

Lecture 11: Intro to Online Learning

Batch Statistical Learning $(x_1, y_1) (x_2, y_2) \dots (x_n, y_n) \cup I$

Realizable or $x \in I, y = h^*(x)$

$\mathcal{X} \rightarrow \{0, 1\}$

$h \in \mathcal{H}$ Hypothesis class

Setting

Goal find h with low error $err_{\mathcal{D}}(h) = E_{x \sim \mathcal{D}} [h(x) \neq h^*(x)] \rightarrow 0$

Online Learning you see the data ~~set~~ point one at a time $(x_1, y_1) \dots (x_n, y_n)$

Alg: $h_1 = \dots$
 $h_2 \in (x_1, y_1), x_2$
 $h_3 \in (x_1, y_1) \cup (x_2, y_2), x_3$
 \vdots
 $h_n \in (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), x_n$

How to evaluate this alg?

Mistake bound. M

Alg A learns class \mathcal{H} if A makes $\leq M$ mistakes on any sequence of examples consistent with $h \in \mathcal{H}$

Example: $\mathcal{X} = \{0, 1\}^d$

\mathcal{H} : the class of all disjunctions $\{x[5] \text{ or } x[8] \text{ or } x[16]\}$

Alg: ERM

1 $V_1 = \mathcal{H}$

$t=2, 2, \dots$

Receive x_t , pick any $h \in V_1$

Predict $\hat{y}_t = h(x_t)$

Receive $y_t = h^*(x_t)$

Update $V_{t+1} = \{h \in V_t : h(x_t) = y_t\}$

\exists sequence of x_t, h^* and sequence of

consistent h sets $M_{ERM} = |I| - 1$
 (ERM)

Bad!

Alg: Halving

1 $V_1 = \mathcal{H}$

$t=2, 2, \dots$

Receive x_t

V_{t+1} using all $h \in V_t$

$y_t = \arg \max_{h \in V_t} \{h(x_t) = y_t\}$

Receive y_t

Update $V_{t+1} = \{h \in V_t : h(x_t) = y_t\}$

Observation: Each trader who makes

Profit: At least $\frac{|V|}{2}$ hyp-tests are wrong

$$| \leq |V_{T+1}| \leq \frac{|V|}{2^M} \Rightarrow M \leq (\log_2 |V|)$$

Is this optimal? Consider a space with n outcomes. M bits of information to describe the unique hypothesis among n

What if Not realizable? Regret = $\sum_{t=1}^n (h_t(x_t) \neq y_t) - \min_{h \in H} \sum_{t=1}^n 1(h_t(x_t) \neq y_t)$

Example - Stock prediction with Experts

	Exp 1 (chay)	Exp 2 (Yugis)	Exp 3 (Omad)	Exp 4 (Oad Paul) the occupy	Out come
Day 1	Down	Up	UP	Down	Down
Day 2	UP	UP	Down	Down	Down
Day 3	UP	Down	UP	UP	UP

Why Weighted Majority:

Weight	Day 1	Day 2	Day 3	Day 4	Day 5
1	1	1	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	0	0
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	0	0

Change Day 1
Choose UP

Claim

Proof: $M = \#$ of mistakes so far

$m = \#$ of mistakes best expert made so far

$W =$ total weight ~~at~~ (state at ~~end~~)
 $n = \#$ of experts

Every mistake \Rightarrow W drops at least 25% (why?)

Weight of best expert is $(1/2)^m$

$$So \quad \left(\frac{1}{2}\right)^m \leq \frac{1}{n} \left(\frac{3}{4}\right)^M, \quad \left(\frac{4}{3}\right)^M \leq \frac{1}{n} 2^m$$

$$\text{take } \log \quad M \leq 2.4(m + \log n)$$

constant ratio (still, pretty good) if M is O the problem n school

Idea: use $1-\epsilon$ instead of $1/2$, choose ϵ , carefully

$$(1-\epsilon)^m \leq \frac{1}{n} \left(1 - \frac{1}{2}(1-\epsilon)\right)^M$$

take log move things around

$$M \leq \frac{m \log(1-\epsilon) + \log n}{\log\left(1 - \frac{1}{2}(1-\epsilon)\right)}$$

choose $\epsilon = \frac{\sqrt{\log n}}{M}$

$$M \leq m + O\left(\frac{M \log n}{\sqrt{\log n}}\right) \leq M + O(\sqrt{\log n})$$

$$m \log(1-\epsilon) \leq \log n + M \log\left(\frac{1}{2} + \frac{\epsilon}{2}\right)$$

$$M \leq \frac{\log(1-\epsilon)}{\log\left(\frac{1}{2} + \frac{\epsilon}{2}\right)} m + \frac{\log n}{\log\left(\frac{1}{2} + \frac{\epsilon}{2}\right)}$$

$$\leq \frac{2}{\epsilon} m + O(\log n)$$

constant

Randomized Weight Majority

1. Set weight $w_i^{(1)} = 1$ for all i

2. At time t , output y_t with probability $\frac{\sum_{i=1}^n w_i^{(t)} \mathbb{1}(y_i = 1)}{\sum_{i=1}^n w_i^{(t)}}$
 0 otherwise

3. $w_i^{(t+1)} \leftarrow w_i \cdot (1 - \epsilon)$ if i makes a mistake.

Analysis At time t , F_t is the fraction of experts that make a mistake

$$W = \prod_{t=1}^T (1 - \epsilon F_t)$$

$$\log(W_{\text{final}}) = \log n + \sum_{t=1}^T \log(1 - \epsilon F_t) \leq \log n - \epsilon \sum_{t=1}^T F_t$$

$$W_{\text{final}} = (1 - F_t) W_t + (\epsilon F_t) W_t$$

When there is a mistake

$$= (1 - \epsilon) W_t$$

$$(\log(1-x) \leq -x)$$

$$\sum_{t=1}^T F_t = \left[\frac{\log(W_{\text{final}})}{\log(1-\epsilon)} \right] \approx \frac{\log(W_{\text{final}})}{\epsilon}$$

$$\log(W_{\text{final}}) \geq \log((1-\epsilon)^M)$$

$$\log(n(1-\epsilon)^M) \geq M \log(1-\epsilon)$$

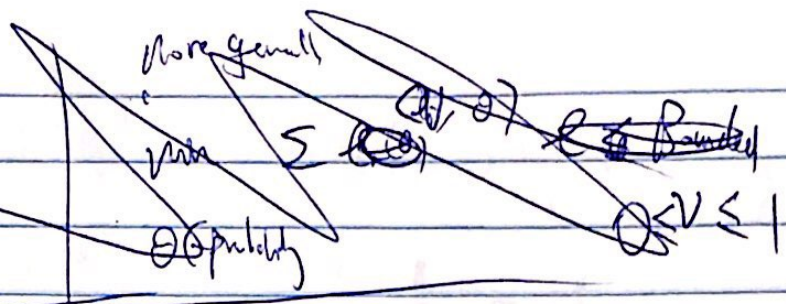
$$M \leq \frac{\log(n(1-\epsilon)^M)}{\log(1-\epsilon)} \approx \frac{\log n}{\epsilon} + \frac{M}{2}$$

$$= \text{OPT} + \epsilon M + \frac{\log n}{\epsilon}$$

$$\text{Regret} = \sum_{t=1}^T \log n \leq \sqrt{T \log n}$$

~~Optimization, Value~~

~~$\min_{\theta} \sum f_i(\theta)$~~
 ~~$\theta \in \text{probability}$~~



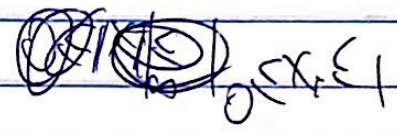
Hedge alg

- 1) Pick expert i with prob $\propto \frac{W_{i,t}}{\sum W_{i,t}}$
- 2) incur loss $\ell_{i,t}$
- 3) $W_{i,t+1} = W_{i,t} e^{-\epsilon \ell_{i,t}}$ for all $i \in [n]$

same analysis

Optimization View $\min_{\theta} \sum f_i(\theta)$ online f_i is linear $\langle x_t, \theta \rangle$
 $\theta \in \text{probability}$

- 1) Make decision θ
- 2) Receive feedback
- 3) incur loss



More generally f_i is convex? f_i is strongly convex?

θ is ~~convex~~ general convex set?
all low rank matrices?

Other Convex Options

- ~~adversarial~~ learn your (w. disturbance assumption)
- Do not need trans. data. learn & make dec. on the fly
- Usually come with theoretical guarantees

More work due on Thursday