Lecture 12 Online (Projected) Gradient Descent

$$\text{regret}_T(A) = \sup_{\{f_1,\cdots,f_T\}\in F} \left\{ \sum_{t=1}^T f_t(x_t) - \min_{x\in K} \sum_{t=1}^T f_t(x) \right\}$$

Player declare algorithm A
Adversary choose $f_1 \cdots f_T$
for $t = 1, 2, \cdots T$:
1. player plays $x_t \in K$
2. incur a loss of $f_t(x_t)$
3. Receive feed back
   - $\nabla f_t(x_t)$ ← Full information
   - $f_t(x_t)$ ← Bandits
   - f_t as a function (Function oracle)

Example Prediction with expert advices

$x \in \{x \mid \sum x = 1, x \geq 0\} =: \Delta_n$

$f_t(x) = \langle l_t, x\rangle = \sum_{i=1}^k x_i l_t[i] = \mathbb{E}_{a \sim x_i}[l_t[a]]$

Feed back mode: $l_t = \nabla f_t(x)$ — Full inform
only $l_t[a]$ — Bandits

Example Online linear models

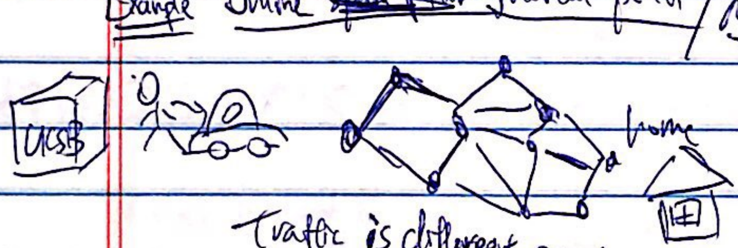
$x_t \in A \subseteq \mathbb{R}^n$, Data $\phi_t \in B \subseteq \mathbb{R}^n$

$f_t = (\langle x_t, \phi_t\rangle - y_t)^2$

or $|\langle x_t, \phi_t\rangle - y_t|$, $\text{logistic}(x_t, y_t)$

Feedback: Subgradient $2\phi_t(x_t^T\phi_t - y_t)$

Example online ~~shortest~~ shortest path / max flow



Traffic is different everyday

$x_t \in \mathbb{R}^{|E|}$ probability of taking each path.
(Flow)

$\sum_{e \in (s,w), w\in V} x_e = 1$, $\sum x_e = 1$, $e = (w,t), w \in V$

$\forall w \in V$ $w \neq s$, $w \neq t$ $\sum_{e \text{ into } w} x_e = \sum_{e \text{ out of } w} x_e$

$\forall e \in E$ $0 \leq x_e \leq 1$

$f_t = \langle x_t, w_t\rangle$ Feed Back: Bandit time on each

Example: ~~online hyperparameter tuning~~ Portfolio selection

r_t: relative ratio ≥ 0

x_t: proportion of investor's wealth $\in \Delta_n$

$\log(r_t^T x_t)$ would like to maximize it

$\max \sum_{t=1}^T \log(r_t^T x_t) - \max_{x} \sum_{t=1}^T \log(r_t^T x^*)$

Hyperparameter Tuning, Best learning Rate in the hind sight $\eta = [- \cdots]$
min

OGD: consider subgradient feedback

Input: Convex set K, Horizon T, Step sizes $\{\eta_t\}$
for $t = 1$ to T do:
play x_t observe $g_t \in \partial f_t(x_t)$
update: a. $y_t = x_t - \eta_t g_t$
$x_{t+1} = \mathrm{Proj}_K(y_t)$

Assumptions:
G-Lipschitz: $f(x) - f(y) \leq G\|x-y\|_2$
or $\|\partial f(x)\|_2 \leq G$
D-Bounded $x \in K$
$\mathrm{diag}(K) \leq D$
$x_1, x_2 \in K$
$\|x_1 - x_2\| \leq D$

Thm: OGD with $\eta_t = \frac{D}{G\sqrt{t}}$, $t = 1, 2, \dots, T$,

$$\mathrm{Regret} = \sum_{t=1}^{T} f_t(x_t) - \min \sum_{t=1}^{T} f_t(x^*) \leq \frac{3}{2} GD\sqrt{T}$$

Proof: convexity of f_t $\quad f(x^*) \geq f(x_t) + \langle g_t, x^* - x_t \rangle \Leftrightarrow f_t(x_t) - f_t(x^*) \leq \langle g_t, x_t - x^* \rangle$

$$\|x_{t+1} - x^*\|^2 = \|\Pi_K(x_t - \eta_t g_t) - x^*\|^2 \underset{\substack{\uparrow \\ \text{nonexpansiveness} \\ \text{of projection}}}{\leq} \|x_t - \eta_t g_t - x^*\|^2 = \|x_t - x^*\|^2 + \eta_t^2 \underbrace{\|g_t\|^2}_{\leq G^2} - 2\eta_t \langle g_t, x_t - x^* \rangle$$

$$2\,\mathrm{Regret}(T) \leq 2\langle g_t, x_t - x^* \rangle \leq \frac{1}{\eta_t}\left(\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2\right) + \eta_t \|g_t\|^2$$

Telescope by summing up: $$2\sum_{t=1}^{T} \left(f(x_t) - f(x^*)\right) \leq \sum_{t=1}^{T} \frac{\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2}{\eta_t} + \sum \eta_t G^2$$

$$= \sum_{t=1}^{T} \|x_t - x^*\|^2 \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}\right) + \sum_{t=1}^{T} \eta_t G^2$$

$$\leq D^2 \sum_{t=1}^{T} \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}\right) + \sum_{t=1}^{T} \eta_t G^2 = D^2 \frac{1}{\eta_T} + G^2 \sum_{t=1}^{T} \eta_t$$

$$= DG\sqrt{T} + 2DG\sqrt{T}$$

$\frac{1}{1} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{T}}$

$= ?$

$\sqrt{T}\left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{T}} + \dots + \frac{1}{T}\right)$

① Induction

② $\sum_{t=1}^{T} \frac{1}{\sqrt{t}} \leq \int_0^T \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_0^T = 2\sqrt{T}$

What if we choose a constant learning rate $\eta_t = \frac{D}{G\sqrt{T}}$

$$\frac{D^2}{\eta} + T\eta G^2 = 2DG\sqrt{T}$$

$\boxed{\text{Constant from } 1.5 \to 1}$

Strongly convex case:

$$\boxed{\underline{\text{Thm}}, \quad \text{Regret} \leq \frac{G^2}{2m}(1+\log T) \quad f_t \text{ is } m\text{-Strongly convex} \quad \text{Take } \eta_t = \frac{1}{\alpha t}}$$

Proof,
$$2(f_t(x_t) - f_t(x^*)) \leq 2 g_t^T (x_t - x^*) \underbrace{- m\|x^* - x_t\|^2}_{\text{The new thing}} \qquad (1)$$

$$\|x_{t+1} - x^*\|^2 = \|\Pi_{\mathcal{X}}(x_t - \eta_t g_t) - x^*\|^2 \leq \|x_t - \eta_t g_t - x^*\|^2$$

$$2 g_t^T (x_t - x^*) \leq \frac{\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2}{\eta_t} + \eta_t G^2 \qquad (2)$$

Sum (1) for all $t = 1 \cdots T$, and apply (2),

$$2\sum_t f_t(x_t) - f_t(x^*) \leq \sum_{t=1}^{T} \|x_t - x^*\|^2 \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} - \alpha\right) + G^2 \sum_{t=1}^{T} \eta_t$$

Take $\eta_t = \frac{1}{\alpha t}$

$$2\,\text{Regret} \leq \sum_{t=1}^{T} \|x_t - x^*\|^2 \cdot \underbrace{(t\alpha - (t-1)\alpha - \alpha)}_{0} + \frac{G^2}{\alpha} \sum_{t=1}^{T} \frac{1}{t}$$

$$\leq \frac{G^2}{\alpha} \frac{(1+\log(T))}{2}$$

Subgradient Descent (Strongly convex)

output $\bar{x}$, $f(\bar{x}) - f(x^*) \leq \frac{1}{T}\sum_{t=1}^{T} f(x_t) - f(x^*) \leq \frac{1}{T}\sum_t \langle g_t, x_t - x^* \rangle$

$$= \frac{1}{T}\sum_t \langle E[\tilde{g}_t | x_t], x_t - x^* \rangle = E\left[\frac{1}{T}\sum_t \langle \tilde{g}_t, x_t - x^* \rangle\right]$$

$$\leq \frac{1}{T}\left(\sum_t f_t(x_t) - f_t(x^*)\right) \leq \frac{1}{T}\text{Regret}$$

Online to Batch Conversion. How does low Regret algorithm solve Statistical Learning

$z_1 \dots z_n \sim D$, $f_1 \dots f_n$

$Risk(x) = E_{z \sim D} f_z(x)$

Condition on $z_1 \dots z_{t-1}$, x_t is evaluation for day.

$$E\left[f_t(x_t)\right] = E_{f(t-1)}[f_t(x_t)] = Risk(x_t)$$

$$Risk(\bar{x}) \le \frac{1}{n} \sum_{t=1}^{n} Risk(x_t) \le Risk(x^*) + \frac{E(Regret)}{m}$$

High probability Bounds

$$Risk(\bar{x}) \le Risk(x^*) + \frac{1}{m} E(R_t) + 2\sqrt{\frac{\log \frac{1}{\delta}}{n}}$$ with prob $1-\delta$

Use Azuma-Hoeffding's lemma