix

Think about its eigenvalue and eigenstate

$$\underline{+1} \left\{ \begin{array}{l} |000\rangle \\ |011\rangle \\ |101\rangle \\ |110\rangle \end{array} \right. \qquad \underline{-1} \left\{ \begin{array}{l} |001\rangle \\ |010\rangle \\ |100\rangle \\ |111\rangle \end{array} \right.$$

parity check

$$\Rightarrow \begin{pmatrix} +1 & & & & & & & \\ & -1 & & & & & & \\ & & -1 & & & & & \\ & & & +1 & & & & \\ & & & & -1 & & & \\ & & & & & +1 & & \\ & & & & & & +1 & \\ & & & & & & & -1 \end{pmatrix}$$

$$Z_1 Z_2 Z_3 = +1 \left(|000\rangle\langle 000| + |011\rangle\langle 011| + \cdots \right) - 1 \left(|001\rangle\langle 001| + |010\rangle\langle 010| + \cdots \right)$$

$e^{iw Z_1 Z_2 Z_3}$ ← function operator
also a diagonal matrix

$$\begin{pmatrix} e^{iw} & & & \\ & e^{-iw} & & \\ & & e^{-iw} & \\ & & & \ddots \end{pmatrix}$$

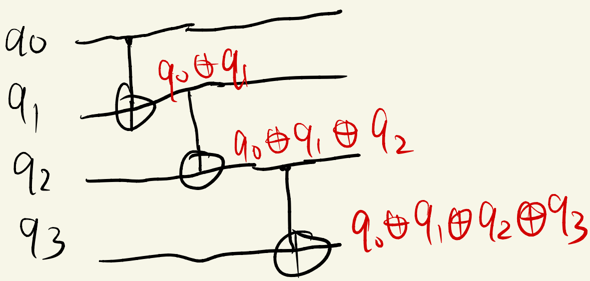
For the "classical" z basis, this gate will apply a e^{iw} phase if the parity is 0, otherwise e^{-iw}

what about $e^{iw Z_1 I_2 Z_3}$?

All applies, but now the parity check is only for qubit 1 and qubit 3.

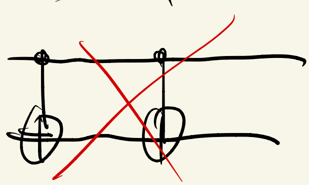
Q: can you now verify the circuit implementation for $\exp(iw\, I_4 Z_3 Z_2 Z_1 Z_0)$

Hint: 1. CNOT tree for parity check in Z basis



The root node of the CNOT tree will encode the parity information

2. CNOT cancel with each othe



$= I$