

Homework 0 of CS 165A (Spring 2023)

University of California, Santa Barbara

To be discussed on Apr 5, 2023 (Wednesday)

Notes:

- The homework is optional. You do not need to submit your solutions anywhere and you will not be evaluated by these.
 - To maximize your learning, you should try understanding the problems and try solving them as much as you can before the discussion class.
 - Feel free to discuss with your peers / form small groups to solve these problems.
 - Feel free to discuss any questions with the instructor and the TA in office hours or on Piazza.
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1 Why should I do this homework?

On the technical level, artificial intelligence is all about designing an agent to “optimize” something. For example, if we are designing a robot cleaner, the goal is to “keep the room clean as much as possible” while “not wasting too much energy”. It is easy to say these two goals to any human beings and we will understand them intuitively, but an agent — a computer program — will require us to define these goals more precisely in a way a computer will understand and know that to do. Specifically, we need to describe these goals using mathematical languages and translate them into codes. This is the job of an AI engineer and being able to do this when presented with a problem is the overarching learning goal of this course.

Since the world is complex (involving many variables), uncertain and probabilistic, such formalization will require us to use elegant notations and languages from linear algebra, probability theory and statistics so that we can express these ideas concisely. In this homework, we will go over the basic notations to help you remember things you learned from your calculus / linear algebra / probability courses (**Problem 1**), and for you to learn basic tools for python / Numpy programming (**Problem 2**).

Note that if you have trouble solving some of the problems, don't worry! That is what this homework is for. Try your best and come to the discussion class with questions. We will teach you.

2 Standard notations

Before starting on the questions, you may wish to review the following standard notations. You can also come back to these whenever you find equations / symbols that you don't understand.

Some standard notations in optimization: $\max, \min, \arg \max, \arg \min$ are commonly used when working with optimizations programs. The subscripts of these operators denote the “argument / variable” that you are optimizing over. For example, in

$$\min_{\theta \in \Theta} f(\theta),$$

f is the objective function or criterion function, θ is the argument or variable that you are optimizing over. Θ is the domain that you can choose θ from (sometimes abbreviated when it is the whole space or clear from context). $\min_{\theta \in \Theta} f(\theta)$ returns the minimum objective function value; $\arg \min_{\theta} f(\theta)$ returns the argument θ^* that achieves the minimum function value.

For example, let's say you are running a convenience store. θ is how much inventory of each item you keep, $f(\theta)$ is the expected business cost minus revenue each month, and Θ describes the constraints, such as size of your storage space, available fund and so on. Your goal is to optimize your inventory so as to maximize your net profit.

Some standard notations in linear algebra: We will denote scalars and vectors by lower case letters, e.g., x, a, v, θ . Whether they are scalars or vectors (a vector is just an array of scalars) are usually clear from context and should be specified by, e.g., $x \in \mathbb{R}^d$ — indicating that $x = [x_1, x_2, \dots, x_d]^T$. The $(\cdot)^T$ denotes the transpose, which makes x a column vector and \mathbb{R} is the space of real numbers, indicating that each entry of x is numerical. x_i denotes the i th element of the vector x in the above, but sometimes it could denote the i th data point in a dataset $\{x_1, \dots, x_n\}$ where each $x_i \in \mathbb{R}^d$ is a d -dimensional vector. We use capital letters, e.g., X, A, Θ to denote matrices. For example, the above dataset can be represented by a matrix of size $d \times n$, i.e., $X \in \mathbb{R}^{d \times n}$ where the i th column of matrix X is the i th data point x_i . When we write $X\Theta$ it refers to matrix multiplication and it is always a good idea to do a dimension check to see that the matrix multiplication is compatible. For example, if $x, a \in \mathbb{R}^d$ we can write $x^T a$ which will return a scalar, because we are multiplying a matrix of $1 \times d$ to a matrix of size $d \times 1$. In comparison, xa is not a valid notation because the inner dimension does not match between the two matrices.

Python codes for most AI agents are usually direct translation of these matrix and vector manipulations. Being able to write down what you are going to code in linear algebraic notations will help you to write efficient, elegant and bug-free code. It will also allow you to more easily learn how to use popular AI software such as sklearn, tensorflow or pytorch.

3 Homework problems

Problem 1. Refreshers on Optimization and probability fundamentals.

- (a) (Continuous optimization) Let x_1, \dots, x_n be real values. Consider a quadratic function $f(\theta) = \sum_{i=1}^n w_i (x_i - \theta)^2$. Assume $w_i > 0$. Derive the optimal solution θ^* that minimizes $f(\theta)$ - denoted by $\theta^* = \arg \min_{\theta} f(\theta)$? What happens if some w_i are negative?

- (b) (counting and combinatorics) Suppose we have a set of 8 AI researchers, and we want to form a committee of 4 members. However, two researchers, Alice and Bob, cannot be part of the same committee due to a conflict of interest. How many ways can the committee be formed under this constraint?
- (c) (Bayes rule) In answering a question on a multiple choice test, a candidate either knows the answer with probability p ($0 \leq p < 1$) or does not know the answer with probability $1 - p$. If she knows the answer, she puts down the correct answer with probability 0.99, whereas if she guesses, the probability of his putting down the correct result is $1/k$ (k choices to the answer). Find the conditional probability that the candidate knew the answer to a question, given that she has made the correct answer.
- (d) (Calculus, gradients) Let $w \in \mathbb{R}^d$ be a column vector. Let $x_1, x_2, \dots, x_n \in \mathbb{R}^d$ be column vectors of the same dimension d and $y_1, \dots, y_n \in \mathbb{R}$ be scalars. Let $\lambda \in \mathbb{R}$ be a non-negative scalar value. Consider function $F(w) = \sum_{i=1}^n (x_i^T w - y_i)^2 + \lambda \sum_{i=1}^d w_i^2$. Calculate the gradient of F . Recall that the gradient with respect to a vector of variables is the vector of partial derivatives with respect to each variable w_i :

$$\nabla f(w) = \left[\frac{\partial F(w)}{\partial w_1}, \dots, \frac{\partial F(w)}{\partial w_d} \right]^T \in \mathbb{R}^d.$$

(Hint: It is easier to work out the partial derivatives one at a time, then concatenate the partial derivatives into a gradient by applying the definition above. The same hint also applies to other instances where you need to calculate the gradient.)

- (e) (Chain-rule and softmax function) Let x_1, \dots, x_n be real values. Let

$$f(x_1, \dots, x_n) = \log \sum_{i=1}^n \exp(x_i).$$

This is called the **softmax** function or the **log-sum-exp** function¹ Calculate the gradient of f w.r.t. vector $x = (x_1, \dots, x_n)^T$.

- (f) (Simple mathematical proof) For the soft-max function in part (f). Prove that $\max_i x_i \leq f(x_1, \dots, x_n) \leq \max_i(x_i) + \log n$.

(Hint: A good practice when writing proof is to write a sequence of inequalities and explain every line in the following form:

“ $f(x_1, \dots, x_n) = \log \sum_{i=1}^n \exp(x_i) \leq \dots \leq \dots \leq \max_i(x_i) + \log n$. The first inequality is by definition, the second inequality is because ..., the third inequality is because ... ”)

¹Note that this softmax function is a scalar function $\mathbb{R}^n \rightarrow \mathbb{R}$ and it is different from the “softmax” transformation typically used in machine learning to convert any score vector to a probability vector, i.e., a vector valued function from $\mathbb{R}^n \rightarrow \mathbb{R}^n$. The latter is a misnomer but has a wide-spread misuse to the point that it becomes a convention. We will refer to the **softmax transform**, i.e., $F(x_1, \dots, x_n) = \frac{[e^{x_1}, \dots, e^{x_n}]}{\sum_{i=1}^n e^{x_i}}$, by **soft-argmax**, because it returns a probability distribution vector that approximates the one-hot representation of an **argmax** output.

Problem 2. Refreshers on time complexity, data structure and python / numpy. The autograder uses python3.7, please write codes for python3.

For easy prototyping, it is a good idea to install “Jupyter Notebook” so that you can easily debug your code.

- (a) In `numpy`, generate two matrices A and B with size 5 by 4 and size 4 by 3 respectively. In matrix A , make sure that the values are 1, 2, 3, ..., 19, 20, from left-to-right then top to bottom. In matrix B , make sure that the values are 1, 2, 3, ..., 11, 12 from top to bottom, then left to right. Write a function that takes matrix A , matrix B as inputs and return the matrix product of AB .

(Hint: Note that $A@B$ denotes matrix multiplication, while $A*B$ aims at doing pointwise multiplication. To make it more explicitly, you can use `numpy.dot` for matrix multiply.)

- (b) (Sparse matrix-vector multiplication) What is the worst case time complexity (in Big O notation) of multiplying a matrix A of dimension $\mathbb{R}^{n \times n}$ with a dense vector $v \in \mathbb{R}^n$? What if matrix A is sparse, denote the number of non-zero elements by $\text{nnz}(A)$? Write a python function that takes a nonnegative integer n and outputs a sparse matrix A of size $(n-1) \times n$, such that for any $x \in \mathbb{R}^n$, $Ax = [x_1 - x_2, \dots, x_{n-1} - x_n]^T$.

(Hint: you can use `numpy.array` to represent the vector x , but use the sparse matrix library in python (`scipy.sparse`) to construct matrix A . In this way, your code will take $O(n)$ rather than $O(n^2)$ time and space.)

- (c) (Manipulating text using python, data structures) Write a python function to take a data file (`data_example.txt`) and return the unique words and the corresponding number of times they each appeared in the file. Each line of the data file (a plain text file) contains a paragraph of text. What are the data-structures you used and what is the time-complexity of your function in Big O notation. What is the space complexity of your code?

(Hint: You first need to read the text file line by line. See <https://stackoverflow.com/questions/3277503/how-to-read-a-file-line-by-line-into-a-list> for an example. The for each line, you should split them into words. Finally, you should try finding the unique words that appeared as well as the number of times they appeared. among the build-in data structures, eg., list, dict, set, etc., which one best suits this problem?)

- (d) (Recursion with bookkeeping in python) The Fibonacci numbers are a sequence of the form $[0, 1, 1, 2, 3, 5, 8, \dots]$, namely, each number is the sum of two numbers before it. Implement a *recursive* python function with a name “fibonacci” that takes a positive integer n and outputs the n th Fibonacci number. A naive recursion-based implementation will have exponential time, but we can keep a global variable (a dictionary) that keeps the result of all input $m \leq n$ so each one of them is computed at most once. What is the time complexity of this version of recursion with book keeping?