CS293S Quantum Computing System

Lecture 2-4: Quantum Computing Preliminaries and Math Foundations

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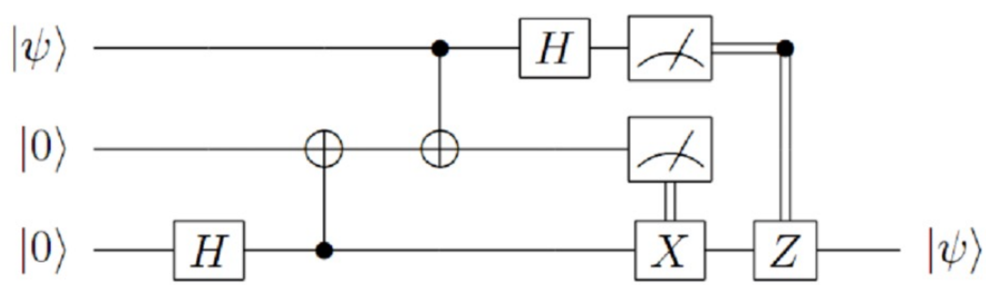
Goal: Understand the Quantum Circuit Mathematically

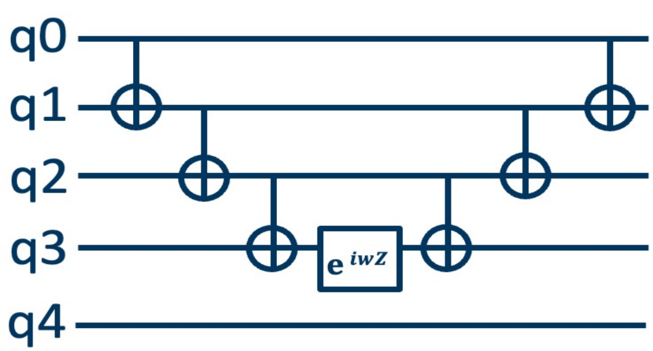


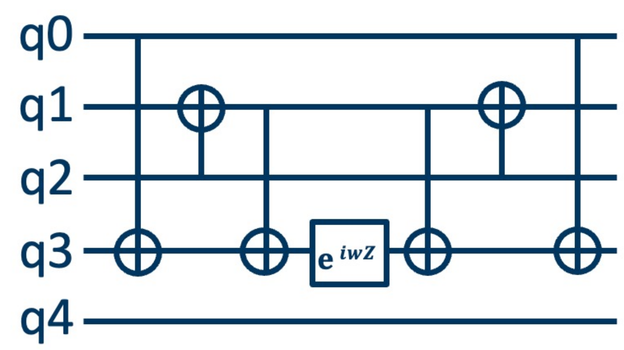
Teleportation Circuit

Goal: Understand the Different Representations

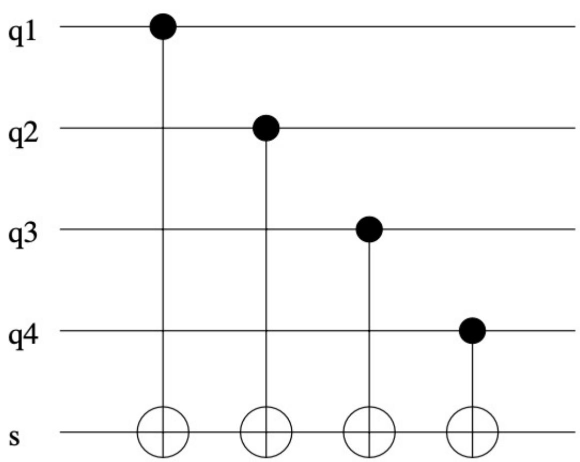
$\exp(iwI_4Z_3Z_2Z_1Z_0)$

- ❖ Five-qubit gate
- ❖ 32 * 32 unitary matrix
- ❖ Its decomposition into 1-qubit and 2-qubit gates (different variants)





Advanced Goal: Understand the Circuit Optimizations



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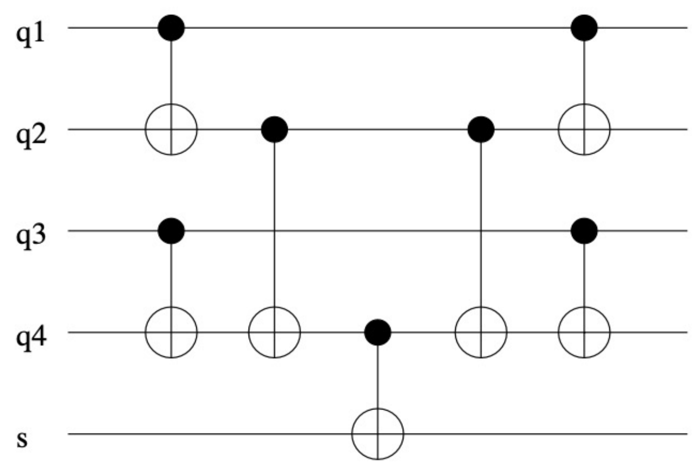


#Gates = N; #Depth = N

#Gates = 2N-1; #Depth = 2logN+1

Trade-off between #depth and #gates

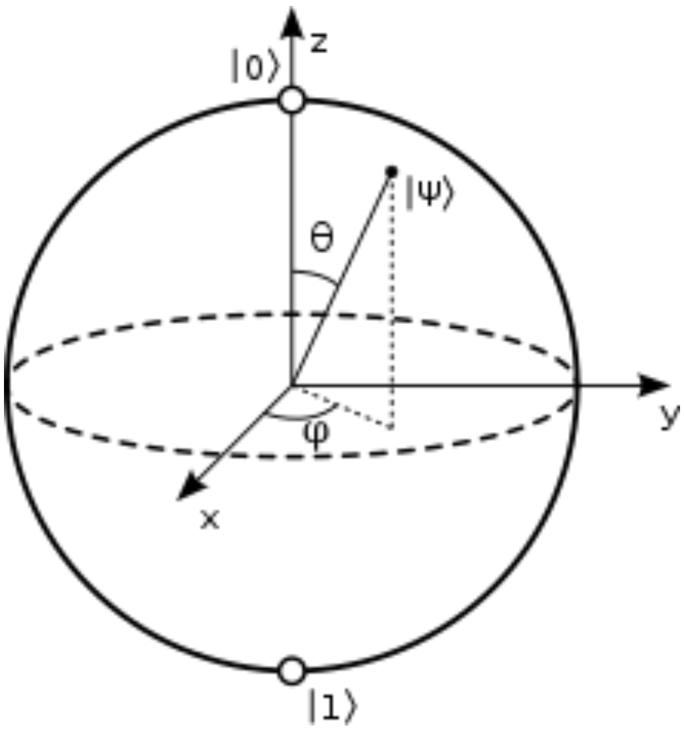
Outline

- Hilbert Space: Qubit State
- Unitary Matrix: Quantum Gates
- Projective Matrix: Quantum Measurements

Note2.pdf (page 1-3)

- Some Math Basics
- Hilbert Space: Qubit State
  - Single-qubit state

Bloch Sphere of a Qubit



- Qubit state is normalized. This suggests a representation in spherical polar coordinates.

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$$

- This is called the Bloch Sphere Representation.
- A single qubit state is a point on the surface of a sphere of unit radius, parametrized by the two coordinates.
- The basis state are then presented by the north and source poles of the sphere, respectively.

Note2.pdf (page 4-5)

- Multi-qubit state
- Product state vs. entangled state

Note2.pdf (page 6-7)

Basics about Linear Operators on Hilbert Space

- Hermitian Matrix: Observables
- **Unitary Matrix: Quantum Gates**
- Normal Matrix: Operator Function

Common Single-Qubit Gates

- Qubit

Wire carrying a single qubit
(time goes left to right)

Hadamard	H	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$	
Pauli-X	X	$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$	NOT gate
Pauli-Y	Y	$\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$	
Pauli-Z	Z	$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$	
Phase	S	$\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}$	
$\pi/8$	T	$\begin{bmatrix}1 & 0\\ 0 & e^{i\pi/4}\end{bmatrix}$	

Practices ☺

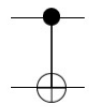
- Check whether these gates are unitary or not?
- Computer their eigenstates and eigenvalues?
- Check how they work on some common state vector bases?
- Prove that Pauli and Hadamard gates are both Unitary and Hermitian?

Note2.pdf (page 8-11)

- Single-qubit gates

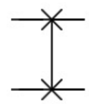
Common Two-Qubit Gates

controlled-NOT



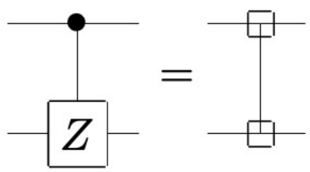
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

swap



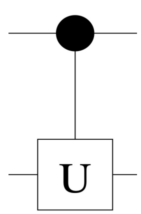
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

controlled-Z



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Controlled-U



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{11} & u_{12} \\ 0 & 0 & u_{21} & u_{22} \end{bmatrix}$$

Practices ☺

- Check whether these gates are unitary or not?
- Check how they work on some common state vector bases?

Details in Note3.pdf (page 1 -- 4)

Practices ☺

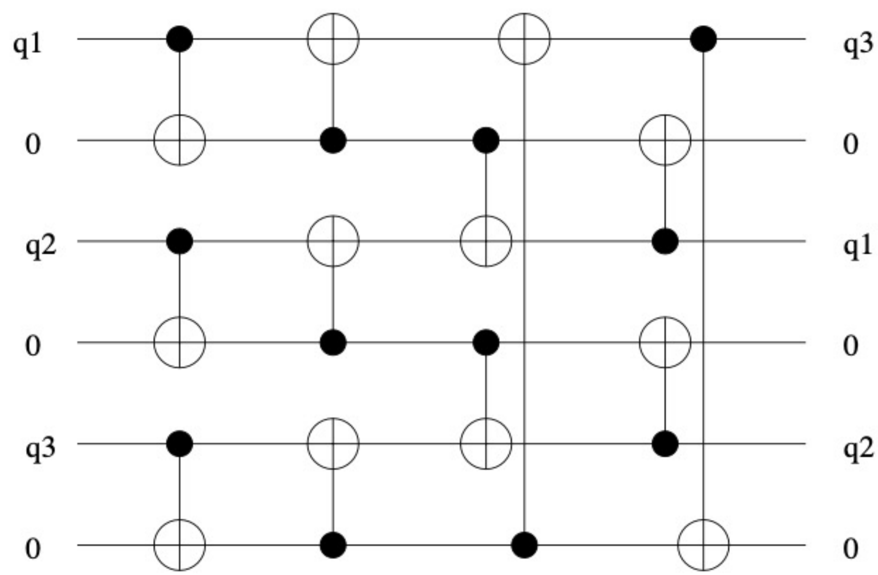
- Check whether these gates are unitary or not?
- How they are related with each other?
- Check how they work on some common state vector bases?

Practices ☺

In a quantum computer, it may be more difficult to move a qubit from place to place. However, we can easily do arbitrary permutations in constant depth:

- **Q:** Can you enable any permutation of n qubits with 4 layers of CNOT gates with n ancilla qubits?
- **A: ?**

Permuting N Qubits in 4 Layers Using N Ancilla Qubits.



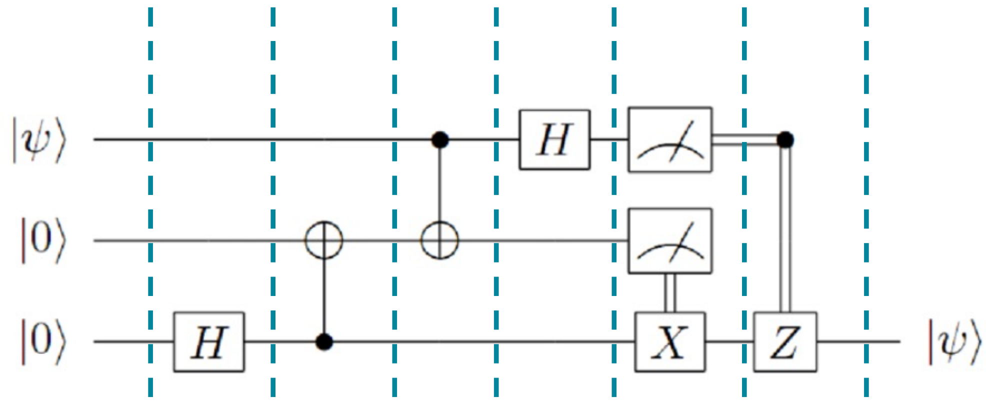
Permuting N qubits in 6 layers without any ancilla qubits. [1]

[1] Moore, Cristopher et. al. "Parallel quantum computation and quantum codes." SIAM journal on computing 31.3 (2001): 799-81

Quantum Measurement

- Note 3.pdf (page 5 - 10)

Practice: Understand the Quantum Circuit Mathematically



Teleportation Circuit

- Note3.pdf page 11-12

Practice: Understand the Different Representations

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- ❖32 * 32 unitary matrix
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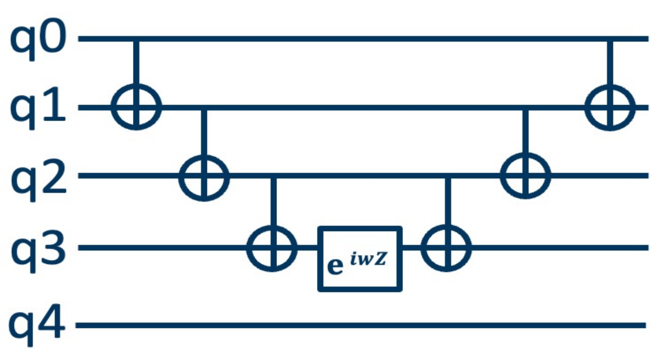
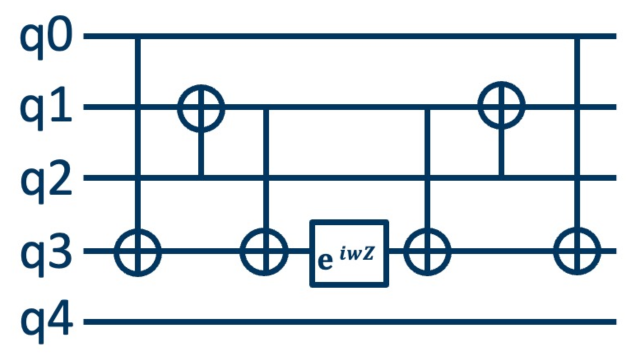
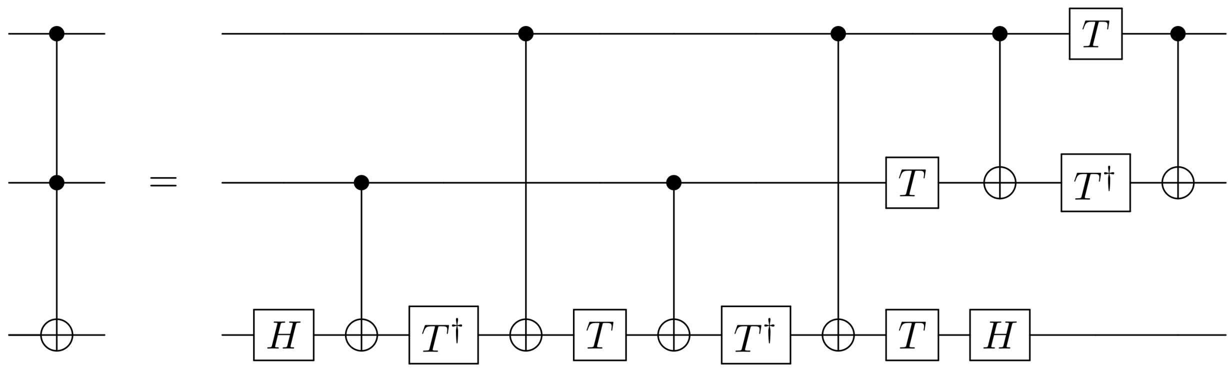




- ❖Note3.pdf page 13-14

Other Common Multi-Qubit Gates

Toffoli
(CCNOT)

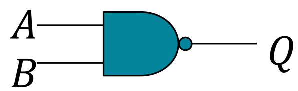


$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The Toffoli gate can be constructed from single qubit T and H gates, and a minimum of six CNOT.

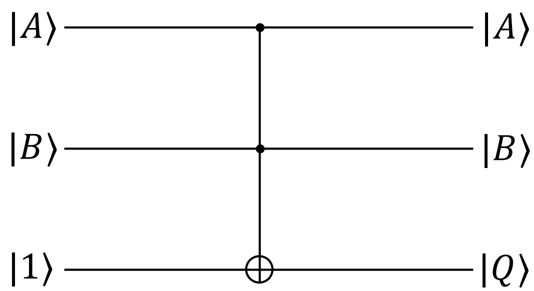
Classical Gates to Quantum Gates

❖ The NAND gate is universal for classical circuits and acts as below.



A	B	Q
0	0	1
0	1	1
1	0	1
1	1	0

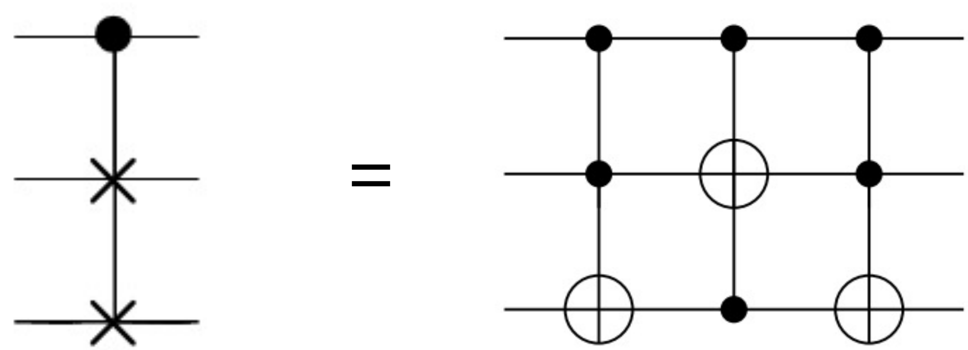
❖ We can perform the same operation using a Toffoli gate.



❖ We can convert any classical algorithm into a quantum algorithm, replacing the NAND gates with Toffolis, and keeping the extra qubits.

Common Multi-Qubit Gates

Fredkin (controlled-swap)



$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

It can be built from the standard swap network by adding an additional control to each gate, turning each controlled-not gate into a Toffoli gate.

Summary

- Hilbert Space: Qubit State
  - Basis states: tensor product from single-qubit basis states
  - Product state vs. entangled state
- Unitary Matrix: Quantum Gates
  - Single-qubit gate: H, T gate; Pauli gate (X, Y, Z)
  - Two-qubit gate: CNOT, Swap
  - Multi-qubit gate: some special gates.
- Projective Matrix: Quantum Measurements
  - Core: Z measurement
  - Composite measurement
  - Partial measurement