Near Optimal Provable Uniform Convergence in Off-Policy Evaluation for Reinforcement Learning

Ming Yin$^{1,3}$, Yu Bai$^2$, and Yu-Xiang Wang$^3$

$^1$Department of Statistics and Applied Probability, UC Santa Barbara
$^2$Salesforce Research
$^3$Department of Computer Science, UC Santa Barbara
ming.yin@ucsb.edu yu.bai@salesforce.com yuxiangw@cs.ucsb.edu

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Abstract

The Off-Policy Evaluation (OPE) aims at estimating the performance of target policy $\pi$ using offline data rolled in by a logging policy $\mu$. Intensive studies have been conducted and the recent marginalized importance sampling (MIS) achieves the sample efficiency for OPE. However, it is rarely known if uniform convergence guarantees in OPE can be obtained efficiently. In this paper, we consider this new question and reveal the comprehensive relationship between OPE and offline learning for the first time. For the global policy class, by using the fully model-based OPE estimator, our best result is able to achieve $\epsilon$-uniform convergence with complexity $\tilde{O}(H^3 \cdot \min(S,H)/d_m\epsilon^2)$, where $d_m$ is an instance-dependent quantity decided by $\mu$. This result is only one factor away from our uniform convergence lower bound up to a logarithmic factor. For the local policy class, $\epsilon$-uniform convergence is achieved with the optimal complexity $\tilde{O}(H^3/d_m\epsilon^2)$ in the off-policy setting. This result complements the work of sparse model-based planning (Agarwal et al., 2019) with generative model. Lastly, one interesting corollary of our intermediate result implies a refined analysis over simulation lemma.
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1 Introduction

Reinforcement learning (RL), is the problem described by an agent interacting with an environment in order to maximize its cumulative rewards through time (Sutton & Barto, 2018). Among the great landscapes of reinforcement learning, off-policy evaluation (OPE) refers to the problem of predicting the performance of a policy with data only collected by a logging/behavioral policy, and is of crucial importance to real-world applications of RL including marketing (Thomas et al., 2017), targeted advertising (Bottou et al., 2013; Tang et al., 2013), finance (Bertoluzzo & Corazza, 2012), robotics (Quillen et al., 2018), and healthcare (Ernst et al., 2006; Raghu et al., 2017, 2018). A central challenge in OPE is the distributional mismatch between the behavioral policy and the target policy, which has been tackled in previous studies using Importance Sampling (IS) based methods (Li et al., 2011; Dudík et al., 2011; Li et al., 2015; Thomas & Brunskill, 2016) or its hybrid versions such as doubly robust estimators (Jiang & Li, 2016; Farajtabar et al., 2018). More recently, a family of estimators based on marginalized importance sampling (MIS) (Liu et al., 2018; Xie et al., 2019; Kallus & Uehara, 2019a,b; Yin & Wang, 2020) have been proposed in order to overcome the “curse of horizon”, which refers to the phenomenon of OPE problem that any unbiased estimator has to suffer the variance which is exponential in horizon for some MDP class (Jiang & Li, 2016; Liu et al., 2018).

However, previous works only consider the OPE problem of a fixed (non data-dependent) target policy π, whereas in practice it is common that we need to evaluate the performance of a data-dependent one. For example, when the target policy is learned from a learning algorithm on a training data D while the evaluation data D1 is not independent of D. Under these situations, statistical tools used in analyzing a fixed target policy π may fail to work well since the statistical independence structure is likely to break down due to the dependence between target policy and evaluation data D1. This phenomenon motivates us to pursue estimators that are amenable to any policy inside certain policy class rather than just a fixed policy. Moreover, in the learning regime, policy evaluation usually serves as the intermediate step (see (Levine et al., 2020) for a tutorial about offline learning), but explicit characterization of the relationship between OPE and offline learning has not been considered yet. Therefore, it is natural to ask the following two questions:

1. **Is sample efficiency possible for uniform convergence in OPE?**
2. **Can we somehow connect OPE with offline learning through a unified perspective?**

In this paper, we present positive answers for both questions. In particular, we consider uniform convergence/local uniform convergence in the OPE problem in the finite horizon setting. Roughly speaking, given a policy class II and a logging policy μ, we want to study how many episodes do we need (from μ) in order to let the OPE estimator \( \hat{v}^{\pi} \) satisfy the
\( \epsilon \)-uniform convergence
\[
\sup_{\pi \in \Pi} |\hat{v}^\pi - v^\pi| \leq \epsilon.
\]

We point out even though uniform convergence is a well studied task in numerous machine learning problems, it has been rarely explored for off-policy evaluation. Moreover, uniform convergence in OPE has a natural implication in offline learning.

**Our contribution.** In reinforcement learning, we study the problem of uniform convergence in off-policy evaluation with finite horizon episodic setting from theoretical perspective. Our contribution can be summarized as follows:

- Perhaps the most paramount significance of this work is the raise of uniform convergence OPE problem. This is because, as we will show, uniform convergence OPE is much stronger argument than ordinary OPE therefore it is generically harder. Efficient uniform convergence OPE guarantees two things: efficient ordinary OPE and the same sample efficiency for the implied *offline learning* algorithm. This provides a unified view of two important tasks.

- For the global policy class (includes all (non-stationary) deterministic policies), we use fully model-based OPEMA estimator to prove two results in parallel. The first result provides \( \hat{O}(H^4/d_m\epsilon^2) \) episode complexity and the second result has \( \hat{O}(H^3S/d_m\epsilon^2) \) episode complexity, where \( d_m \) is minimal marginal state-action probability depend on logging policy \( \mu \). Combing them together we end up with \( \hat{O}(H^3 \cdot \min(S, H)/d_m\epsilon^2) \) for achieving \( \epsilon \)-uniform convergence.

- For a local policy class with all policies are in the neighborhood of the empirical optimal policy (all concrete definitions can be found in Section 2.1), we derive \( \epsilon \)-uniform convergence with sample complexity \( \hat{O}(H^3/d_m\epsilon^2) \), which is essentially optimal (Theorem 3.7).

- Towards investigating the optimality of our upper bounds, we prove a information-theoretical lower bound of \( \Omega(H^3SA/\epsilon^2) \), this shows our result for global case is near optimal.

- Another main take-away is above results provide direct sample efficient *offline learning* algorithm with the same complexity respectively, assuming the optimal policy is within the class. Those results nearly match the lower bounds in the existing *policy learning* literature.

- Last but not least, our analysis reveals without complex construction, simple model plug-in estimator OPEMA is able to achieve optimality. This analysis help to correct a commonly held misunderstanding that purely model plug-in estimator can only provide loose theoretical complexity due to *simulation lemma*.
To the best of our knowledge, this is the first work that derives such uniform convergence results for off-policy evaluation in reinforcement learning.

**Related work.** Importance sampling estimators, originated from Inverse Propensity Scoring (IPS) in statistics (Horvitz & Thompson, 1952; Powell & Swann, 1966), serves as the mainstream tool for handling off-policy evaluation since Precup et al. (2000) as it helps correct the mismatch between the distributions of the behavior policy and target policy. Later, plenty of variant estimators of IS are invented, *e.g.* Weighted Importance Sampling (WIS), Per-Decision Importance Sampling (PDIS). Typically, importance sampling methods usually possess the property of unbiasedness and often work well in the problem with short horizon like contextual bandits (Wang et al., 2017; Kallus & Uehara, 2019c). Doubly robust estimators usually couple direct fitting/regression methods with importance sampling methods to try to keep the unbiasedness and lower the variance at the same time, *e.g.* (Dudík et al., 2011; Jiang & Li, 2016; Farajtabar et al., 2018). Given its popularity, IS estimator usually suffers the variance exponential in horizon because of the cumulative weights, which is referred as “the curse of horizon” (Liu et al., 2018).

Recently, a new family of estimators based on marginalized importance sampling (MIS) (Liu et al., 2018; Xie et al., 2019; Kallus & Uehara, 2019a,b; Uehara & Jiang, 2019) overcome the curse of horizon with mild model assumptions. In particular, Yin & Wang (2020) design the Tabular-MIS estimator which achieves the asymptotic Cramer-Rao lower bound constructed by Jiang & Li (2016) and they have a brief discussion on uniform convergence of off-policy evaluation which is part of the motivation of the current work. Essentially, the method we use, off-policy empirical model approximator (OPEMA), is consistent with Tabular-MIS, see the MIS interpretation of OPEMA in Section 2.3.

## 2 Preliminaries and Method

Reinforcement learning problem consists of an unknown environment and an agent who interacts with the underlying unknown dynamics. The environment is modeled as a *Markov Decision Process* (MDP) which is denoted by a tuple $M = (S, A, r, P, d_1, H)$. The MDP consists of a state space $S$, an action space $A$ and a transition kernel $P_t : S \times A \times S \mapsto [0, 1]$ with $P_t(s'|s, a)$ representing the probability transition from state $s$, action $a$ to next state $s'$ at time $t$. In particular here we consider non-stationary transition dynamics as the subscript $t$ in $P_t$ indicates that the transition dynamic at different time step could be different. Besides, $r_t : S \times A \mapsto \mathbb{R}$ is the expected reward function and given $(s_t, a_t)$, $r_t(s_t, a_t)$ specifies the average reward obtained at time $t$. $d_1$ is the initial state distribution and $H$ is the horizon. Moreover, we focus on the case where state space $S$ and the action space $A$ are finite, *i.e.* $S := |S| < \infty, A := |A| < \infty$. A (non-stationary) policy
is formulated by \( \pi := (\pi_1, \pi_2, \ldots, \pi_H) \), where \( \pi_t \) assigns each state \( s_t \in S \) a probability distribution over actions at each time \( t \) (i.e. \( \pi_t(s_t) \in \mathbb{R}^{|A|} \) and \( \sum_{a_t} \pi_t(a_t|s_t) = 1 \)). Any fixed policy \( \pi \) together with MDP \( M \) induce a distribution over trajectories of the form \( (s_1, a_1, r_1, s_2, \ldots, s_H, a_H, r_H, s_{H+1}) \) where \( s_1 \sim d_1, a_t \sim \pi_t(\cdot|s_t), s_{t+1} \sim P_t(\cdot|s_t, a_t) \) and \( r_t \) has mean \( r_t(s_t, a_t) \) for \( t = 1, \ldots, H. \)

In addition, we denote \( d_t^\pi(s_t, a_t) \) the induced marginal state-action distribution and \( d_t^\pi(s_t) = d_t^\pi(s_t, a_t) = d_t^\pi(s_t) \cdot \pi(a_t|s_t) \). Moreover, \( d_t^\pi = d_1 \forall \pi \). We use the notation \( P_t^\pi \in \mathbb{R}^{S \times A} \) to represent the state-action transition \( (P_t^\pi)_{(s,a),(s',a')} := P_t(s'|s, a)\pi_t(a'|s'), \) then the marginal state-action vector \( d_t^\pi(\cdot, \cdot) \in \mathbb{R}^{S \times A} \) satisfies the expression \( d_{t+1}^\pi = P_{t+1}^\pi d_t^\pi \). We define the quantity \( v^\pi(s) = \mathbb{E}_\pi[\sum_{t=1}^H r_t|s_t = s] \) and the Q-function \( Q_t^\pi(s, a) = \mathbb{E}_\pi[\sum_{t'=t}^H r_{t'}|s_t = s, a_t = a] \) for all \( t = 1, \ldots, H. \) The ultimate measure of the performance of policy \( \pi \) is the value function:

\[
  v^\pi = \mathbb{E}_\pi \left[ \sum_{t=1}^H r_t \right] .
\]

Lastly, for the standard OPE problem, the goal is to estimate \( v^\pi \) for a given \( \pi \) while assuming that \( n \) episodic data \( D = \{(s_t^{(i)}, a_t^{(i)}, r_t^{(i)}, s_{t+1}^{(i)})\}_{i=1}^n \) are rolling from a different behavior policy \( \mu. \)

### 2.1 Uniform convergence problems

Uniform convergence OPE extends the ordinary OPE concept to arbitrary policy class. Specifically, for any policy class \( \Pi \) that is of interest, we want to ask can we obtain

\[
  \sup_{\pi \in \Pi} |\hat{v}^\pi - v^\pi| < \epsilon,
\]

in a sample efficient way. Here we consider two representative classes.

**The global policy class.** The policy class \( \Pi \) we considered here consists of all the non-stationary deterministic policies. By the standard results in tabular reinforcement learning, there exists at least one deterministic policy that is optimal (Sutton & Barto, 2018). Therefore, the deterministic policy class is rich enough for evaluating any learning algorithm (e.g. Q-value iteration in (Sidford et al., 2018)) that wants to learn to the optimal policy.

**The local policy class: in the neighborhood of empirical optimal policy.** Given empirical MDP \( \hat{M} \) (i.e. the transition kernel is replaced by \( \hat{P}_t(s_{t+1}|s_t, a_t) := n_{s_{t+1}, s_t, a_t}/n_{s_t, a_t} \)

\(^1\)Here \( r_t \) without any argument is random reward and \( \mathbb{E}[r_t|s_t, a_t] = r_t(s_t, a_t) \).
if \( n_{s_t, a_t} > 0 \) and 0 otherwise, where \( n_{s_t, a_t} \) is the number of visitations to \((s_t, a_t)\) among all \( n \) episodes\(^2\), it is convenient to learn the empirical optimal policy \( \hat{\pi}^* := \arg\max_{\pi} \hat{v}^\pi \) since the full empirical transition \( \hat{P} \) is known. Standard methods like Policy Iteration (PI) and Value Iteration (VI) can be leveraged for finding \( \hat{\pi}^* \). This observation allows us to consider the following interesting policy class: 
\[
\Pi_1 := \{ \pi : s.t. ||\hat{v}_t^\pi - \hat{v}_{t}^{\hat{\pi}^*}||_\infty \leq \epsilon_{opt}, \forall t = 1, ..., H \}
\]
with \( \epsilon_{opt} \geq 0 \) a pre-specified parameter. Here we consider \( \hat{\pi}^* \) (instead of \( \pi^* \)) since by defining with empirical optimal policy, we can use data \( D \) to really check class \( \Pi_1 \), therefore this definition is more practical.

### 2.2 Assumptions

Next we present some mild necessary regularity assumptions for uniform convergence OPE problem.

**Assumption 2.1** (Bounded rewards). \( \forall t = 1, ..., H \) and \( i = 1, ..., n \), \( 0 \leq r_t^{(i)} \leq 1 \).

This is not particular to uniform convergence OPE problem but a common assumption made for RL problems.

**Assumption 2.2** (Exploration requirement). Logging policy \( \mu \) obeys that \( \min_{t, s_t} d_{\mu}^t(s_t) > 0 \), for any state \( s_t \) that is “accessible”. Moreover, we define quantity \( d_m := \min\{d^\mu_t(s_t, a_t) : d^\mu_t(s_t, a_t) > 0\} \).

State \( s_t \) is “accessible” means there exists a policy \( \pi \) so that \( d^\pi_t(s_t) > 0 \). If for any policy \( \pi \) we always have \( d^\pi_t(s_t) = 0 \), then state \( s_t \) can never be visited in the given MDP. Assumption 2.2 simply says logging policy \( \mu \) have the right to explore all “accessible” states. This assumption is required for the consistency of uniform convergence estimator since we have “sup_{\pi \in \Pi}” (recall (1)). As a short comparison, off-policy learning problems (e.g. off-policy policy optimization in (Liu et al., 2019)) only require \( d^\mu_t(s_t) > 0 \) for any state \( s_t \) satisfies \( d^{\pi^*}_t(s_t) > 0 \). Last but not least, even though our target policy class is deterministic, by above assumptions the logging policy \( \mu \) is always stochastic.

### 2.3 Method: Off-Policy Empirical Model Approximator

The method we use for doing OPE in uniform convergence problem is the off-policy empirical model approximator OPEMA. OPEMA uses off-policy data to build the empirical estimators for both the transition dynamic and the expected reward and then substitute the related components in real value function by its empirical counterparts. First recall for any target policy \( \pi \), by definition: 
\[
\hat{v}^\pi = \sum_{t=1}^{H} \sum_{s_t} d^\pi_t(s_t) \sum_{a_t} r_t(s_t, a_t) \pi(a_t|s_t) = \sum_{t=1}^{H} \sum_{s_{t+1}} d^\pi_{t+1}(s_{t+1}) \sum_{a_t} d^\pi_t(s_t, a_t) r_t(s_t, a_t),
\]
where the marginal state-action transitions satisfy \( d^\pi_{t+1} = \hat{P}^\pi_{t+1} d^\pi_t \). OPEMA then directly
construct empirical estimates for \( \hat{P}_{t+1}(s_{t+1}|s_t, a_t) \) and \( \hat{r}_t(s_t, a_t) \) as:

\[
\hat{P}_{t+1}(s_{t+1}|s_t, a_t) = \frac{1}{n_{s_t,a_t}} \sum_{i=1}^{n} 1[(s_{t+1}^{(i)}, a_t^{(i)}, s_t^{(i)}) = (s_{t+1}, s_t, a_t)], \quad \hat{r}_t(s_t, a_t) = \frac{1}{n_{s_t,a_t}} \sum_{i=1}^{n} r_t^{(i)} 1[(s_{t+1}^{(i)}, a_t^{(i)}) = (s_{t+1}, s_t, a_t)],
\]

and \( \hat{P}_{t+1}(s_{t+1}|s_t, a_t) = 0 \) and \( \hat{r}_t(s_t, a_t) = 0 \) if \( n_{s_t,a_t} = 0 \) (recall \( n_{s_t,a_t} \) is the empirical visitation frequency to state-action \((s_t, a_t)\) at time \( t \)) and then the estimates for state-action transition \( \hat{P}_{\pi}^{t+1} \) is defined as: \( \hat{P}_{\pi}^{t+1}(s_{t+1}, a_{t+1}|s_t, a_t) = \hat{P}_{t+1}(s_{t+1}|s_t, a_t) \pi(a_{t+1}|s_{t+1}) \).

The initial distribution is also constructed using empirical estimator \( \hat{d}_{\pi}^1(s_1) = n_{s_1}/n \). Based on the construction, the empirical marginal state-action transition follows \( \hat{d}_{\pi}^{t+1} = \hat{P}_{\pi}^{t+1} \hat{d}_{\pi}^t \) and the final estimator for \( v_{\pi} \) is:

\[
\hat{v}_{\text{OPEMA}} = \sum_{t=1}^{H} \sum_{s_t, a_t} \hat{d}_{\pi}^t(s_t, a_t) \hat{r}_t(s_t, a_t).
\]

**Why such simple OPE method can have tight provably bound?** OPEMA is model-based method as it uses plug-in estimators \( \hat{d}_t^\pi \) and \( \hat{r}_t \) for each model components \( d_t^\pi \) and \( r_t \). It seems hard to understand why such straightforward method can provide tight bounds. Critically, the reason behind the scene comes from OPEMA has *Marginalized importance sampling* (MIS) interpretation:

\[
\hat{v}_{\text{OPEMA}} = \frac{1}{n} \sum_{i=1}^{n} \sum_{t=1}^{H} \frac{\hat{d}_{\pi}(s_t^{(i)})}{\hat{d}_t^\pi(s_t^{(i)})} \hat{r}_t^{(i)}(s^{(i)}),
\]

which is first noted by Xie et al. (2019); Yin & Wang (2020). Recent success of MIS partially explains why OPEMA could work, even for the harder uniform convergence OPE problem.

### 3 Main Results for Uniform Convergence in OPE

In this section we present our results for off-policy uniform convergence problems raised in Section 2.1. For brevity, we use \( \hat{v}_{\pi} \) to denote \( \hat{v}_{\text{OPEMA}} \) throughout the rest of paper. Proofs of all technical results are deferred to the appendix. We start with the following Lemma:

**Lemma 3.1.** For any \( 0 < \delta < 1 \), there exists an absolute constant \( c_1 \) such that when total episode \( n > c_1 \cdot 1/d_m \cdot \log(HSA/\delta) \), then with probability \( 1 - \delta \),

\[
n_{s_t,a_t} \geq n \cdot d_t^\pi(s_t, a_t)/2, \quad \forall s_t, a_t.
\]
If state \( s_t \) is not accessible, then \( n_{s_t, a_t} = d_t^D(s_t, a_t) = 0 \) so the lemma holds trivially.\(^3\) Now we define: \( N := \min_{t, s_t, a_t} n_{s_t, a_t} \), then above implies \( N \geq nd_m/2 \) (recall \( d_m \) in Assumption 2.2).

Now we aggregate only the first \( N \) pieces of data in each state-action \((s_t, a_t)\) of off-policy data \( D \) and they consist of a new dataset \( D' = \{(s_t, a_t, s_{t+1}^{(i)}, r_t^{(i)}): i = 1, ..., N; t \in [H]; s_t \in S, a_t \in A\} \), and is a subset of \( D \). For the rest of paper, we will use either \( D' \) or the original \( D \) to create \( \hat{\nu}^v \) (only for theoretical analysis purpose). Whether \( D \) or \( D' \) is used will be stated clearly in each context.

**Remark 3.2.** It is worth mentioning that when use \( D' \) to construct \( \hat{\nu}^v \), \( n_{s_t, a_t}^{D'} = N \) for all \( s_t, a_t \). Also, \( N := \min n_{s_t, a_t}^D \) (note \( n_{s_t, a_t}^D \) is the count from \( D \) itself is a random variable and in the extreme case we could have \( N = 0 \) and if that happens \( \hat{\nu}^v = 0 \) (since in that case \( \hat{P}_t \equiv 0 \) and \( \hat{d}^v_t \) is degenerated). However, there is only tiny probability \( N \) will be small, as guaranteed by Lemma 3.1.

### 3.1 Uniform convergence OPE for global policy class

We derive two results Theorem 3.3 and Theorem 3.5 in parallel using \( D' \) and \( D \) separately. Combing the best of these two we end up with complexity \( \tilde{O}(H^3 \cdot \min(S, H)/nd_m) \). We start with \( D' \).

**Theorem 3.3** (Uniform convergence bound with \( D' \)). Let \( \Pi \) consists of all deterministic policies and \( \hat{\nu}^v \) is constructed using \( D' \), then there exists an absolute constant \( c \) such that if \( n > c \cdot 1/d_m \cdot \log(HSA/\delta) \), then with probability \( 1 - \delta \), we have:

\[
\sup_{\pi \in \Pi} |\hat{\nu}^v - v^\pi| \leq c \sqrt{\frac{H^4}{d_m} \cdot \log\left(\frac{HSA}{\delta}\right)} + \sqrt{S^2 \log(SA)}
\]

In particular, if failure probability \( \delta < 1/e^{S^2} \), then for a new absolute constant \( c_1 \) we further have

\[
\sup_{\pi \in \Pi} |\hat{\nu}^v - v^\pi| \leq c_1 \sqrt{\frac{H^4}{d_m \cdot n} \log\left(\frac{HSA}{\delta}\right)}
\]

Note even though in general Theorem 3.3 implies a rate of \( \tilde{O}(\sqrt{H^4S^2/nd_m}) \), in the small failure probability regime we can further reduce it to \( \tilde{O}(\sqrt{H^4/nd_m}) \). It happens since if \( \delta < 1/e^{S^2} \), then \( \sqrt{S^2 \log(SA)} < \sqrt{\log(HSA/\delta)} \) in (4). In such case parameters \( S \) and \( A \) achieve optimal dependence, see Section 3.3. This is meaningful since we usually want to derive results with high confidence, i.e. \( \delta \) small. Moreover, exquisite readers who read our

\(^3\)In general, non-accessible state will not affect our results so to make our presentation succinct we will not mention non-accessible state for the rest of paper unless necessary.

\(^4\)Note we can do this since by definition \( N \leq n_{s_t, a_t} \) for all \( s_t, a_t \).
proofs in appendix carefully can notice we did not use \( \Pi \) is deterministic policy class. This means Theorem 3.3 actually holds uniformly for all policies, see Remark D.3 in appendix.

Next we present our results using full data \( \mathcal{D} \).

**Lemma 3.4 (Convergence for a fixed policy).** Fix any policy \( \pi \), and construct \( \hat{v}^\pi \) using \( \mathcal{D} \). Then there exists absolute constants \( c, c_1, c_2, c_3 \) such that if \( n > c \cdot 1/d_m \cdot \log(HSA/\delta) \), then with probability \( 1 - \delta \), we have:

\[
|\hat{v}^\pi - v^\pi| \leq c_1 \sqrt{\frac{H^2 \log(c_3 HSA/\delta)}{n \cdot d_m}} + c_2 \frac{H^2 \sqrt{SA} \log(c_3 H^2 S^2 A^2/\delta) \log(c_3 HSA/\delta)}{n \cdot d_m}.
\]

Note if we ignore the higher order term, our result implies sample complexity of \( \tilde{O}(H^2/d_m \epsilon^2) \) for evaluating any fixed target policy \( \pi \), stochastic or deterministic. Based on this result, we have the following second uniform convergence bound.

**Theorem 3.5 (Uniform convergence bound with \( \mathcal{D} \)).** Let \( \Pi \) consists of all deterministic policies and \( \hat{v}^\pi \) is constructed using \( \mathcal{D} \), then there exists absolute constants \( c, c_1, c_2, c_3 \) such that if \( n > c \cdot 1/d_m \cdot \log(HSA/\delta) \), then with probability \( 1 - \delta \), we have:

\[
\sup_{\pi \in \Pi} |\hat{v}^\pi - v^\pi| \leq c_1 \sqrt{\frac{H^3 S \log(c_3 HSA/\delta)}{n \cdot d_m}} + c_2 \frac{H^3 S \sqrt{SA} \log(c_3 H^2 S^2 A^2/\delta) \log(c_3 HSA/\delta)}{n \cdot d_m}.
\]

In particular, combing this with Theorem 3.3, we have when \( \delta < 1/e^2 \), \( \sup_{\pi \in \Pi} |\hat{v}^\pi - v^\pi| \leq \tilde{O}(\sqrt{H^3 \cdot \min(S, H)}/nd_m) \).

**Remark 3.6.** We point out that Yin & Wang (2020) (Section 3.2) provide a similar \( \tilde{O}(H^3 S^2 A/\epsilon^2) \) uniform convergence bound by using data-splitting estimators. Their results require the number of episodes in each splitted data \( M \) to satisfy \( \tilde{O}(\sqrt{nSA}) > M > O(HSA) \). In order to achieve data efficiency, they need \( n \approx \Theta(H^2 SA/\epsilon^2) \) and by that condition \( M \) has to satisfy \( M \approx C \cdot HSA \). However, it is not clear how large the constant \( C \) should be in order to implement data-splitting TMIS efficiently. In contrast, OPEMA provides clear formulation about how to calculate the estimator, with data to be either \( \mathcal{D} \) or \( \mathcal{D}' \). In conclusion, rigorously, their results only prove the existence of the estimator that can give \( \tilde{O}(H^3 S^2 A/\epsilon^2) \) bound and they do not provide a concrete formulation for implementation. Arbitrarily chosen \( C \) may cause harm for either loss of accuracy or efficiency.

### 3.2 Uniform convergence OPE for the local policy class

For local policy case, we use \( \mathcal{D}' \) for OPEMA estimator and we have the following main result:

\(^5\)For completeness, we include both OPEMA and the data-splitting version in appendix Section J.
**Theorem 3.7** (Optimal sample complexity for local uniform convergence). Suppose \( \epsilon_{opt} \leq \sqrt{H/S} \) and \( \Pi_1 := \{ \pi : s.t. \|\tilde{v}_t^\pi - \hat{v}_t^\star\|_\infty \leq \epsilon_{opt}, \forall t = 1, \ldots, H \} \). Then there exists constant \( c_1, c_2 \) such that for any \( 0 < \delta < 1 \), when \( n > c_1 H^2 \log(HSA/\delta)/d_m \), we have with probability \( 1 - \delta \),

\[
\sup_{\pi \in \Pi_1} \left\| \hat{Q}_1^\pi - Q_1^\pi \right\|_\infty \leq c_2 \frac{H^3 \log(HSA/\delta)}{n \cdot d_m}.
\]

This uniform convergence result is presented with \( l_\infty \) norm over \((s, a)\). A direct corollary is \( \sup_{\pi \in \Pi_1} \|\hat{v}_1^\pi - v_1^\pi\|_\infty \) achieves the same rate. Theorem 3.7 provides the sample complexity of \( O(H^3 \log(HSA/\delta)/d_m \epsilon^2) \) and the dependence of all parameters are optimal up to the logarithmic term. Note that our bound does not explicitly depend on \( \epsilon_{opt} \), which is an improvement over (Agarwal et al., 2019) as they have an additional \( O(\epsilon_{opt}/(1 - \gamma)) \) error in the infinite horizon setting. Besides, our assumption on \( \epsilon_{opt} \) is quite weak since the required upper bound is proportional to \( \sqrt{H} \). Last but not least, this result implies an \( O(\epsilon + \epsilon_{opt}) \)-optimal policy for offline/batch learning of the same order \( O(H^3 \log(HSA/\delta)/d_m \epsilon^2) \), as shown in Theorem 4.2.

### 3.3 Information theoretical lower bound for Uniform convergence OPE

**Theorem 3.8** (Uniform convergence lower bound). There exists universal constants \( c_1, c_2, c_3, p \) (with \( H, S, A \geq c_1 \) and \( 0 < \epsilon < c_2 \)) such that for any estimator \( \hat{v} \) and any \( n \leq c_3 H^3 SA/\epsilon^2 \), there exists a non-stationary \( H \)-horizon MDP with probability at least \( p \), the estimator satisfies \( \sup_{\pi \in \Pi} |\hat{v}^\pi - v^\pi| \geq \epsilon \), where \( \Pi \) consists of all deterministic policies.

**On optimality.** To see how optimal our results are, consider a MDP such that the induced marginal distribution \( d^\mu(s_t, a_t) > 0 \) for all \( s_t, a_t \). Use direct computation \( 1 = \sum s_t, a_t d^\mu(s_t, a_t) \geq \sum s_t, a_t d_m = SA \cdot d_m \), so \( 1/d_m \geq SA \). If the logging policy is “good enough” such that \( 1/d_m = \Theta(SA) \), then Theorem 3.5 gives \( \tilde{O}(H^3 \min(S, H) \cdot SA/\epsilon^2) \) is one factor away from the lower bound and Theorem 3.7 is optimal. Of course, if the instance-dependent \( d_m \) is too small then the required sample could be very large, which is, not surprisingly, due to the lack of exploration by logging policy \( \mu \).

### 4 Main Results for Offline Learning

The significance of uniform convergence OPE is not confined to that the same OPE bound holds for fixed policy within the policy class. More importantly, it also provides efficient offline learning algorithm of the same rate (see Theorem 4.1 below). This new finding helps unify OPE problem with offline learning through the concept of uniform convergence. Indeed, when uniform convergence in OPE is guaranteed, we can derive an offline learning algorithm.
Table 1: Implied offline learning results

<table>
<thead>
<tr>
<th>Method</th>
<th>Guarantee</th>
<th>Sample complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>PI/VI for empirical optimal(^a)</td>
<td>(\epsilon)-optimal policy</td>
<td>(\tilde{O}(H^3/d_m\epsilon^2) \min{S,H})</td>
</tr>
<tr>
<td>PI/VI for (\epsilon_{\text{opt}})-empirical optimal</td>
<td>((\epsilon + \epsilon_{\text{opt}}))-optimal policy</td>
<td>(\tilde{O}(H^3/d_m\epsilon^2))</td>
</tr>
<tr>
<td>Minimax lower bound (Theorem G.1)</td>
<td>——</td>
<td>(\Omega(H^3SA/\epsilon^2))</td>
</tr>
</tbody>
</table>

\(^a\) PI/VI is not essential and can be replaced by any empirical MDP solver.

algorithm in a straightforward way by choosing \(\hat{\pi}^* := \arg\max_{\pi \in \Pi} \hat{v}_\pi\). We summarize it into the following theorem:

**Theorem 4.1** (Offline learning algorithm). Suppose a OPE method \(\hat{\theta}\) achieves \(\epsilon_{\text{uni}}\)-uniform convergence, i.e. with high probability, \(\sup_{\pi \in \Pi} |\hat{v}_\pi - v_\pi| \leq \epsilon_{\text{uni}}\). Assuming the realizability that \(\pi^* \in \Pi\). Then by selecting \(\hat{\pi}^* := \arg\max_{\pi \in \Pi} \hat{v}_\pi\), we obtain with high probability,

\[
v^{\pi^*} - \hat{v}^{\pi^*} \leq 2\epsilon_{\text{uni}}.
\]

The proof is just one line: \(v^{\pi^*} - \hat{v}^{\pi^*} = v^{\pi^*} - \hat{\pi}^* + \hat{\pi}^* - \hat{v}^{\pi^*} \leq v^{\pi^*} - \hat{\pi}^* + \hat{\pi}^* - \hat{v}^{\pi^*} \leq 2\epsilon_{\text{uni}}\).

If \(|\Pi|\) is relatively small, we can find \(\hat{\pi}^*\) using brute force with policy search: we traverse all the policies \(\pi \in \Pi\) and calculate \(\hat{v}_\pi\), and learn the empirical optimal policy \(\hat{\pi}^*\). If \(|\Pi|\) is large, then \(\hat{\pi}^* := \arg\max_{\pi \in \Pi} \hat{v}_\pi\) can be solved in a more efficient way by applying standard Policy Iteration (PI) or Value Iteration (VI) methods on empirical MDP \(\hat{M}\) (Sutton & Barto, 2018) since \(\hat{M}\) is fully known. In particular, our Theorem 3.5 guarantees offline learning with complexity \(\tilde{O}(H^3/d_m\epsilon^2) \cdot \min\{S,H\}\), which is one factor away from optimality in learning context (compare to Theorem G.1 in appendix). Moreover, the local uniform convergence result in Theorem 3.7 also guarantees a \(O(\epsilon_{\text{opt}} + \epsilon)\)-optimal policy for any policy \(\pi\) in the neighborhood of \(\hat{\pi}^*\). Formally, we have the following:

**Theorem 4.2.** Suppose \(\epsilon_{\text{opt}} \leq \sqrt{H}/S\) and \(\hat{\pi}\) satisfies \(\|Q_1^* - Q_1^{\hat{\pi}}\|_\infty \leq \epsilon_{\text{opt}}\) and \(\|\hat{v}_t^* - \hat{\pi}^*\|_\infty \leq \epsilon_{\text{opt}}, \forall t = 2, ..., H\). Then there exists constant \(c_1, c_2\) such that when \(n > c_1H^3\log(HSA/\delta)/d_m\epsilon^2\) we have probability \(1 - \delta\),

\[
Q_1^{\pi^*} - Q_1^{\hat{\pi}} \leq c_2 \cdot \epsilon \cdot 1 + \epsilon_{\text{opt}} \cdot 1.
\]

In the case where \(\epsilon_{\text{opt}} = 0\), we have \(\hat{\pi} = \hat{\pi}^*\) achieves the optimal sample complexity \(O(H^3 \log(HSA/\delta)/d_m\epsilon^2)\). This analysis has an improvement over (Agarwal et al., 2019) as they have a \(O(\epsilon_{\text{opt}}/(1 - \gamma))\) term which should be translated as \(O(\epsilon_{\text{opt}}H)\) in the finite horizon setting. We summarize these results in Table 1 for clear reference.
5 Proof Overview

Our uniform convergence analysis in Section 3.1, relies on creating an unbiased version of \( \hat{v}_{OPEMA} \) (which we call it \( \check{v}_{OPEMA} \)) artificially and use Lemma 3.1 to guarantee \( \check{v}_{OPEMA} \) is identical to \( \hat{v}_{OPEMA} \) in most situations. By doing so we can reduce our analysis from \( \sup_{\pi \in \Pi} |\hat{v}^\pi - v^\pi| \) to \( \sup_{\pi \in \Pi} |\check{v}^\pi - v^\pi| \). Specifically, \( \check{v}^\pi \) replaces \( \hat{v}^\pi \) in \( \hat{v}^\pi \) by its fictitious counterparts \( \check{P}_t, \check{r}_t \), defined as:

\[
\check{r}_t(s_t, a_t) = \check{r}_t(s_t, a_t) \mathbb{1}(E_t) + r_t(s_t, a_t) \mathbb{1}(E_t^c), \quad \check{P}_{t+1}(\cdot|s_t, a_t) = \check{P}_{t+1}(\cdot|s_t, a_t) \mathbb{1}(E_t) + P_{t+1}(\cdot|s_t, a_t) \mathbb{1}(E_t^c).
\]

where \( E_t \) denotes the event \( \{n_{s_t, a_t} \geq ndt(s_t, a_t)/2\} \). For OPEMA estimator with dataset \( D' \), the fictitious estimator can be created in a similar way based on \( E := \{N \geq ndm/2\} \). Let \( T_{h+1} \in \mathbb{R}^{S \times (SA)} \) be the one step transition matrix, \textit{i.e.} \( T_{s_{h+1}(s_h, a_h)} = P_{h+1}(s_{h+1}|s_h, a_h) \), then we have the following critical decomposition:

\[
\check{v}^\pi - v^\pi = \sum_{h=2}^{H} \langle v_h^\pi, (\check{T}_h - T_h)\check{d}_{h-1}^\pi \rangle + \langle v_1^\pi, \check{d}_1^\pi - d_1^\pi \rangle,
\]

where the inner product is taken w.r.t states (see Theorem C.3 in appendix). On one hand, by using the structure of \( \sup_{\pi \in \Pi} |v_h^\pi, (\check{T}_h - T_h)\check{d}_{h-1}^\pi \| \) we can relax it into a Rademacher-type complexity and the \( \sqrt{S^2 \log(SA)} \) in Theorem 3.3 is part of the upper bound of the Rademacher complexity, see Theorem D.2. The fact \( \sqrt{S^2 \log(SA)} \) does not depend on \( \delta \) help complete Theorem 3.3. On the other hand, This decomposition has a natural martingale structure so martingale concentration inequalities can be appropriately applied, \textit{i.e.} Theorem 3.4. In addition, each term \( \langle v_h^\pi, (\check{T}_h - T_h)\check{d}_{h-1}^\pi \rangle \) separates the non-stationary policy into two parts with empirical distribution only depends on \( \pi_{1:h-1} \) that governs the data “roll in” and the long term value function \( v_h^\pi \) only depends on \( \pi_{h:H} \) that governs how the reward “roll out”. For local uniform convergence, by Bellman equations we can obtain a similar decomposition on \( Q \)-function:

\[
\check{Q}_t^\pi - Q_t^\pi = \sum_{h=t+1}^{H} \Gamma_{t+1:h-1}^\pi (\check{P}_h - P_h)\check{v}_h^\pi,
\]

where \( \Gamma_{t:h}^\pi = \prod_{i=t}^{h} P_i^\pi \) is the multi-step state-action transition and \( \Gamma_{t+1:h}^\pi := I \). Since \( \pi \) is any policy in \( \Pi \) which may dependent on \( D' \) so we cannot directly apply concentration inequalities on \( (\check{P}_h - P_h)\check{v}_h^\pi \). Instead, we overcome this hurdle by doing concentration on \( (\check{P}_h - P_h)\check{v}_h^\pi \) since \( \check{v}_h^\pi \) and \( \check{P}_h \) are independent, and we connect \( \check{v}_h^\pi \) back to \( v_h^\pi \) by using they are \( \epsilon_{opt} \) close (Theorem 3.7). This idea helps avoiding the technicality of absorbing MDP used in (Agarwal et al., 2019) for infinite horizon case because of our non-stationary transition setting. For the uniform convergence lower bound, our analysis relies on reducing
the problem to identifying \( \epsilon \)-optimal policy and proving any algorithm that learns a \( \epsilon \)-optimal policy requires at least \( \Omega(H^3SA/\epsilon^2) \) episodes in the non-stationary episodic setting. There are quite a few studies that provide \( \epsilon \)-optimal policy learning lower bound in different setting (Dann & Brunskill, 2015; Jiang et al., 2017; Krishnamurthy et al., 2016; Jin et al., 2018; Sidford et al., 2018). The most related work is (Jiang et al., 2017) which proves the \( \Omega(HSA/\epsilon^2) \) lower bound with assumption \( \sum_{i=1}^{H} r_i \leq 1 \). Our proof uses a modified version of their hard-to-learn MDP instance to achieve the desired result. To produce extra \( H^2 \) dependence, we leverage the Assumption 2.1 that \( \sum_{i=1}^{H} r_i \) may be of order \( O(H) \). We only present the high-level ideas here due the space constraint, detailed proofs are explicated in order in Appendix D, E, F, G.

6 Discussion: the ordinary OPE analysis as refined simulation lemma

One intrinsic challenge for sequential decision making problem is decisions made over long horizon may become imprecise due to the error propagation. Therefore, accurate planning over long horizon is core for the success of RL and getting optimal dependence on \( H \) is highly non-trivial, e.g. see COLT open problem (Jiang & Agarwal, 2018). How much improvement/novelty does our analysis made in this task? For example, the standard analysis tool for model plug-in estimates is simulation lemma (Kearns & Singh, 2002) (also see an exposition in (Jiang, 2018)), it follows (assuming deterministic reward):\(^6\)

\[
\|\hat{\pi} - v^\pi\|_\infty \leq H^2 \sup_{t,s,a} \left\| \hat{P}(\cdot|s_t, a_t) - P(\cdot|s_t, a_t) \right\|_1 \leq O \left( \frac{H^4S^2 \log(HSA/\delta)}{nd_m} \right),
\]

which has provable complexity of \( \tilde{O}(H^4S^2/d_m\epsilon^2) \). Meanwhile, our Lemma 3.4 only has order \( \tilde{O}(H^2/d_m\epsilon^2) \), which has \( H^2S^2 \) improvement over the vanilla simulation lemma. Moreover, this result essentially matches the Corollary 1 of (Duan & Wang, 2020) and is minimax optimal under our Assumption 2.2 (our \( d_m = \min_{s,a} \bar{\mu}_h(s,a) \) in their case). Our analysis reveals, without complex construction e.g. Fitted Q-iteration in (Duan & Wang, 2020), direct model-based plug-in estimator is also tight, which helps to correct the commonly held misunderstanding that purely model plug-in estimator can only provide loose bound due to simulation lemma.

7 Conclusion

This work systematically considers uniform convergence in off-policy evaluation for the first time and derives near optimal results for two representative policy classes. By viewing

\(^6\)For detailed verification of the computation, see Appendix I.
off-policy evaluation from the uniform convergence perspective, we are able to unify two central topics in offline RL, OPE and offline learning, in a consistent way. We hope this work shed the light for research in uniform convergence in OPE and more interesting methods can be invented.

References


Appendix

A  Technical lemmas

**Lemma A.1** (Multiplicative Chernoff bound Chernoff et al. (1952)). Let $X$ be a Binomial random variable with parameter $p,n$. For any $\delta > 0$, we have that

$$
P[X < (1 - \delta)pn] < \left( \frac{e^{-\delta}}{(1 - \delta)^{1-\delta}} \right)^{np}.
$$

A slightly looser bound that suffices for our propose:

$$
P[X < (1 - \delta)pn] < e^{-\frac{\delta^2pn}{2}}.
$$

**Lemma A.2** (Hoeffding’s Inequality Sridharan (2002)). Let $x_1,...,x_n$ be independent bounded random variables such that $\mathbb{E}[x_i] = 0$ and $|x_i| \leq \xi_i$ with probability 1. Then for any $\epsilon > 0$ we have

$$
P \left( \frac{1}{n} \sum_{i=1}^{n} x_i \geq \epsilon \right) \leq e^{-\frac{2\epsilon^2}{\sum_{i=1}^{n} \xi_i^2}}.
$$

**Lemma A.3** (Bernstein’s Inequality). Let $x_1,...,x_n$ be independent bounded random variables such that $\mathbb{E}[x_i] = 0$ and $|x_i| \leq \xi$ with probability 1. Let $\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} \text{Var}[x_i]$, then with probability $1 - \delta$ we have

$$
\frac{1}{n} \sum_{i=1}^{n} x_i \leq \sqrt{\frac{2\sigma^2 \cdot \log(1/\delta)}{n}} + \frac{2\xi}{3n} \log(1/\delta)
$$

**Lemma A.4** (McDiarmid’s Inequality (Sridharan, 2002)). Let $x_1,...,x_n$ be independent random variables and $S : X^n \rightarrow \mathbb{R}$ be a measurable function which is invariant under permutation and let the random variable $Z$ be given by $Z = S(x_1,x_2,...,x_n)$. Assume $S$ has bounded difference: i.e.

$$
\sup_{x_1,...,x_n,x_i'} |S(x_1,...,x_i,...,x_n) - S(x_1,...,x_i',...,x_n)| \leq \xi_i,
$$

then for any $\epsilon > 0$ we have

$$
P(|Z - \mathbb{E}[Z]| \geq \epsilon) \leq 2e^{-\frac{2\epsilon^2}{\sum_{i=1}^{n} \xi_i^2}}.$$
Lemma A.5 (Rademacher complexity of $L_1$ ball). Consider the $L_1$ norm-constrained hypothesis class $F = \{ Z \mapsto \langle w, Z \rangle : ||w||_1 \leq B \}$ where $Z \in \mathbb{R}^d$ and the corresponding Rademacher complexity

$$\mathcal{R}(F) = \mathbb{E} \left[ \sup_{w \in F} \frac{1}{n} \sum_{i=1}^{n} \sigma_i \langle w, Z_i \rangle \right].$$

If $||Z_i||_\infty \leq C_\infty$ with probability 1 for all $i = 1, \ldots, n$, then

$$\mathcal{R}(F) \leq BC_\infty \sqrt{\frac{2 \log(2d)}{\sqrt{n}}}.$$ 

Lemma A.6 (Azuma-Hoeffding inequality). Suppose $X_k, k = 1, 2, 3, \ldots$ is a martingale and $|X_k - X_{k-1}| \leq c_k$ almost surely. Then for all positive integers $N$ and any $\epsilon > 0$,

$$\mathbb{P}(|X_N - X_0| \geq \epsilon, W \leq \sigma^2) \leq 2e^{-\frac{\epsilon^2}{2\sum_{i=1}^{n} c_i^2}}.$$ 

Lemma A.7 (Freedman’s inequality Tropp et al. (2011)). Let $X$ be the martingale associated with a filter $\mathcal{F}$ (i.e. $X_i = \mathbb{E}[X|\mathcal{F}_i]$) satisfying $|X_i - X_{i-1}| \leq M$ for $i = 1, \ldots, n$. Denote $W := \sum_{i=1}^{n} \text{Var}(X_i|\mathcal{F}_{i-1})$ then we have

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq \epsilon, W \leq \sigma^2) \leq 2e^{-\frac{\epsilon^2}{2(\sigma^2 + M\epsilon/3)}}.$$ 

Or in other words, with probability $1 - \delta$,

$$|X - \mathbb{E}[X]| \leq \sqrt{8\sigma^2 \cdot \log(1/\delta)} + \frac{2M}{3} \cdot \log(1/\delta), \quad \text{Or} \quad W \geq \sigma^2.$$ 

Lemma A.8 (Best arm identification lower bound Krishnamurthy et al. (2016)). For any $A \geq 2$ and $\tau \leq \sqrt{1/8}$ and any best arm identification algorithm that produces an estimate $\hat{a}$, there exists a multi-arm bandit problem for which the best arm $a^*$ is $\tau$ better than all others, but $\mathbb{P}[\hat{a} \neq a^*] \geq 1/3$ unless the number of samples $T$ is at least $\frac{A}{\tau^2 \cdot \epsilon^2}$. 

B On error metric for OPE

In this section, we give some brief discussions on the metric considered for this work. Indeed, most works directly use Mean Square Error (MSE) $\mathbb{E}[(\hat{v}^\pi - v^\pi)^2]$ as the criterion for measuring OPE methods e.g. Thomas & Brunskill (2016); Thomas (2015); Thomas et al. (2017); Farajtabar et al. (2018), or equivalently, by proposing unbiased estimators and discussing its variance e.g. Jiang & Li (2016). Alternately, we consider bounding the absolute difference between $v^\pi$ and $\hat{v}^\pi$ with high probability, i.e. $|\hat{v}^\pi - v^\pi| \leq \epsilon_{\text{prob}} \text{ w.h.p.}$

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Generally speaking, high probability bound can be seen as a stricter criterion compared to MSE since

\[
E[(\hat{v}^\pi - v^\pi)^2] = E[(\hat{v}^\pi - v^\pi)^2 1_E] + E[(\hat{v}^\pi - v^\pi)^2 1_{E^c}] 
\leq \epsilon_{\text{prob}}(\delta)^2 \cdot (1 - \delta) + H^2 \cdot \delta,
\]

where \( E \) is the event that \( \epsilon_{\text{prob}} \) error holds and \( \delta \) is the failure probability. As a result, if both \( \delta \) and \( \epsilon_{\text{prob}}(\delta) \) can be controlled small, then the high probability bound implies a result for MSE bound. This is realistic, since \( \delta \) usually appears inside the logarithmic term of \( \epsilon_{\text{prob}}(\delta) \) so the second term can be scaled to sufficiently small without affecting the polynomial dependence for the first term.

\[ C \quad \text{Some preparations} \]

In this section we present some results that are critical for proving the main theorems.

We first give the proof of Lemma 3.1.

**Proof of Lemma 3.1.** Define \( E := \{ \exists t, s_t, a_t \text{ s.t. } n_{s_t,a_t} < nd_t^\mu(s_t,a_t)/2 \} \). Then combining the multiplicative Chernoff bound (Lemma A.1 in the Appendix) and a union bound over each \( t, s_t \) and \( a_t \), we obtain

\[
P[E] \leq \sum_t \sum_{s_t} \sum_{a_t} P[n_{s_t,a_t} < nd_t^\mu(s_t,a_t)/2] 
\leq HSA \cdot e^{-n_{\min_t, s_t, a_t} \frac{d_t^\mu(s_t,a_t)}{s}} = HSA \cdot e^{-n_{\min_t, s_t, a_t} \frac{d_t^\mu(s_t,a_t)}{s}} := \delta
\]

solving this for \( n \) then provides the stated result. \( \blacksquare \)

\[ C.1 \quad \text{Fictitious OPEMA estimator.} \]

Similar to Xie et al. (2019); Yin & Wang (2020), we introduce an unbiased version of \( \hat{v}^\pi \) to fill in the gap at \( (s_t, a_t) \) where \( n_{s_t,a_t} \) is small. Concretely, every component in \( \hat{v}^\pi \) is substituted with a fictitious counterpart, \( i.e. \hat{v}^\pi := \sum_{t=1}^H \{\hat{d}_t^\pi, \hat{r}_t^\pi\} \), with \( \hat{d}_t^\pi = \tilde{P}_t^\pi \tilde{d}_{t-1}^\pi \) and \( \tilde{P}_t^\pi(s_t|s_{t-1}) = \sum_{a_t-1} \tilde{P}_t(s_t|s_{t-1}, a_{t-1}) \pi(a_{t-1}|s_{t-1}) \). In particular, consider the high probability event in Lemma 3.1, \( i.e. \) let \( E_t \) denotes the event \( \{n_{s_t,a_t} \geq nd_t^\mu(s_t,a_t)/2\} \), then

\[ ^7 \text{More rigorously, } E_t \text{ depends on the specific pair } s_t, a_t \text{ and should be written as } E_t(s_t, a_t). \text{ However, for brevity we just use } E_t \text{ and this notation should be clear in each context.} \]
we define
\[ \tilde{r}_t(s_t, a_t) = \tilde{r}_t(s_t, a_t)1(E_t) + r_t(s_t, a_t)1(E_t^c) \]
\[ P_{t+1}(\cdot|s_t, a_t) = \tilde{P}_{t+1}(\cdot|s_t, a_t)1(E_t) + P_{t+1}(\cdot|s_t, a_t)1(E_t^c). \]

Similarly, for the OPEMA estimator uses data \( D' \), the fictitious estimator is set to be
\[ \tilde{r}_t(s_t, a_t) = \tilde{r}_t(s_t, a_t)1(E) + r_t(s_t, a_t)1(E^c) \]
\[ \tilde{P}_{t+1}(\cdot|s_t, a_t) = \tilde{P}_{t+1}(\cdot|s_t, a_t)1(E) + P_{t+1}(\cdot|s_t, a_t)1(E^c) \]

where \( E \) denote the event \( \{N \geq nd_m/2\} \).

\( \hat{v}^\pi \) creates a bridge between \( \hat{v}^\pi \) and \( v^\pi \) because of its unbiasedness and it is also bounded by \( H \) (see Lemma B.3 and Lemma B.5 in Yin & Wang (2020) for those preliminary results). Also, \( \hat{v}^\pi \) is identical to \( \hat{v}^\pi \) with high probability, as stated by the following lemma.

**Lemma C.1.** For any \( 0 < \delta < 1 \), there exists an absolute constant \( c_1 \) such that when total episode \( n > c_1d_m \cdot \log(HSA/\delta) \), then with probability \( 1 - \delta \),
\[ \sup_{\pi \in \Pi} |\hat{v}^\pi - \tilde{v}^\pi| = 0. \]

**Proof.** This Lemma is a direct corollary of Lemma 3.1 by considering the event \( E_1 := \{\exists t, s_t, a_t \text{ s.t. } n_{s_t, a_t} < nd_t^\mu(s_t, a_t)/2\} \) or \( \{N < nd_m/2\} \) since \( \hat{v}^\pi \) and \( \tilde{v}^\pi \) are identical on \( E_t^c \).

Note \( \hat{v}^\pi \) and \( \tilde{v}^\pi \) even equal to each other uniformly over all \( \pi \) in \( \Pi \). This is not surprising since only logging policy \( \mu \) will decide if they are equal or not. This lemma shows how close \( \hat{v}^\pi \) and \( \tilde{v}^\pi \) are. Therefore in the following it suffices to consider the uniform convergence of \( \sup_{\pi \in \Pi} |\hat{v}^\pi - v^\pi| \).

Next by using a fictitious version state-action expression in equation (2), we have:
\[ \sup_{\pi \in \Pi} |\hat{v}^\pi - v^\pi| = \sup_{\pi \in \Pi} \left| \sum_{t=1}^{H} \langle d_t^\pi, \tilde{r}_t \rangle - \sum_{t=1}^{H} \langle d_t^\pi, r_t \rangle \right| \]
\[ = \sup_{\pi \in \Pi} \left| \sum_{t=1}^{H} \langle d_t^\pi, \tilde{r}_t \rangle - \sum_{t=1}^{H} \langle \tilde{d}_t^\pi, r_t \rangle + \sum_{t=1}^{H} \langle \tilde{d}_t^\pi, r_t \rangle - \sum_{t=1}^{H} \langle d_t^\pi, r_t \rangle \right| \]
\[ \leq \sup_{\pi \in \Pi} \left| \sum_{t=1}^{H} \langle d_t^\pi - \tilde{d}_t^\pi, r_t \rangle \right| + \sup_{\pi \in \Pi} \left| \sum_{t=1}^{H} \langle \tilde{d}_t^\pi, \tilde{r}_t - r_t \rangle \right| \]

We first deal with \((**\)) by the following lemma.
Lemma C.2. We have with probability $1 - \delta$:

$$\sup_{\pi \in \Pi} \left| \sum_{t=1}^{H} \langle \tilde{d}_t^\pi, \tilde{r}_t - r_t \rangle \right| \leq O\left( \sqrt{\frac{H^2 \log(HSA/\delta)}{n \cdot d_m}} \right)$$

Proof of Lemma C.2. Since $|\langle \tilde{d}_t^\pi, \tilde{r}_t - r_t \rangle| \leq \|\tilde{d}_t^\pi\|_1 \cdot \|\tilde{r}_t - r_t\|_\infty$, we obtain

$$|\sum_{t=1}^{H} \langle \tilde{d}_t^\pi, \tilde{r}_t - r_t \rangle| \leq \sum_{t=1}^{H} \|\tilde{d}_t^\pi\|_1 \cdot \|\tilde{r}_t - r_t\|_\infty = \sum_{t=1}^{H} \|\tilde{r}_t - r_t\|_\infty,$$

where we used $\tilde{d}_t^\pi(\cdot)$ is a probability distribution. Therefore above expression further indicates $\sup_{\pi \in \Pi} \left| \sum_{t=1}^{H} \langle \tilde{d}_t^\pi, \tilde{r}_t - r_t \rangle \right| \leq \sum_{t=1}^{H} \|\tilde{r}_t - r_t\|_\infty$. Now by a union bound and Hoeffding inequality (Lemma A.2),

$$\mathbb{P}(\sup_t |\tilde{r}_t - r_t|_\infty > \epsilon) = \mathbb{P}(\sup_{t,s_t,a_t} |\tilde{r}_t(s_t, a_t) - r_t(s_t, a_t)| > \epsilon) \leq HSA \cdot \mathbb{P}(|\tilde{r}_t(s_t, a_t) - r_t(s_t, a_t)|_1 > \epsilon) \leq 2HSA \cdot \mathbb{E}[\mathbb{E}[e^{-2n_{s_t,a_t}\epsilon^2} | E_t]] \leq 2HSA \cdot \mathbb{E}[\mathbb{E}[e^{-nd_m\epsilon^2} | E_t]] = 2HSA \cdot e^{-nd_m\epsilon^2} := \frac{\delta}{2},$$

where we use $\mathbb{P}(A) = \mathbb{E}[\mathbf{1}_A] = \mathbb{E}[\mathbb{E}[\mathbf{1}_A|X]]$. Solving for $\epsilon$, then it follows:

$$\sup_{\pi \in \Pi} \left| \sum_{t=1}^{H} \langle \tilde{d}_t^\pi, \tilde{r}_t - r_t \rangle \right| \leq \sum_{t=1}^{H} \|\tilde{r}_t - r_t\|_\infty \leq O\left( \sqrt{\frac{H^2 \log(HSA/\delta)}{n \cdot d_m}} \right)$$

with probability $1 - \delta$. The case $E = \{N \geq nd_m/2\}$ can be proved easily in a similar way.

Note that in order to measure the randomness in reward, sample complexity $n$ only has dependence of order $H^2$, this result implies random reward will only cause error of lower order dependence in $H$. Therefore, in a lot of RL literature deterministic reward is directly assumed. Next we consider $(\ast)$ in (5) by decomposing $\sum_{t=1}^{H} \langle \tilde{d}_t^\pi, \tilde{r}_t \rangle$ into a martingale type representation. This is the key for our proof since with it we can use either uniform concentration inequalities or martingale concentration inequalities to prove efficiency.
C.2 Decomposition of $\sum_{t=1}^{H} \langle \tilde{d}_t^\pi - d_t^\pi, r_t \rangle$

Let $\tilde{d}_t^\pi \in \mathbb{R}^{S \times A}$ denote the marginal state-action probability vector, $\pi_t \in \mathbb{R}^{(S \times A) \times S}$ is the policy matrix with $(\pi_t)_{(s_t,a_t),s_t} = \pi_t(a_t|s_t)$ and $(\pi_t)_{(s_t,a_t),s} = 0$ for $s \neq s_t$. Moreover, let state-action transition matrix $T_t \in \mathbb{R}^{S \times (S \times A)}$ to be $(T_t)_{s_t,(s_{t-1},a_{t-1})} = P_t(s_t|s_{t-1},a_{t-1})$, then we have

$$\tilde{d}_t^\pi = \pi_t \tilde{T}_t \tilde{d}_{t-1}^\pi$$

(6)

$$d_t^\pi = \pi_t T_t d_{t-1}^\pi.$$  

(7)

Therefore we have

$$\tilde{d}_t^\pi - d_t^\pi = \pi_t (\tilde{T}_t - T_t) \tilde{d}_{t-1}^\pi + \pi_t T_t (\tilde{d}_{t-1}^\pi - d_{t-1}^\pi)$$  

(8)

recursively apply this formula, we have

$$\tilde{d}_t^\pi - d_t^\pi = \sum_{h=2}^{t} \Gamma_{h:t} \pi_h (\tilde{T}_h - T_h) \tilde{d}_{h-1}^\pi + \Gamma_{1:t} (\tilde{d}_1^\pi - d_1^\pi)$$

(9)

where $\Gamma_{h:t} = \prod_{v=h}^{t} \pi_v T_v$ and $\Gamma_{t+1:t} := 1$. Now let $X = \sum_{t=1}^{H} \langle r_t, \tilde{d}_t^\pi - d_t^\pi \rangle$, then we have the following:

Theorem C.3 (martingale decomposition of $X$). We have:

$$X = \sum_{h=2}^{H} \langle v_h^\pi(s), ((\tilde{T}_h - T_h) \tilde{d}_{h-1}^\pi)(s) \rangle + \langle v_1^\pi(s), (\tilde{d}_1^\pi - d_1^\pi)(s) \rangle,$$

where the inner product is taken w.r.t states.

Proof of Theorem C.3. By applying (9) and the change of summation, we have

$$X = \sum_{t=1}^{H} \left( \sum_{h=2}^{t} \langle r_t, \Gamma_{h+1:t} \pi_h (\tilde{T}_h - T_h) \tilde{d}_{h-1}^\pi \rangle + \langle r_t, \Gamma_{1:t} (\tilde{d}_1^\pi - d_1^\pi) \rangle \right)$$

$$= \sum_{t=1}^{H} \left( \sum_{h=2}^{t} \langle r_t, \Gamma_{h+1:t} \pi_h (\tilde{T}_h - T_h) \tilde{d}_{h-1}^\pi \rangle \right) + \sum_{h=1}^{H} \langle r_t, \Gamma_{1:t} (\tilde{d}_1^\pi - d_1^\pi) \rangle$$

$$= \sum_{t=2}^{H} \left( \sum_{h=2}^{t} \langle r_t, \Gamma_{h+1:t} \pi_h (\tilde{T}_h - T_h) \tilde{d}_{h-1}^\pi \rangle \right) + \sum_{h=1}^{H} \langle r_t, \Gamma_{1:t} (\tilde{d}_1^\pi - d_1^\pi) \rangle$$

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\[= \sum_{h=2}^{H} \left( \sum_{t=h}^{H} \langle r_t, \Gamma_{h+1:t} \pi_h(\widehat{T}_h - T_h)\bar{d}_{h-1}^{\pi} \rangle \right) + \sum_{h=1}^{H} \langle (\pi_1^T \Gamma_{1:t}^T r_t)(s), (\bar{d}_1^{\pi} - d_1^t)(s) \rangle \]
\[= \sum_{h=2}^{H} \left( \sum_{t=h}^{H} \left\langle \pi_h^T \Gamma_{h+1:t}^T r_t, (\widehat{T}_h - T_h)\bar{d}_{h-1}^{\pi} \right\rangle \right) + \left\langle \left( \sum_{h=1}^{H} \pi_1^T \Gamma_{1:t}^T r_t \right)(s), (\bar{d}_1^{\pi} - d_1^t)(s) \right\rangle \quad (10) \]

\section{Proof of uniform convergence in OPE problem with standard uniform concentration tools: Theorem 3.3}

As a reminder for the reader, the OPEMA estimator used in this section is with data subset $\mathcal{D}'$. Also, by Lemma C.2 we only need to consider $\sup_{\pi \in \Pi} | \sum_{t=1}^{H} \langle \bar{d}_{t}^{\pi} - d_{t}^{\pi}, r_t \rangle |$.

**Theorem D.1.** There exists an absolute constant $c$ such that if $n > c \cdot \frac{1}{d_m} \cdot \log(HSA/\delta)$, then with probability $1 - \delta$, we have:

\[
\sup_{\pi \in \Pi} \left| \sum_{t=1}^{H} \langle \bar{d}_{t}^{\pi} - d_{t}^{\pi}, r_t \rangle \right| \leq O\left( \sqrt{\frac{H^{4} \log(HSA/\delta)}{nd_m}} \right) + \mathbb{E} \left[ \sup_{\pi \in \Pi} \left| \sum_{t=1}^{H} \langle \bar{d}_{t}^{\pi} - d_{t}^{\pi}, r_t \rangle \right| \right]
\]

**Proof of Theorem D.1.** Note in data $\mathcal{D}' = \{(s_t, a_t, s_{t+1}^{(i)}) : i = 1, ..., N; t = 1, ..., H; s_t \in S, a_t \in A \}$\(^8\), not only $s_{t+1}^{(i)}$ but also $N$ are random variables.

We first conditional on $N$, then $(s_t, a_t, s_{t+1}^{(i)})$'s are independent samples for all $i, s_t, a_t$ since any sample will not contain information about other samples. Therefore we can regroup $\mathcal{D}'$ into $N$ independent samples with $\mathcal{D}' = \{X^{(i)} : i = 1, ..., N\}$ where $X^{(i)} = \{(s_t, a_t, s_{t+1}^{(i)}), t = 1, ..., H; s_t \in S, a_t \in A \}$. Now for any $i_0$, change $X^{(i_0)}$ to $X_{n(i_0)}^{(i_0)} = \{(s_t, a_t, s_{t+1}^{(i_0)}), t = 1, ..., H; s_t \in S, a_t \in A \}$ and keep the rest $N - 1$ data the same, use this data to create new

\(^8\)Here we do not include $r_{t}^{(i)}$ since the quantity $\sup_{\pi \in \Pi} | \sum_{t=1}^{H} \langle \bar{d}_{t}^{\pi} - d_{t}^{\pi}, r_t \rangle |$ only contains the mean reward function $r_t$. 

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estimator with state-action transition $\tilde{d}^π$, then we have

$$\sup_{\pi \in \Pi} \left| \sum_{t=1}^{H} \langle \tilde{d}^π_{t} - d^π_{t}, r_t \rangle \right| - \sup_{\pi \in \Pi} \left| \sum_{t=1}^{H} \langle \tilde{d}^π_{t} - d^π_{t}, r_t \rangle \right|$$

where the last equation comes from the trick that substitutes $d^π_t$ by $\tilde{d}^π_t$ in Theorem C.3. By definition, the above equals to

$$= \sup_{\pi \in \Pi} \left| \sum_{h=2}^{H} \langle \tilde{v}^π_{h}, \tilde{d}^π_{h-1} \rangle + \langle \tilde{v}^π_{1}, \tilde{d}^π_{1} - \tilde{d}^π_{1} \rangle \right| \cdot \mathbb{1}(E)$$

$$\leq \sup_{\pi \in \Pi} \left( \sum_{h=2}^{H} \| (\tilde{T}_h - \tilde{T}_h')^T \tilde{v}^π_{h} \|_1 \| \tilde{d}^π_{h-1} \|_1 + \| \langle \tilde{v}^π_{1}, \tilde{d}^π_{1} - \tilde{d}^π_{1} \rangle \|_1 \right) \cdot \mathbb{1}(E)$$

Note the change of a single $X^{(t)}$ will only change two entries of each row of $(\tilde{T}_h - \tilde{T}_h')^T$ by $1/N$ since with data $D'$, $n_{s_t, a_t} = N$ for all $s_t, a_t$. Or in other words, given $E$,

$$\tilde{T}_h^T - \tilde{T}_h'^T = \begin{bmatrix} 0 & ... & 0 & \frac{1}{N} & 0 & ... & -\frac{1}{N} & ... & 0 \\ 0 & ... & 0 & \frac{1}{N} & 0 & ... & -\frac{1}{N} & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ -\frac{1}{N} & 0 & ... & 0 & ... & 0 & ... & \frac{1}{N} \end{bmatrix}$$

where the locations of $1/N, -1/N$ in above expression are random as it depends on how different is $X^{(t)}$ from $X^{(t')}$. However, based on this fact, it is enough for us to guarantee

$$\| (\tilde{T}_h - \tilde{T}_h')^T \tilde{v}^π_{h} \|_1 \leq \frac{2}{N} \| \tilde{v}^π_{h} \|_1 \leq \frac{2}{N} (H - h + 1) \leq \frac{2}{N} H$$

and same result holds for $\| \langle \tilde{v}^π_{1}, \tilde{d}^π_{1} - \tilde{d}^π_{1} \rangle \|_1 \leq 2H/N$ given $N$. 

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Combine all the results above, for a single change of $X^{(i_0)}$ we have

$$\left| \sup_{\pi \in \Pi} \sum_{t=1}^{H} \langle \tilde{d}_{t}^{\pi} - d_{t}^{\pi}, r_t \rangle - \sup_{\pi \in \Pi} \sum_{t=1}^{H} \langle \tilde{d}_{t}^{\pi} - d_{t}^{\pi}, r_t \rangle \right| \leq 2 \frac{H^2}{N} \mathbb{P}(\epsilon) \leq 2 \frac{H^2}{N}$$

for any fixed $N$. If we let $Z = S(X^{(1)}, ..., X^{(N)}) = \sup_{\pi \in \Pi} \sum_{t=1}^{H} \langle \tilde{d}_{t}^{\pi} - d_{t}^{\pi}, r_t \rangle$, then for a given $N$ by independence and above bounded difference condition we can apply McDiarmid inequality Lemma A.4 (where $\xi = 2H^2/N$) to obtain

$$\mathbb{P}(|Z - \mathbb{E}[Z]| \geq \epsilon |N) \leq 2e^{-\frac{N\epsilon^2}{2H^2}} := \frac{\delta}{2} \quad (11)$$

Now note when $n > O(\frac{1}{n^2} \cdot \log(HSA/\delta))$, by Lemma 3.1 we can obtain $N > nd_m/2$ with probability $1 - \delta/2$, combining this result and solving $\epsilon$ in (11), we have

$$\sup_{\pi \in \Pi} \left| \sum_{t=1}^{H} \langle \tilde{d}_{t}^{\pi} - d_{t}^{\pi}, r_t \rangle \right| \leq O\left( \frac{H^4 \log(HSA/\delta)}{n \cdot d_m} \right) + \mathbb{E} \left[ \sup_{\pi \in \Pi} \sum_{t=1}^{H} \langle \tilde{d}_{t}^{\pi} - d_{t}^{\pi}, r_t \rangle \right]$$

with probability $1 - \delta$.

Next we bound $\mathbb{E} \left[ \sup_{\pi \in \Pi} \sum_{t=1}^{H} \langle \tilde{d}_{t}^{\pi} - d_{t}^{\pi}, r_t \rangle \right]$ using Rademacher complexity.

**Theorem D.2.**

$$\mathbb{E} \left[ \sup_{\pi \in \Pi} \sum_{t=1}^{H} \langle \tilde{d}_{t}^{\pi} - d_{t}^{\pi}, r_t \rangle \right] \leq O\left( \frac{H^4S^2 \log(SA)}{n \cdot d_m} \right)$$

**Proof.** Again, by martingale decomposition, we have

\[
\mathbb{E} \left[ \sup_{\pi \in \Pi} \sum_{t=1}^{H} \langle \tilde{d}_{t}^{\pi} - d_{t}^{\pi}, r_t \rangle \right] \\
= \mathbb{E} \left[ \sup_{\pi \in \Pi} \sum_{h=2}^{H} \langle v_{h-1}^{\pi}, \overline{T}_h - T_h \rangle \tilde{d}_{h-1}^{\pi} + \langle v_{1}^{\pi}, \tilde{d}_{1}^{\pi} - d_{1}^{\pi} \rangle \right] \\
= \mathbb{E} \left[ \sup_{\pi \in \Pi} \sum_{h=2}^{H} \langle v_{h-1}^{\pi}, \overline{T}_h - T_h \rangle \tilde{d}_{h-1}^{\pi} + \langle v_{1}^{\pi}, \tilde{d}_{1}^{\pi} - d_{1}^{\pi} \rangle \cdot \mathbb{1}(E) \right] \\
\leq \sum_{h=2}^{H} \mathbb{E} \left[ \sup_{\pi \in \Pi} \langle v_{h-1}^{\pi}, \overline{T}_h - T_h \rangle \tilde{d}_{h-1}^{\pi} \cdot \mathbb{1}(E) \right] + \mathbb{E} \left[ \sup_{\pi \in \Pi} \langle v_{1}^{\pi}, \tilde{d}_{1}^{\pi} - d_{1}^{\pi} \rangle \cdot \mathbb{1}(E) \right]
\]
Now we fix $N$ and consider $\mathbb{E} \left[ \sup_{\pi \in \Pi} \left| \langle v^\pi_h, (\hat{T}_h - T_h) \hat{d}^\pi_{h-1} \rangle \right| \right]$. Note given $N$, $\hat{T}_h$ has the form

$$
\hat{T}_h(s_h|s_{h-1}, a_{h-1}) = \frac{1}{N} \sum_{i=1}^{N} 1[s^{(i)}_{h,s_{h-1},a_{h-1}} = s_h]
$$

(13)

where we use subscript $s_{h-1}, a_{h-1}$ to specify that $(i)$-th data is sampled from true transition $P_h(s_{h-1}, a_{h-1})$. Next we unroll matrix $T_h \in \mathbb{R}^{S \times (S_2 A)}$ into a $(S_2 A) \times 1$ vector, i.e the vector $\pi \in \mathbb{R}^{S_2 A \times 1}$ with $(T_h)(s_h|s_{h-1}, a_{h-1}) = P_h(s_{h-1}, a_{h-1})$ and similarly $\hat{T}_h \in \mathbb{R}^{S_2 A \times 1}$ with $(\hat{T}_h)(s_h|s_{h-1}, a_{h-1}) = \hat{T}_h(s_{h-1}, a_{h-1})$. Also, we define $w^\pi_h(s_h, s_{h-1}, a_{h-1}) = v^\pi_h(s_h)\hat{d}^\pi_{h-1}(s_{h-1}, a_{h-1})$. By this definition, we have

$$
\mathbb{E} \left[ \sup_{\pi \in \Pi} \left| \langle v^\pi_h, (\hat{T}_h - T_h) \hat{d}^\pi_{h-1} \rangle \right| \right] = \mathbb{E} \left[ \sup_{\pi \in \Pi} \left| \sum_{s_{h-1}} \sum_{a_{h-1}} v^\pi_h(s_h)\hat{T}_h - T_h(s_{h-1}, a_{h-1})\hat{d}^\pi_{h-1}(s_{h-1}, a_{h-1}) \right| \right]
$$

(14)

Note $w^\pi_h$ satisfies

$$
||w^\pi_h||_1 = \sum_{s_h,s_{h-1},a_{h-1}} v^\pi_h(s_h)\hat{d}^\pi_{h-1}(s_{h-1}, a_{h-1}) \leq H \sum_{s_h,s_{h-1},a_{h-1}} \hat{d}^\pi_{h-1}(s_{h-1}, a_{h-1}) \leq HS,
$$

(15)

so let $F = \{ w : w \in \mathbb{R}^{S_2 A} ||w||_1 \leq HS \}$, then by (14) we can relax

$$
\mathbb{E} \left[ \sup_{\pi \in \Pi} \left| \langle w^\pi_h, \hat{T}_h - T_h \rangle \right| \right] \leq \mathbb{E} \left[ \sup_{w \in F} \left| \langle w, \hat{T}_h - T_h \rangle \right| \right]
$$

Next by (13) we have

$$
\hat{T}_h = \frac{1}{N} \sum_{i=1}^{N} e^{(i)}, \text{ where } e^{(i)}(s_h, s_{h-1}, a_{h-1}) = 1[s^{(i)}_{h,s_{h-1},a_{h-1}} = s_h]
$$

Note $e^{(i)}$ is unbiased estimator of $T_h$, by standard symmetrization technique using an independent copy of $e^{(i)}$’s, we have

$$
\mathbb{E} \left[ \sup_{w \in F} \left| \langle w, \hat{T}_h - T_h \rangle \right| \right] = \mathbb{E} \left[ \sup_{w \in F} \left| \langle w, \frac{1}{N} \sum_{i=1}^{N} e^{(i)} - \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^{N} e^{(i)} \right] \right| \right]
$$

$$
\leq \mathbb{E} \left[ \sup_{w \in F} \left| \frac{1}{N} \sum_{i=1}^{N} \langle w, e^{(i)} - e^{(i)} \rangle \right| \right]
$$

(16)

$$
\leq 2\mathbb{E} \left[ \sup_{w \in F} \left| \frac{1}{N} \sum_{i=1}^{N} \langle w, e^{(i)} \rangle \right| \right]
$$
Note \(||e^{(i)}||_\infty \leq 1\), by Lemma A.5, we have

\[
E \left[ \sup_{w \in F} \left| \frac{1}{N} \sum_{i=1}^{N} \langle w, e^{(i)} \rangle \right| \right] \leq 2HS \sqrt{\frac{2 \log(S^2A)}{N}}
\]

To sum up, by above and (14), (16), we have shown

\[
E \left[ \sup_{\pi \in \Pi} \left| \langle v^\pi_h, (\hat{T}_h - T_h)\hat{d}^\pi_{h-1} \rangle \right| \right] \leq 4HS \sqrt{\frac{2 \log(S^2A)}{N}}
\]

where we have conditional on \(N\).

Finally, by law of total expectation, we have

\[
E \left[ \sup_{\pi \in \Pi} \left| \langle v^\pi_h, (\hat{T}_h - T_h)\hat{d}^\pi_{h-1} \rangle \right| \right] \cdot 1(E) = E \left[ E \left[ \sup_{\pi \in \Pi} \left| \langle v^\pi_h, (\hat{T}_h - T_h)\hat{d}^\pi_{h-1} \rangle \right| \cdot 1(E) \right] \right] \leq E \left[ 1(E) \cdot 4HS \sqrt{\frac{2 \log(S^2A)}{N}} \right]
\]

\[
\leq E \left[ 1(E) \cdot 4HS \sqrt{\frac{4 \log(S^2A)}{nd_m}} \right]
\]

Plug this into (12), we get the stated result.

---

**Proof of Theorem 3.3.** The proof of Theorem 3.3 comes from combing Lemma C.1, C.2 and Theorem D.1, Theorem D.2 and use a scale factor for \(\delta\).

**Remark D.3.** For readers who checked all above proofs carefully might notice the use of Mcdiarmid inequality and Rademacher complexity do not require \(\Pi\) is deterministic policy class. This means our result of Theorem 3.3 holds uniformly over all policies, which is a policy class of infinite cardinality.

We end this section by presenting the following result.
**Corollary D.4** (The second part of Theorem 3.3). *In the small failure probability regime, i.e. \( \delta < 1/e^{S^2} \), then under the same condition of Theorem 3.3, we have*

\[
\sup_{\pi \in \Pi} |\hat{v}^\pi - v^\pi| \leq c_1 \sqrt{\frac{H^4 \log(\text{HSA}/\delta)}{n \cdot d_m}}.
\]

**Proof.** If \( \delta < 1/e^{S^2} \), then \( \sqrt{S^2 \log(SA)} < \sqrt{\log(\text{HSA}/\delta)} \) in (4). By Theorem 3.3 we directly have the stated result. \( \blacksquare \)

**E  Proof of uniform convergence in OPE problem with martingale concentration inequalities: Theorem 3.5**

A reminder that all results in this section use data \( D \) for OPEMA estimator \( \hat{v}^\pi \).

**E.1 Martingale concentration result on \( \sum_{t=1}^H \langle \tilde{d}_t^\pi - d_t^\pi, r_t \rangle \).**

Let \( X = \sum_{t=1}^H \langle \tilde{d}_t^\pi - d_t^\pi, r_t \rangle \) and \( D_h := \{ s_t^{(i)}, a_t^{(i)} : t = 1, ..., h \}_{i=1}^n \). Since \( D_h \) forms a filtration, then by law of total expectation we have \( X_t = \mathbb{E}[X|D_t] \) is martingale. Moreover, we have

**Lemma E.1.**

\( X_t := \mathbb{E}[X|D_t] = \sum_{h=2}^t \langle v_h^\pi, (\tilde{T}_h - T_h)\tilde{d}_{h-1}^\pi \rangle + \langle v_1^\pi, \tilde{d}_1^\pi - d_1^\pi \rangle \).

**Proof of Lemma E.1.** By martingale decomposition Theorem C.3 and note \( \tilde{T}_i, \tilde{d}_i^\pi \) are measurable w.r.t. \( D_t \) for \( i = 1, ..., t \), so we have

\[
\mathbb{E}[X|D_t] = \sum_{h=t+1}^H \mathbb{E} \left[ \langle v_h^\pi, (\tilde{T}_h - T_h)\tilde{d}_{h-1}^\pi \rangle \bigg| D_t \right] + \sum_{h=2}^t \langle v_h^\pi, (\tilde{T}_h - T_h)\tilde{d}_{h-1}^\pi \rangle + \langle v_1^\pi, (\tilde{d}_1^\pi - d_1^\pi) \rangle.
\]

Note for \( h \geq t + 1 \), \( D_t \subset D_{h-1} \), so by total law of expectation (tower property) we have

\[
\mathbb{E} \left[ \langle v_h^\pi, (\tilde{T}_h - T_h)\tilde{d}_{h-1}^\pi \rangle \bigg| D_t \right] =\mathbb{E} \left[ \mathbb{E} \left[ \langle v_h^\pi, (\tilde{T}_h - T_h)\tilde{d}_{h-1}^\pi \rangle \bigg| D_{h-1} \right] \bigg| D_t \right] = \mathbb{E} \left[ \mathbb{E} \left[ (\tilde{T}_h - T_h) \bigg| D_{h-1} \right] \tilde{d}_{h-1}^\pi \right] \bigg| D_t \right] = 0
\]

where the last equality uses \( \tilde{T}_h \) is unbiased of \( T_h \) given \( D_{h-1} \). This gives the desired result. \( \blacksquare \)
Next we show martingale difference $|X_t - X_{t-1}|$ is bounded with high probability.

**Lemma E.2.** With probability $1 - \delta$,

$$\sup_t |X_t - X_{t-1}| \leq O(\sqrt{\frac{H^2 \log(\text{HSA}/\delta)}{n \cdot d_m}}).$$

**Proof.**

$$|X_t - X_{t-1}| = \langle v_t^\pi, (\tilde{T}_t - T_t)d_{t-1}^\pi \rangle \leq ||(\tilde{T}_t - T_t)^Tv_t^\pi||_\infty \cdot \tilde{d}_{t-1}^\pi \leq ||(\tilde{T}_t - T_t)^Tv_t^\pi||_\infty.$$

For any fixed pair $(s_t, a_t)$, we have

$$(\tilde{T}_t - T_t)^Tv_t^\pi(s_{t-1}, a_{t-1})$$

$$= \mathbb{I}(E_{t-1}) \cdot ((\tilde{T}_t - T_t)^Tv_t^\pi)(s_{t-1}, a_{t-1})$$

$$= \mathbb{I}(E_{t-1}) \cdot \sum_{s_t} v_t^\pi(s_t)(\tilde{T}_t - T_t)(s_t|s_{t-1}, a_{t-1})$$

$$= \mathbb{I}(E_{t-1}) \cdot \left( \sum_{s_t} v_t^\pi(s_t)\tilde{T}_t(s_t|s_{t-1}, a_{t-1}) - \mathbb{E}[v_t^\pi|s_{t-1}, a_{t-1}] \right)$$

$$= \mathbb{I}(E_{t-1}) \cdot \left( \sum_{s_t} v_t^\pi(s_t)\frac{1}{n_{s_{t-1}, a_{t-1}}} \sum_{i=1}^n \mathbb{I}(s_t^{(i)} = s_t, s_{t-1}^{(i)} = s_{t-1}, a_{t-1}^{(i)} = a_{t-1}) - \mathbb{E}[v_t^\pi|s_{t-1}, a_{t-1}] \right)$$

$$= \mathbb{I}(E_{t-1}) \left( \frac{1}{n_{s_{t-1}, a_{t-1}}} \sum_{i=1}^n v_t^\pi(s_t^{(i)} \mathbb{I}(s_t^{(i)} = s_t, s_{t-1}^{(i)} = s_{t-1}, a_{t-1}^{(i)} = a_{t-1}) - \mathbb{E}[v_t^\pi|s_{t-1}, a_{t-1}] \right)$$

$$= \mathbb{I}(E_{t-1}) \left( \frac{1}{n_{s_{t-1}, a_{t-1}}} \sum_{s_t^{(i)}, a_{t-1}^{(i)}} v_t^\pi(s_t^{(i)}) - \mathbb{E}[v_t^\pi|s_{t-1}, a_{t-1}] \right),$$

where the fourth line uses the definition of $\tilde{T}_t$ and the fifth line uses the fact $\sum s_t v_t^\pi(s_t)\mathbb{I}(s_t^{(i)} = s_t, s_{t-1}^{(i)} = s_{t-1}, a_{t-1}^{(i)} = a_{t-1}) = v_t^\pi(s_t^{(i)})\mathbb{I}(s_t^{(i)} = s_t, s_{t-1}^{(i)} = s_{t-1}, a_{t-1}^{(i)} = a_{t-1}).$

Note $||v_t^\pi(\cdot)||_\infty \leq H$ and also conditional on $E_t$, $n_{s_t, a_t} \geq nd_t^\mu(s_t, a_t)/2$, therefore by Hoeffding’s inequality and a Union bound we obtain with probability $1 - \delta$

$$\sup_t |X_t - X_{t-1}| \leq O\left( \sqrt{\frac{H^2 \log(\text{HSA}/\delta)}{n \cdot \min_t s_t, a_t d_t^\mu(s_t, a_t)}} \right) = O\left( \sqrt{\frac{H^2 \log(\text{HSA}/\delta)}{n \cdot d_m}} \right).$$

Next we calculate the conditional variance of $\text{Var}[X_{t+1}|D_t]$. 

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Lemma E.3. We have the following decomposition of conditional variance:

$$\text{Var}[X_{t+1}|D_t] = \sum_{s_t,a_t} \tilde{d}_t^2(s_t,a_t) \cdot \mathbb{I}(E_t) \cdot \text{Var}[v_{t+1}(s_{t+1})|s_t^{(1)} = s_t, a_t^{(1)} = a_t]$$

Proof. Indeed,

$$\text{Var}[X_{t+1}|D_t] = \text{Var} \left[ \sum_{s_t,a_t} v_{t+1}(s_{t+1})(\tilde{T} - T)(s_{t+1}|s_t,a_t)\tilde{d}_t^2(s_t,a_t) \bigg| D_t \right]$$

$$= \sum_{s_t,a_t} \text{Var} \left[ \sum_{s_{t+1}} v_{t+1}(s_{t+1}) (\tilde{T} - T)(s_{t+1}|s_t,a_t) \bigg| D_t \right] \tilde{d}_t^2(s_t,a_t)^2$$

$$= \sum_{s_t,a_t} \mathbb{I}(E_t) \cdot \text{Var} \left[ \sum_{s_{t+1}} v_{t+1}(s_{t+1}) \tilde{T}(s_{t+1}|s_t,a_t) \bigg| D_t \right] \tilde{d}_t^2(s_t,a_t)^2$$

$$= \sum_{s_t,a_t} \mathbb{I}(E_t) \cdot \text{Var} \left[ \sum_{s_{t+1}} v_{t+1}(s_{t+1}) \frac{1}{n_{s_t,a_t}} \sum_{i=1}^n \mathbb{I}(s_{t+1}^{(i)} = s_t + 1, s_t^{(i)} = s_t, a_t^{(i)} = a_t) \bigg| D_t \right] \tilde{d}_t^2(s_t,a_t)^2$$

$$= \sum_{s_t,a_t} \frac{\tilde{d}_t^2(s_t,a_t)^2}{n_{s_t,a_t}^2} \mathbb{I}(E_t) \cdot \text{Var}[v_{t+1}(s_{t+1})|s_t^{(1)} = s_t, a_t^{(1)} = a_t]$$

(18)

where the second equal sign comes from the fact that when conditional on $D_t$, we can separate $n$ episodes into $SA$ groups and episodes from different groups are independent of each other. The third uses $\mathbb{I}(E_t)$ is measurable w.r.t $D_t$. Similarly, the last equal sign again uses $n_{s_t,a_t}$ episodes are independent given $D_t$. 

Lemma E.4 (Yin & Wang (2020) Lemma 3.4). For any policy $\pi$ and any MDP.

$$\text{Var}_\pi \left[ \sum_{t=1}^H r_t^{(1)} \right] = \sum_{t=1}^H \left( \mathbb{E}_\pi \left[ \text{Var}[r_t^{(1)} + V_t^{\pi}(s_{t+1})|s_t^{(1)}, a_t^{(1)}] \right] \right)$$

$$+ \mathbb{E}_\pi \left[ \text{Var}[\mathbb{E}[r_t^{(1)} + V_t^{\pi}(s_{t+1})|s_t^{(1)}, a_t^{(1)}]|s_t^{(1)}] \right].$$

This Lemma suggests if we can bound $\tilde{d}_t^2$ by $O(d_t^2)$ with high probability, then by Lemma E.3
we have w.h.p
\[ \sum_{t=1}^{H} \Var[X_{t+1} | D_t] \leq O \left( \frac{1}{nd_m} \cdot \sum_{t=1}^{H} \E[\Var[v_{t+1}^{(1)}(s_{t+1}^{(1)}, a_{t}^{(1)})]] \right) \leq O \left( \frac{H^2}{nd_m} \right) \]

Note this gives only $H^2$ dependence for $\sum_{t=1}^{H} \Var[X_{t+1} | D_t]$ instead of a naive bound with $H^3$ and helps us to find a $H$ factor.

Next we show how to bound $\tilde{d}_t^\pi$.

### E.2 Bounding $\tilde{d}_t^\pi(s_t, a_t) - d_t^\pi(s_t, a_t)$

Our analysis is based on using martingale structure to derive bound on $\tilde{d}_t^\pi(s_t, a_t) - d_t^\pi(s_t, a_t)$ for fixed $t, s_t, a_t$ with probability $1 - \delta/HSA$, then use a union bound to get a bound for all $\tilde{d}_t^\pi(s_t, a_t) - d_t^\pi(s_t, a_t)$ with probability $1 - \delta$.

Concretely, in (9) if we only extract the specific $(s_t, a_t)$, then we have
\[ \tilde{d}_t^\pi(s_t, a_t) - d_t^\pi(s_t, a_t) = \sum_{h=2}^{t} (\Gamma_{h+1:t}\pi_h(\bar{T}_h - T_h)d_{h-1}^\pi)(s_t, a_t) + (\Gamma_{1:t}(\tilde{d}_1^\pi - d_1^\pi))(s_t, a_t) \]

where $\tilde{d}_t^\pi(s_t, a_t) - d_t^\pi(s_t, a_t)$ already forms a martingale with filtration $\mathcal{F}_t = \sigma(D_t)$ and $(\Gamma_{h+1:t}\pi_h(\bar{T}_h - T_h)d_{h-1}^\pi)(s_t, a_t)$ is the corresponding martingale difference since
\[ \E[(\Gamma_{h+1:t}\pi_h(\bar{T}_h - T_h)d_{h-1}^\pi)(s_t, a_t) | \mathcal{F}_{h-1}] = (\Gamma_{h+1:t}\pi_h \E[(\bar{T}_h - T_h) | \mathcal{F}_{h-1}]d_{h-1}^\pi)(s_t, a_t) = 0. \]

Now we fix specific $(s_t, a_t)$. Then denote $(\Gamma_{h+1:t}\pi_h)(s_t, a_t) := \Gamma'_{h:t} \in \mathbb{R}^{1 \times S}$, then we have
\[ |(\Gamma_{h+1:t}\pi_h(\bar{T}_h - T_h)d_{h-1}^\pi)(s_t, a_t)| = |\Gamma'_{h:t}(\bar{T}_h - T_h)d_{h-1}^\pi| = |(\bar{T}_h - T_h)^T \Gamma'_{h:t}d_{h-1}^\pi| \leq \|\Gamma'_{h:t}(\bar{T}_h - T_h)\|_\infty \cdot 1. \]

Note here $\Gamma'_{h:t}(\bar{T}_h - T_h)$ is a row vector with dimension $SA$.

**Bounding** $\|\Gamma'_{h:t}(\bar{T}_h - T_h)\|_\infty$.
In fact, for any given \((s_{h-1}, a_{h-1})\), we have
\[
\Gamma'_{h:t}(\tilde{T}_h - T_h)(s_{h-1}, a_{h-1}) = \mathbb{1}(E_t) \cdot \Gamma'_{h:t}(\tilde{T}_h - T_h)(s_{h-1}, a_{h-1})
\]
\[
= \mathbb{1}(E_t) \cdot \Gamma'_{h:t}\left(\frac{1}{n_{s_{t-1}, a_{t-1}}} \sum_{i:s_{h-1} = s_{h-1}, a_{h-1} = a_{h-1}} \mathbb{E}[e_{s_h}^{(i)} | s_{h-1}^{(1)} = s_{h-1}, a_{h-1}^{(1)} = a_{h-1}]\right)
\]
\[
= \mathbb{1}(E_t) \left(\frac{1}{n_{s_{t-1}, a_{t-1}}} \sum_{i:s_{h-1} = s_{h-1}, a_{h-1} = a_{h-1}} \Gamma'_{h:t}(s_{h}^{(i)}) - \mathbb{E}[\Gamma'_{h:t}(s_{h}^{(i)}) | s_{h-1} = s_{h-1}, a_{h-1} = a_{h-1}]\right)
\]
Note by definition \(\Gamma'_{h:t}(s_{h}^{(i)}) \leq 1\), since \((\Gamma_{h+1:t} \pi_h)(s_t, a_t) := \Gamma'_{h:t} \in \mathbb{R}^{1 \times S}\) and \(\Gamma_{h+1:t}, \pi_h\) are just probability transitions. Therefore by Hoeffding’s inequality and law of total expectation, we have
\[
\mathbb{P}\left(\left|\Gamma'_{h:t}(\tilde{T}_h - T_h)(s_{h-1}, a_{h-1})\right| > \epsilon\right) = \mathbb{P}\left(\left|\Gamma'_{h:t}(\tilde{T}_h - T_h)(s_{h-1}, a_{h-1})\right| > \epsilon | E_t\right)
\]
\[
\leq \mathbb{E}\left[\exp\left(-\frac{2n_{s_{h-1}, a_{h-1}} \cdot \epsilon^2}{1}\right) | E_t\right] \leq \exp\left(-\frac{n \min_{h, s_{h-1}, a_{h-1}} d^{\mu}_{h-1}(s_{h-1}, a_{h-1}) \cdot \epsilon^2}{1}\right)
\]
and apply a union bound to get
\[
P(\sup_h ||\Gamma'_{h:t}(\tilde{T}_h - T_h)||_\infty > \epsilon) \leq H \cdot \sup_h P(||\Gamma'_{h:t}(\tilde{T}_h - T_h)||_\infty > \epsilon)
\]
\[
\leq HSA \cdot \sup_{h, s_{h-1}, a_{h-1}} \mathbb{P}\left(\left|\Gamma'_{h:t}(\tilde{T}_h - T_h)(s_{h-1}, a_{h-1})\right| > \epsilon\right)
\]
\[
\leq HSA \cdot \exp\left(-\frac{n \min_{h, s_{h-1}, a_{h-1}} d^{\mu}_{h-1}(s_{h-1}, a_{h-1}) \cdot \epsilon^2}{1}\right) = \frac{\delta}{HSA}
\]
(19)

Let the right hand side of (19) to be \(\delta/HSA\), then we have w.p. \(1 - \delta/HSA\),
\[
\sup_h ||\Gamma'_{h:t}(\tilde{T}_h - T_h)||_\infty \leq O\left(\sqrt{\frac{1}{n \cdot d_m} \log \frac{H^2 S^2 A^2}{\delta}}\right).
\]
(20)
Go back to bounding $\tilde{d}_t^\pi(s_t, a_t) - d_t^\pi(s_t, a_t)$. By Azuma-Hoeffding’s inequality (Lemma A.6), we have

$$\mathbb{P}(|\tilde{d}_t^\pi(s_t, a_t) - d_t^\pi(s_t, a_t)| > \epsilon) \leq \exp\left(-\frac{\epsilon^2}{\sum_{i=1}^{t}(\sup_h ||\Gamma'_{h,t}(T_h - T_h)||_\infty)^2}\right) := \delta / HSA,$$

where $\sum_{i=1}^{t}(\sup_h ||\Gamma'_{h,t}(T_h - T_h)||_\infty)^2$ is the sum of bounded square differences in Azuma-Hoeffding’s inequality. Therefore we have w.p. $1 - \delta / HSA$,

$$|\tilde{d}_t^\pi(s_t, a_t) - d_t^\pi(s_t, a_t)| \leq O\left(\sqrt{t \cdot (\sup_h ||\Gamma'_{h,t}(T_h - T_h)||_\infty)^2 \log \frac{HSA}{\delta}}\right),$$

(21)

combining (20) with above we further have that w.p. $1 - 2\delta / HSA$,

$$|\tilde{d}_t^\pi(s_t, a_t) - d_t^\pi(s_t, a_t)| \leq O\left(\sqrt{t \cdot (\sum_{i=1}^{t}(sup_h ||\Gamma'_{h,t}(T_h - T_h)||_\infty)^2 \log \frac{HSA}{\delta}}\right)$$

Lastly, by a union bound and simple scaling (from $2\delta$ to $\delta$) we have w.p. $1 - \delta$

$$\sup_t ||\tilde{d}_t^\pi - d_t^\pi||_\infty \leq O\left(\sqrt{\frac{H}{nd_m} \log \frac{H^2S^2A^2}{\delta} \log \frac{HSA}{\delta}}\right).$$

This implies that w.p. $1 - \delta$, $\forall t, s_t, a_t$,

$$\tilde{d}_t^\pi(s_t, a_t)^2 \leq 2d_t^\pi(s_t, a_t)^2 + O\left(\frac{H}{nd_m} \log \frac{H^2S^2A^2}{\delta} \log \frac{HSA}{\delta}\right).$$

(22)

Combining (22) with Lemma E.4 and Lemma E.3, we obtain:

**Lemma E.5.** With probability $1 - \delta$,

$$\sum_{t=1}^{H} \text{Var}[X_{t+1}|D_t] \leq O\left(\frac{H^2}{nd_m}\right) + O\left(\frac{H^4 SA}{n^2 d_m^2} \cdot \log \frac{H^2 S^2 A^2}{\delta} \log \frac{HSA}{\delta}\right)$$

(23)

---

9To be more precise here we actually use a weaker version of Azuma-Hoeffding’s inequality, see Remark E.7.
Proof of Lemma E.5. By (22) and Lemma E.3, we have \( \forall t \), with probability \( \delta \), at least 1, that
\[
\text{Var}[X_{t+1} | D_t] \leq \sum_{s_t, a_t} O\left( \frac{d_t^p(s_t, a_t)^2}{nd_m} \right) \cdot \text{Var}[v_{t+1}^{(1)}(s_{t+1}) | s_t = s_t, a_t = a_t]
\]
\[
\leq \sum_{s_t, a_t} O\left( \frac{1}{nd_m} \right) \left( 2d_t^p(s_t, a_t)^2 + O\left( \frac{H^2 S^2 A^2}{\delta} \log \frac{HSA}{\delta} \right) \right) \cdot \text{Var}[v_{t+1}^{(1)}(s_{t+1}) | s_t = s_t, a_t = a_t]
\]
\[
\leq \sum_{s_t, a_t} O\left( \frac{1}{nd_m} \right) \left( 2d_t^p(s_t, a_t) + O\left( \frac{H^2 S^2 A^2}{\delta} \log \frac{HSA}{\delta} \right) \right) \cdot \text{Var}[v_{t+1}^{(1)}(s_{t+1}) | s_t = s_t, a_t = a_t]
\]
\[
\leq O\left( \frac{1}{nd_m} \right) E \left[ \text{Var}[v_{t+1}^{(1)}(s_{t+1}) | s_t, a_t] \right] + O\left( \frac{1}{nd_m} \cdot H\cdot \frac{H^2 S^2 A^2}{\delta} \log \frac{HSA}{\delta} \cdot H^2 SA \right)
\]
\[
= O\left( \frac{1}{nd_m} \right) E \left[ \text{Var}[v_{t+1}^{(1)}(s_{t+1}) | s_t, a_t] \right] + O\left( \frac{H^4 SA}{n^2 d_m^2} \cdot \frac{H^2 S^2 A^2}{\delta} \log \frac{HSA}{\delta} \right)
\]
then sum over \( t \) and apply Lemma E.4 gives the stated result. \( \square \)

Combining all the results, we are able to prove:

**Theorem E.6.** With probability \( 1 - \delta \), we have
\[
\left| \sum_{t=1}^H \langle \bar{d}_t^p - d_t^p, r_t \rangle \right| \leq O\left( \sqrt{\frac{H^2 \log(HSA/\delta)}{nd_m}} + \sqrt{\frac{H^4 S^2 A^2}{n^2 d_m^2} \log(HSA/\delta)} \right)
\]
where \( O(\cdot) \) absorbs only the absolute constants.

**Proof of Theorem E.6.** Recall \( X = \sum_{t=1}^H \langle \bar{d}_t^p - d_t^p, r_t \rangle \) and by law of total expectation it is easy to show \( E[X] = 0 \). Next denote \( \sigma^2 = O\left( \frac{H^2}{nd_m} \right) + O\left( \frac{H^4 SA}{n^2 d_m} \cdot \log \frac{H^2 S^2 A^2}{\delta} \log \frac{HSA}{\delta} \right) \) as in Lemma E.5 and also let \( M = \sup_t |X_t - X_{t-1}| \). Then by Freedman inequality (Lemma A.7), we have with probability \( 1 - \delta/3 \),
\[
|X - E[X]| \leq \sqrt{8\sigma^2 \cdot \log(3/\delta)} + \frac{2M}{3} \cdot \log(3/\delta), \quad \text{Or} \quad W \geq \sigma^2.
\]
where \( W = \sum_{t=1}^H \text{Var}[X_{t+1} | D_t] \). Next by Lemma E.5, we have \( P(W \geq \sigma^2) \leq 1/3\delta \), this implies with probability \( 1 - 2\delta/3 \),
\[
|X - E[X]| \leq \sqrt{8\sigma^2 \cdot \log(3/\delta)} + \frac{2M}{3} \cdot \log(3/\delta).
\]
Finally, by Lemma E.2, we have \( P(M \geq O\left( \sqrt{\frac{H^2 \log(HSA/\delta)}{nd_m}} \right)) \leq \delta/3 \). Also use \( E[X] = 0 \), we have with probability \( 1 - \delta \),
\[
|X| \leq \sqrt{8\sigma^2 \cdot \log(3/\delta)} + O\left( \sqrt{\frac{H^2 \cdot \log(HSA/\delta)}{nd_m}} \log(3/\delta) \right).
\]
Plugging back the expression of $\sigma^2 = O \left( \frac{H^2}{n^2 \delta_m^2} \right) + O \left( \frac{H^4 \alpha \beta^2}{n^2 \delta_m^2} \cdot \log \frac{H^2 \alpha \beta^2}{\delta_m} \log \frac{H \alpha}{\delta_m} \right)$ and assimilating the same order terms give the desired result.

**Remark E.7.** Rigorously, standard Azuma-Hoeffding’s inequality Lemma A.6 does not apply to (21) since $\sup_h ||\Gamma_{h,t}(\bar{T}_h - T_h)||_{\infty}$ is not a deterministic upper bound, we only have the difference bound with high probability sense, see (20). Therefore, strictly speaking, we need to apply Theorem 32 in Chung & Lu (2006) which is a weaker Azuma-Hoeffding’s inequality allowing bounded difference with high probability. The same logic applies for a weaker Freedman’s inequality consisting of Theorem 34 and Theorem 37 in Chung & Lu (2006) since our martingale difference $M = \sup_t |X_t - X_{t-1}|$ in the proof of Theorem E.6 is bounded with high probability. We avoid explicitly using them in order to make our proofs more readable for our readers.

We end this section by giving the proofs of Theorem 3.4 and Theorem 3.5.

**Proof of Theorem 3.4 and Theorem 3.5.** The proof of Lemma 3.4 comes from Lemma C.1, Lemma C.2 and Theorem E.6. The proof of Theorem 3.5 relies on applying a union bound over $\Pi$ in Theorem 3.4 (recall all non-stationary deterministic policies have $||\Pi|| = A_{HS}$), then extra dependence of $\sqrt{\log(||\Pi||)} = \sqrt{HS \log(A)}$ pops out. Note that the higher order term has two trailing log terms (see the right hand side of (23)), so when replacing $\delta$ by $\delta/||\Pi||$ with a union bound, both terms will give extra $\sqrt{HS}$ dependence so in higher order term we have extra $HS$ dependence but not just $\sqrt{HS}$.

**F Proof of uniform convergence problem with local policy class.**

In this section, we consider using OPEMA estimator with data $D'$. Also, WLOG we only consider deterministic reward (as implied by Lemma C.2 random reward only causes lower order dependence). Also, we fix $N > 0$ for the moment. First recall for all $t = 1, ..., H$

\[
v^*_t(s_t) = E_\pi \left[ \sum_{t'=t}^H r_{t'}(s_{t'}^{(1)}, a_{t'}^{(1)}) \left| s_t^{(1)} = s_t, a_t^{(1)} = a_t \right. \right]
\]

\[
Q^*_t(s_t, a_t) = E_\pi \left[ \sum_{t'=t}^H r_{t'}(s_{t'}^{(1)}, a_{t'}^{(1)}) \left| s_t^{(1)} = s_t, a_t^{(1)} = a_t \right. \right]
\]

where $r_t(s, a)$ are deterministic rewards and $s_t^{(1)}, a_t^{(1)}$ are random variables. Consider $v_t^*, Q_t^*$ as vectors, then by standard Bellman equations we have for all $t = 1, ..., H$ (define
We now analyze \(v_{H+1} = Q_{H+1} = 0\)
\[
Q^\pi_t = r_t + P^\pi_{t+1}Q^\pi_{t+1} = r_t + P^\pi_{t+1}v^\pi_{t+1},
\]
where \(P^\pi_t \in \mathbb{R}^{(SA) \times (SA)}\) is the state-action transition and \(P_t(\cdot, \cdot) \in \mathbb{R}^{(SA) \times S}\) is the transition probabilities defined in Section 2. Also, we have bellman optimality equations:
\[
Q^\pi_t = r_t + P^\pi_{t+1}v^\pi_{t+1}, \quad v^\pi_t(s_t) := \max_{a_t} Q^\pi_t(s_t, a_t) \forall s_t (25)
\]
where \(\pi^*\) is one optimal deterministic policy. The corresponding Bellman equations and Bellman optimality equations for empirical MDP \(\widehat{M}\) are defined similarly. Since we consider deterministic rewards, by Bellman equations we have
\[
\widehat{Q}^\pi_t - Q^\pi_t = \hat{P}^\pi_{t+1}\widehat{Q}^\pi_{t+1} - P^\pi_{t+1}Q^\pi_{t+1} = (\hat{P}^\pi_{t+1} - P^\pi_{t+1})\widehat{Q}^\pi_{t+1} + P^\pi_{t+1}(\widehat{Q}^\pi_{t+1} - Q^\pi_{t+1})
\]
for \(t = 1, \ldots, H\). By writing it recursively, we have \(\forall t = 1, \ldots, H - 1\)
\[
\widehat{Q}^\pi_t - Q^\pi_t = \sum_{h=t+1}^{H} \Gamma^\pi_{t+1;h-1}(\hat{P}^\pi_{h} - P^\pi_{h})\hat{v}^\pi_{h}
\]
where \(\Gamma^\pi_{t;h} = \prod_{i=1}^{h} P^\pi_{i}\) is the multi-step state-action transition and \(\Gamma^\pi_{t+1;t} := I\).

Note \(\hat{\pi}^*\) to be the empirical optimal policy over \(\widehat{M}\), we are interested in how to obtain uniform convergence for any policy \(\pi\) that is close to \(\hat{\pi}^*\). More precisely, in this section we consider the policy class \(\Pi_1\) to be:
\[
\Pi_1 := \{\pi : s.t. ||\hat{v}^\pi_t - \hat{v}^\pi_*||_\infty \leq \epsilon_{opt}, \forall t = 1, \ldots, H\}
\]
where \(\epsilon_{opt} \geq 0\) is a parameter decides how large the policy class is. We now assume \(\hat{\pi}\) to be any policy within \(\Pi_1\) throughout this section. Also, \(\hat{\pi}\) may be a policy learned from a learning algorithm using the data \(D\). In this case, \(\hat{\pi}\) may not be independent of \(\hat{P}\).

We start with the following simple calculation:\footnote{Since all quantities in the calculation are vectors, so the absolute value \(|\cdot|\) used is point-wise operator.}
\[
\left|\hat{Q}^\pi_t - Q^\pi_t\right| \leq \sum_{h=t+1}^{H} \Gamma^\pi_{t+1;h-1} \left|\hat{P}_h - P_h\right| \hat{v}^\pi_h \\
\leq \sum_{h=t+1}^{H} \Gamma^\pi_{t+1;h-1} \left|\hat{P}_h - P_h\right| \hat{v}^\pi_h + \sum_{h=t+1}^{H} \Gamma^\pi_{t+1;h-1} \left|\hat{P}_h - P_h\right| \left(\hat{v}^\pi_h - v^\pi_h\right) \tag{26}
\]

We now analyze \((***)\) and \((****)\).
F.1 Analyzing $\sum_{h=t+1}^{H} \Gamma_{t+1:h-1}^\pi \left| (\hat{P}_h - P_h)(\hat{v}_h^\pi - \bar{v}_h^\pi) \right|$.

First, by vector induced matrix norm we have

$$\left\| \sum_{h=t+1}^{H} \Gamma_{t+1:h-1}^\pi \left| (\hat{P}_h - P_h)(\hat{v}_h^\pi - \bar{v}_h^\pi) \right| \right\|_{\infty} \leq H \cdot \sup_h \left\| \Gamma_{t+1:h-1}^\pi \right\|_{\infty} \left\| (\hat{P}_h - P_h)(\hat{v}_h^\pi - \bar{v}_h^\pi) \right\|_{\infty} \leq H \cdot \sup_h \left\| (\hat{P}_h - P_h)(\hat{v}_h^\pi - \bar{v}_h^\pi) \right\|_{\infty}$$

where the last equal sign uses multi-step transition $\Gamma_{t+1:h-1}^\pi$ is row-stochastic. Note given $N, \hat{P}_t(\cdot, \cdot)$ all have $N$ in the denominator. Therefore, by Hoeffding inequality and a union bound we have with probability $1 - \delta$,

$$\sup_{t,s,t-1,a_{t-1}} |\hat{P}_t(s_t|s_{t-1},a_{t-1}) - P_t(s_t|s_{t-1},a_{t-1})| \leq O\left(\sqrt{\log(HSA/\delta)} / N\right),$$

this indicates

$$\sup_h \left\| (\hat{P}_h - P_h)(\hat{v}_h^\pi - \bar{v}_h^\pi) \right\|_{\infty} \leq \epsilon_{\text{opt}} \cdot \sup_h \left\| \hat{P}_h - P_h \right\|_{\infty} \leq \epsilon_{\text{opt}} \cdot O(S\sqrt{\log(HSA/\delta)} / N),$$

where $1 \in \mathbb{R}^S$ is all-one vector. To sum up, we have

**Lemma F.1.** Fix $N > 0$, we have with probability $1 - \delta$, for all $t = 1, ..., H - 1$

$$\sum_{h=t+1}^{H} \Gamma_{t+1:h-1}^\pi \left| (\hat{P}_h - P_h)(\hat{v}_h^\pi - \bar{v}_h^\pi) \right| \leq \epsilon_{\text{opt}} \cdot O\left(\sqrt{H^2S^2\log(HSA/\delta)} / N \cdot 1\right)$$

Now we consider $(***)$.

F.2 Analyzing $\sum_{h=t+1}^{H} \Gamma_{t+1:h-1}^\pi \left| (\hat{P}_h - P_h)\hat{v}_h^\pi \right|$.

**Lemma F.2.** Given $N$, we have with probability $1 - \delta$, $\forall t = 1, ..., H - 1$

$$\sum_{h=t+1}^{H} \Gamma_{t+1:h-1}^\pi \left| (\hat{P}_h - P_h)\hat{v}_h^\pi \right| \leq \sum_{h=t+1}^{H} \Gamma_{t+1:h-1}^\pi \left( 4 \sqrt{\log(HSA/\delta) / N} \sqrt{\text{Var}(\hat{v}_h^\pi)} + \frac{4(H-t)}{3N} \log(HSA/\delta) \cdot 1 \right)$$

where $\text{Var}(v_h^\pi) \in \mathbb{R}^{SA}$ and $\text{Var}(v_h^\pi)(s_{t-1},a_{t-1}) = \text{Var}_{s_t}[v_h^\pi(\cdot)|s_{t-1},a_{t-1}]$ and $\cdot, \cdot, \sqrt{\cdot}$ are point-wise operator.

---

11 For $A$ a matrix and $x$ a vector we have $\|Ax\|_\infty \leq \|A\|_\infty \|x\|_\infty$. 

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Proof of Lemma F.2. The key point is to guarantee \( \hat{P}_h \) is independent of \( \hat{v}_h^{\pi^*} \) so that we can apply Bernstein inequality w.r.t the randomness in \( \hat{P}_h \). In fact, note given \( N \) all data pairs in \( D' \) are independent of each other, and \( \hat{P}_h \) only uses data from \( h - 1 \) to \( h \). Moreover, \( \hat{v}_h^{\pi^*} \) only uses data from time \( h \) to \( H \) since \( \hat{v}_h^{\pi^*} \) uses data from \( h \) to \( H \) by bellman equation (24) for any \( \pi \) and optimal policy \( \hat{\pi}_h^{*} \) also only uses data from \( h \) to \( H \) by bellman optimality equation (25).

Then by Bernstein inequality (Lemma A.3), with probability \( 1 - \delta \)

\[
\left| (\hat{P}_h - P_h)\hat{v}_h^{\pi^*} \right| (s_{t-1}, a_{t-1}) \leq 4\sqrt{\frac{\log(1/\delta)}{N} \text{Var}(\hat{v}_h^{\pi^*})} (s_{t-1}, a_{t-1}) + \frac{4(H - t)}{3N} \log(\frac{1}{\delta})
\]

apply a union bound and take the sum we get the stated result. ■

Now combine Lemma F.1 and Lemma F.2 we obtain with probability \( 1 - \delta \), for all \( t = 1, ..., H - 1 \)

\[
\left| \hat{Q}_t - Q_t \right| \leq \sum_{h=t+1}^{H} \Gamma_{t+1:h-1} \left( 4\sqrt{\frac{\log(HSA/\delta)}{N} \text{Var}(\hat{v}_h^{\pi^*})} (s_{t-1}, a_{t-1}) + \frac{4(H - t)}{3N} \log(\frac{HSA}{\delta}) \right) + c_1\epsilon_{opt} \cdot \frac{1}{\sqrt{N}}
\]

Next note \( \sqrt{\text{Var}(\cdot)} \) is a norm, therefore by norm triangle inequality we have

\[
\sqrt{\text{Var}(\hat{v}_h^{\pi^*})} \leq \sqrt{\text{Var}(\hat{v}_h^{\pi^*} - \hat{v}_h^{\pi})} + \sqrt{\text{Var}(\hat{v}_h^{\pi} - v_h^{\pi})} + \sqrt{\text{Var}(v_h^{\pi})} \\
\leq \left\| \hat{v}_h^{\pi^*} - v_h^{\pi} \right\|_\infty \cdot 1 + \left\| \hat{v}_h^{\pi} - v_h^{\pi} \right\|_\infty \cdot 1 + \sqrt{\text{Var}(v_h^{\pi})} \\
\leq \epsilon_{opt} \cdot 1 + \left\| \hat{Q}_h - Q_h^{\pi} \right\|_\infty \cdot 1 + \sqrt{\text{Var}(v_h^{\pi})}
\]

(27)
Plug this into (27) to obtain
\[
\left| \hat{Q}_t^{\pi} - Q_t^{\pi} \right| \leq 4\sqrt{\frac{\log(HSA/\delta)}{N}} \sum_{h=t+1}^{H} \left( \Gamma_{t+1:h-1}^{\pi} \sqrt{\text{Var}(v_h^{\pi})} + \left\| \hat{Q}_h^{\pi} - Q_h^{\pi} \right\|_{\infty} \cdot 1 \right) + \frac{4H^2}{3N} \log\left( \frac{HSA}{\delta} \right) \cdot \mathbf{1} \\
+ c_2 \epsilon_{\text{opt}} \cdot \sqrt{\frac{H^2S^2 \log(HSA/\delta)}{N}} \cdot \mathbf{1}.
\]

(28)

Next lemma helps us to bound \( \sum_{h=t+1}^{H} \Gamma_{t+1:h-1}^{\pi} \sqrt{\text{Var}(v_h^{\pi})} \).

**Lemma F.3.** A conditional version of Lemma E.4 holds:

\[
\text{Var}_{\pi} \left[ \sum_{t=h}^{H} r_t^{(1)} \left| s_h^{(1)} = s_h, a_h^{(1)} = a_h \right. \right] = \sum_{t=h}^{H} \left( \mathbb{E}_{\pi} \text{Var} \left[ r_t^{(1)} + v_{t+1}^{\pi}(s_{t+1}^{(1)}, a_{t+1}^{(1)}) \mid s_t^{(1)} = s_h, a_t^{(1)} = a_h \right] \right) + \mathbb{E}_{\pi} \left[ \text{Var} \left[ \mathbb{E}[r_t^{(1)} + v_{t+1}^{\pi}(s_{t+1}^{(1)}, a_{t+1}^{(1)}) \mid s_{t+1}^{(1)} = s_h, a_{t+1}^{(1)} = a_h] \right] \right],
\]

and by using (29) we can show

\[
\sum_{h=t+1}^{H} \Gamma_{t+1:h-1}^{\pi} \sqrt{\text{Var}(v_h^{\pi})} \leq \sqrt{(H - t)^3} \cdot \mathbf{1}.
\]

**Proof.** The proof of (29) uses the identical trick as Lemma E.4 except the total law of variance is replaced by the total law of conditional variance.

Moreover, recall \( \Gamma_{t+1:h-1}^{\pi} = \prod_{i=t+1}^{h-1} P_i^{\pi} \) is the multi-step transition, so for any pair \( (s_t, a_t) \),

\[
\sum_{h=t+1}^{H} \Gamma_{t+1:h-1}^{\pi} \sqrt{\text{Var}(v_h^{\pi})} (s_t, a_t)
\]

\[
= \sum_{h=t+1}^{H} \sum_{s_{h-1}, a_{h-1}} \sqrt{\text{Var}[v_{h-1}^{\pi} \mid s_{h-1}, a_{h-1}]} d_v^{\pi}(s_{h-1}, a_{h-1} \mid s_t, a_t)
\]

\[
= \sum_{h=t+1}^{H} \sum_{h=1} \sum_{s_{h-1}, a_{h-1}} \sqrt{\text{Var}[v_{h-1}^{\pi} \mid s_{h-1}, a_{h-1}]} d_v^{\pi}(s_{h-1}, a_{h-1} \mid s_t, a_t) \cdot \sqrt{d_v^{\pi}(s_{h-1}, a_{h-1} \mid s_t, a_t)}
\]

\[
\leq \sum_{h=t+1}^{H} \sum_{s_{h-1}, a_{h-1}} \sqrt{\text{Var}[v_{h-1}^{\pi} \mid s_{h-1}, a_{h-1}]} d_v^{\pi}(s_{h-1}, a_{h-1} \mid s_t, a_t) \cdot \sum_{s_{h-1}, a_{h-1}} d_v^{\pi}(s_{h-1}, a_{h-1} \mid s_t, a_t)
\]

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Apply Lemma F.3 to bound (28), and use ∞ norm on both sides, we obtain

**Theorem F.4.** Conditional on \( N > 0 \), then with probability \( 1 - \delta \), we have for all \( t = 1, ..., H - 1 \)

\[
\|\hat{Q}^\pi_t - Q^\pi_t\|_\infty \leq 4\sqrt{\frac{H^3 \log(HSA/\delta)}{N}} + 4\sqrt{\frac{\log(HSA/\delta)}{N}} \sum_{h=t+1}^{H} \|\hat{Q}^\pi_h - Q^\pi_h\|_\infty + \frac{4H^2}{3N} \log\left(\frac{HSA}{\delta}\right)
\]

\[\leq (9 + c_2\epsilon_{opt}) \sqrt{\frac{H^3 S^2 \log(HSA/\delta)}{N}}.\]

Then by using backward induction and Theorem F.4, we have the following:

**Theorem F.5.** Suppose \( N \geq 64H^2 \cdot \log(HSA/\delta) \) and \( \epsilon_{opt} \leq \sqrt{H}/S \), then we have with probability \( 1 - \delta \),

\[
\|\hat{Q}^\pi_t - Q^\pi_t\|_\infty \leq 2(9 + c_2) \sqrt{\frac{H^3 \log(HSA/\delta)}{N}}
\]

where \( c_2 \) is the same constant in Theorem F.4.

**Proof.** Under the condition, by Theorem F.4 it is easy to check for all \( t = 1, ..., H - 1 \) with probability \( 1 - \delta \),

\[
\|\hat{Q}^\pi_t - Q^\pi_t\|_\infty \leq (5 + c_2) \sqrt{\frac{H^3 \log(HSA/\delta)}{N}} + 4\sqrt{\frac{\log(HSA/\delta)}{N}} \sum_{h=t+1}^{H} \|\hat{Q}^\pi_h - Q^\pi_h\|_\infty.
\]
which we conditional on.

For $t = H - 1$, we have

$$
\left\| \hat{Q}_{H-1}^\ast - Q_{H-1}^\ast \right\|_\infty \leq (5 + c_2) \sqrt{\frac{H^3 \log(HSA/\delta)}{N}} + 4 \sqrt{\frac{\log(HSA/\delta)}{N}} \left\| \hat{Q}_H^\ast - Q_H^\ast \right\|_\infty \\
\leq (5 + c_2) \sqrt{\frac{H^3 \log(HSA/\delta)}{N}} + 4 \sqrt{\frac{H^2 \log(HSA/\delta)}{N}} \\
\leq (5 + c_2) \sqrt{\frac{H^3 \log(HSA/\delta)}{N}}
$$

Suppose $\left\| \hat{Q}_h^\ast - Q_h^\ast \right\|_\infty \leq 2(9 + c_2) \sqrt{\frac{H^3 \log(HSA/\delta)}{N}}$ holds for all $h = t + 1, \ldots, H$, then for $h = t$, we have

$$
\left\| \hat{Q}_t^\ast - Q_t^\ast \right\|_\infty \leq (5 + c_2) \sqrt{\frac{H^3 \log(HSA/\delta)}{N}} + 4 \sqrt{\frac{\log(HSA/\delta)}{N}} \sum_{h=t+1}^{H} \left\| \hat{Q}_h^\ast - Q_h^\ast \right\|_\infty \\
\leq (9 + c_2) \sqrt{\frac{H^3 \log(HSA/\delta)}{N}} + 4 \sqrt{(H - 1)^2 \log(HSA/\delta)} \cdot 2(9 + c_2) \sqrt{\frac{H^3 \log(HSA/\delta)}{N}} \\
\leq 2(9 + c_2) \sqrt{\frac{H^3 \log(HSA/\delta)}{N}}
$$

where the last line uses the condition $N \geq 64H^2 \cdot \log(HSA/\delta)$. By induction, we have the result.

\[ \Box \]

\textbf{Proof of Theorem 3.7.} By Theorem F.5 we have for $N \geq c \cdot H^2 \cdot \log(HSA/\delta)$,

$$
P \left( \left\| \hat{Q}_1^\ast - Q_1^\ast \right\|_\infty \geq 2(9 + c_2) \sqrt{\frac{H^3 \log(HSA/\delta)}{N}} \right) \leq \delta
$$

The only thing left is to use Lemma 3.1 to bound the event that $\{N < nd_m/2\}$ has small probability.

Last but not least, the condition $n > c_1H^2 \log(HSA/\delta)/d_m$ is sufficient for applying Lemma 3.1 and it also implies $N \geq c \cdot H^2 \cdot \log(HSA/\delta)$ (the condition of Theorem F.5) when $N \geq nd_m/2$ since:

$$
n > c_1H^2 \log(HSA/\delta)/d_m \Rightarrow nd_m/2 \geq c_2H^2 \log(HSA/\delta)
$$

which implies $N \geq c_2 \cdot H^2 \cdot \log(HSA/\delta)$ when $N \geq nd_m/2$.

\[ \Box \]
G Proof of uniform convergence lower bound.

In this section we prove a uniform convergence OPE lower bound of $\Omega(H^3SA/\epsilon^2)$. Conceptually, uniform convergence lower bound can be derived by a reduction to the lower bound of identifying the $\epsilon$-optimal policy. There are quite a few literature that provide information theoretical lower bounds in different setting, e.g. Dann & Brunskill (2015); Krishnamurthy et al. (2016); Jin et al. (2018); Sidford et al. (2018). However, to the best of our knowledge, there is no result proven for the non-stationary transition finite horizon episodic setting with bounded rewards. For example, Sidford et al. (2018) prove the result sample complexity lower bound of $\Omega(H^3SA/\epsilon^2)$ with stationary MDP and their proof cannot be directly applied to non-stationary setting as they reduce the problem to infinite horizon discounted setting which always has stationary transitions. Dann & Brunskill (2015) prove the episode complexity of $\bar{\Omega}(H^2SA/\epsilon^2)$ for the stationary transition setting. Jin et al. (2018) prove the $\Omega(\sqrt{H^2SAT})$ regret lower bound for non-stationary finite horizon online setting but it is not clear how to translate the regret to PAC-learning setting by keeping the same sample complexity optimality. Jiang et al. (2017) prove the $\Omega(HSA/\epsilon^2)$ lower bound for the non-stationary finite horizon offline episodic setting where they assume $\sum_{i=1}^{H} r_i \leq 1$ and this is also different from our setting since we have $0 \leq r_t \leq 1$ for each time step.

Our proof consists of two steps. We will first show the lower bound for learning $\epsilon$-optimal policy is $\Omega(H^3SA/\epsilon^2)$ and then prove the uniform convergence OPE lower bound of the same rate.

G.1 Information theoretical lower sample complexity bound for identifying $\epsilon$-optimal policy.

In fact, a modified construction of Theorem 5 in Jiang et al. (2017) is our tool for obtaining $\Omega(H^3SA/\epsilon^2)$ lower bound. We can get the additional $H^2$ factor by using $\sum_{i=1}^{H} r_i$ can be of order $O(H)$.

**Theorem G.1.** Given $H \geq 2$, $A \geq 2$, $0 < \epsilon < \frac{1}{48\sqrt{8}}$ and $S \geq c_1$ where $c_1$ is a universal constant. Then there exists another universal constant $c$ such that for any algorithm and any $n \leq cH^3SA/\epsilon^2$, there exists a non-stationary $H$ horizon MDP with probability at least $1/12$, the algorithm outputs a policy $\hat{\pi}$ with $v^* - v^{\hat{\pi}} \geq \epsilon$.

Like in Jiang et al. (2017), the proof relies on embedding $\Theta(HS)$ independent multi-arm bandit problems into a hard-to-learn MDP so that any algorithm that wants to output a near-optimal policy needs to identify the best action in $\Omega(HS)$ problems. In our construction, the first half of the hard-to-learn MDP instance is identical to the case in Jiang et al. (2017) and the latter half uses a “naive” copy construction which is uninformative. The uninformative extension will help to produce the additional $H^2$ factor.
Proof of Theorem G.1. We construct a non-stationary MDP with $S$ states per level, $A$ actions per state and has horizon $2H$. At each time step, states are categorized into four types with three special states $w_h$, $g_h$, $b_h$ and the remaining $S - 3$ “bandit” states denoted by $s_{h,i}$, $i \in [S - 3]$. Each bandit state has an unknown best action $a_{h,i}^*$ that provides the highest expected reward comparing to other actions.

The transition dynamics are defined as follows:

- for $h = 1, \ldots, H - 1$,
  - For waiting states $w_h$, all actions have equivalent state-action transitions, i.e. for all $a \in A$, we have
    \[
    P(\cdot | w_h, a) = \begin{cases} 
      1 - \frac{1}{H} & \text{if } \cdot = w_{h+1} \\
      \frac{1}{H(S-3)} & \text{if } \cdot = s_{h+1,i}, i = 1, \ldots, S - 3 
    \end{cases}
    \]
    - $g_h$ always transitions to $g_{h+1}$ and $b_h$ always transitions to $b_{h+1}$, i.e. for all $a \in A$, we have
      \[
      P(g_{h+1} | g_h, a) = 1, \quad P(b_{h+1} | b_h, a) = 1.
      \]
    - For each bandit state $s_{h,i}$, the corresponding optimal action $a_{h,i}^*$ follows:
      \[
      P(\cdot | s_{h,i}, a_{h,i}^*) = \begin{cases} 
        1/2 + \tau, & \text{if } \cdot = g_{h+1} \\
        1/2 - \tau, & \text{if } \cdot = b_{h+1},
      \end{cases}
      \]
      and all other actions $a \in A$ follow:
      \[
      P(\cdot | s_{h,i}, a) = \begin{cases} 
        1/2, & \text{if } \cdot = g_{h+1}, \\
        1/2, & \text{if } \cdot = b_{h+1}.
      \end{cases}
      \]

    We will determine parameter $\tau$ at the end of the proof.

- for $h = H, \ldots, 2H - 1$, all states will always transition to the same type of states for the next step, i.e. $\forall a \in A$,
  \[
  P(g_{h+1} | g_h, a) = P(b_{h+1} | b_h, a) = P(w_{h+1} | w_h, a) = P(s_{h+1,i} | s_{h,i}, a) = 1, \; \forall i \in [S - 3].
  \] (30)

- The initial distribution is decided by:
  \[
  P(w_1) = 1 - 1/H, \quad P(s_{1,i}) = \frac{1}{H(S-3)}, \; \forall i \in [S - 3].
  \]
State $s$ will receive reward 1 if and only if $s = g_h$ and $h \geq H$. The reward at all other states is zero. By this construction the optimal policy must take $a_{h,i}^*$ for each bandit state $s_{h,i}$ for at least the first half of the MDP, i.e. need to take $a_{h,i}^*$ for $h \leq H$. In other words, this construction embeds at least $H(S - 3)$ independent best arm identification problems that are identical to the stochastic multi-arm bandit problem in Lemma A.8 into the MDP. The reason why these problems are independent is because the samples collected from one provides no information about any others.

Notice in our construction, for any bandit state $s_{h,i}$ with $h \leq H$, the difference of the expected reward between optimal action $a_{h,i}^*$ and other action is:

$$(\frac{1}{2} + \tau) \cdot \mathbb{E}[r_{(h+1):2H} | g_{h+1}] + (\frac{1}{2} - \tau) \cdot \mathbb{E}[r_{(h+1):2H} | b_{h+1}] - \frac{1}{2} \cdot \mathbb{E}[r_{(h+1):2H} | g_{h+1}] + \frac{1}{2} \cdot \mathbb{E}[r_{(h+1):2H} | b_{h+1}]$$

$= \left(\frac{1}{2} + \tau\right) \cdot H + \left(\frac{1}{2} - \tau\right) \cdot 0 - \frac{1}{2} \cdot H + \frac{1}{2} \cdot 0 = \tau H$ \hspace{1cm} (31)

so it seems by Lemma A.8 one suffices to use the least possible $\frac{A}{\tau^2H^2}$ samples to identify the best action $a_{h,i}^*$. However, note the construction of the latter half of the MDP (30) uses mindless reproduction of previous steps and therefore provides no additional information about the best action once the state at time $H$ is known. In other words, observing $t_H r_h = H$ is equivalent as observing $t^H H r_t = 1$. Therefore, for the bandit states in the first half the samples that provide information for identifying the best arm is up to time $H$.

As a result, the difference of the expected reward between optimal action $a_{h,i}^*$ and other action for identifying the best arm should be corrected as:

$$(\frac{1}{2} + \tau) \cdot \mathbb{E}[r_{(h+1):H} | g_{h+1}] + (\frac{1}{2} - \tau) \cdot \mathbb{E}[r_{(h+1):H} | b_{h+1}] - \frac{1}{2} \cdot \mathbb{E}[r_{(h+1):H} | g_{h+1}] + \frac{1}{2} \cdot \mathbb{E}[r_{(h+1):H} | b_{h+1}]$$

$= \left(\frac{1}{2} + \tau\right) \cdot 1 + \left(\frac{1}{2} - \tau\right) \cdot 0 - \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \tau.$

Now by Lemma A.8, for each bandit state $s_{h,i}$ satisfying $h \leq H$, unless $\frac{A}{\tau^2H^2}$ samples are collected from that state, the learning algorithm fails to identify the optimal action $a_{h,i}^*$ with probability at least $1/3$.

After running any algorithm, let $C$ be the set of $(h,s)$ pairs for which the algorithm identifies the correct action. Let $D$ be the set of $(h,s)$ pairs for which the algorithm collects fewer than $\frac{A}{\tau^2H^2}$ samples. Then by Lemma A.8 we have

$$\mathbb{E}[|C|] = \mathbb{E} \left[ \sum_{(h,s)} \mathbb{1}[a_{h,s} = a_{h,s}^*] \right] \leq (S - 3)H - |D| + \mathbb{E} \left[ \sum_{(h,s) \in D} \mathbb{1}[a_{h,s} = a_{h,s}^*] \right]$$

$$\leq (S - 3)H - |D| + \frac{2}{3} |D| = (S - 3)H - \frac{1}{3} |D|.$$
If we have \( n \leq \frac{H(S-3)}{2} \times \frac{A}{127} \), by pigeonhole principle the algorithm can collect \( \frac{A}{127} \) samples for at most half of the bandit problems, i.e. \( |D| \geq H(S-3)/2 \). Therefore we have

\[
\mathbb{E}[|C|] \leq (S-3)H - \frac{1}{3} |D| \leq \frac{5}{6}(S-3)H.
\]

Then by Markov inequality

\[
P\left(|C| \geq \frac{11}{12}H(S-3) \right) \leq \frac{5/6}{11/12} = \frac{10}{11}
\]

so the algorithm failed to identify the optimal action on 1/12 fraction of the bandit problems with probability at least 1/11. Note for each failure in identification, the reward is different by \( \tau H \) (see (31)), therefore under the event \( \{|C'| \geq \frac{1}{12}H(S-3)\} \), following the similar calculation of Jiang et al. (2017) the suboptimality of the policy produced by the algorithm is

\[
\epsilon := v^* - v^\pi = \mathbb{P}\text{[visit } C'] \times \tau H + \mathbb{P}\text{[visit } C] \times 0 = \mathbb{P}\left[ \bigcup_{(h,i) \in C'} \text{visit}(h,i) \right] \times \tau H
\]

\[
= \sum_{(h,i) \in C'} \mathbb{P}\text{[visit}(h,i)] \times \tau H = \sum_{(h,i) \in C'} \frac{1}{H(S-3)}(1 - 1/H)^{h-1} \tau H
\]

\[
\geq \sum_{(h,i) \in C'} \frac{1}{H(S-3)}(1 - 1/H)^{1/H} \tau H \geq \sum_{(h,i) \in C'} \frac{1}{H(S-3)} \frac{1}{4} \tau H
\]

\[
\geq \frac{H(S-3)}{12} \frac{1}{H(S-3)} \frac{1}{4} \tau H = \frac{\tau H}{48}.
\]

where the third equal sign uses all best arm identification problems are independent. Now we set \( \tau = \min(\sqrt{1/8}, 48\epsilon/H) \) and under condition \( n \leq cH^3SA/\epsilon^2 \), we have

\[
n \leq cH^3SA/\epsilon^2 \leq cHSA/(\epsilon/H)^2 \leq c48^2HSA/\tau^2 = c48^2 \cdot 72HS \cdot \frac{A}{72\tau^2} := c'HS \cdot \frac{A}{72\tau^2} \leq \frac{H(S-3)}{2} \cdot \frac{A}{72\tau^2},
\]

the last inequality holds as long as \( S \geq 3/(1-2c') \). Therefore in this situation, with probability at least 1/11, \( v^* - v^\pi \geq \epsilon \). Finally, we can use scaling to reduce the horizon from \( 2H \) to \( H \).

Remark G.2. A directly corollary is that the sample complexity in Theorem 4.2 is optimal. Indeed, for the case \( \epsilon_{\text{opt}} = 0 \), Theorem 4.2 implies \( \hat{\pi} \) is the \( \epsilon \)-optimal policy learned with sample complexity \( O(H^3 \log(HSA/\delta)/d_m \epsilon^2) \). Theorem G.1 implies this sample complexity cannot be further reduced up to the logarithmic factor.

Remark G.2. A directly corollary is that the sample complexity in Theorem 4.2 is optimal. Indeed, for the case \( \epsilon_{\text{opt}} = 0 \), Theorem 4.2 implies \( \hat{\pi} \) is the \( \epsilon \)-optimal policy learned with sample complexity \( O(H^3 \log(HSA/\delta)/d_m \epsilon^2) \). Theorem G.1 implies this sample complexity cannot be further reduced up to the logarithmic factor.
G.2 Information theoretical lower sample complexity bound for uniform convergence in OPE.

By applying Theorem G.1, we can now prove Theorem 3.8.

**Proof of Theorem 3.8.** We prove it by contradiction. Suppose there is one off-policy evaluation method \( \hat{v}^{\pi} \) such that

\[
\sup_{\pi \in \Pi} |\hat{v}^{\pi} - v^{\pi}| \leq o \left( \sqrt{\frac{H^3SA}{n}} \right),
\]

where \( o(\cdot) \) represents the standard small \( o \)-notation. Then by Theorem 4.1 this OPE method implies a \( \epsilon \)-optimal policy learning algorithm with sample complexity \( o \left( \frac{H^3SA}{\epsilon^2} \right) \) which is smaller than the information theoretical lower bound obtained in Theorem G.1. Contradiction!

**H Proofs of Theorem 4.2**

**Proof of Theorem 4.2.** First note by definition of empirical optimal policy we have \( \hat{Q}^{\pi^*} \leq \hat{Q}^{\pi^*} \), so we have the following:

\[
Q_{1}^{\pi^*} - Q_{1} = Q_{1}^{\pi^*} - \hat{Q}_{1}^{\pi^*} + \hat{Q}_{1}^{\pi^*} - \hat{Q}_{1}^{\pi} + \hat{Q}_{1}^{\pi} - Q_{1}^{\pi}
\leq Q_{1}^{\pi^*} - \hat{Q}_{1}^{\pi^*} + \hat{Q}_{1}^{\pi^*} - \hat{Q}_{1}^{\pi} + \hat{Q}_{1}^{\pi} - Q_{1}^{\pi}
\leq Q_{1}^{\pi^*} - \hat{Q}_{1}^{\pi^*} + \epsilon_{opt} \cdot 1 + \hat{Q}_{1}^{\pi} - Q_{1}^{\pi}
\]

and \( \hat{Q}_{1}^{\pi} - Q_{1}^{\pi} \) can be bounded by Theorem 3.7 using local uniform convergence. \( Q_{1}^{\pi^*} - \hat{Q}_{1}^{\pi^*} \) can be bounded by \( O(\sqrt{\frac{H^3 \log(HSA/\delta)}{n \delta m}}) \) using the similar technique in Section F even without introducing \( \epsilon_{opt} \) since \( \pi^* \) is a fixed policy.

**I On improvement over vanilla simulation lemma**

**Vanilla simulation lemma, Lemma 1 of Jiang (2018).** Without loss of generality, assuming reward is determinsitic function over state-action. By definition of Bellman equation, we have the following:

\[
\hat{v}_{t}^{\pi} = r + \hat{P}_{t+1} \hat{v}_{t+1}^{\pi}, \quad v_{t}^{\pi} = r + P_{t+1} v_{t+1}^{\pi},
\]

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define $\epsilon_P = \sup_{t,s,t,a} ||\hat{P}_t(\cdot|s_t,a_t) - P_t(\cdot|s_t,a_t)||_1$, then by Hoeffding’s inequality and union bound, with probability $1 - \delta$,

$$
\epsilon_P \leq S \cdot \sup_{t,s,t,a} ||\hat{P}_t(\cdot|s_t,a_t) - P_t(\cdot|s_t,a_t)||_\infty \leq S \cdot \sup_{t,s,t,a} O \left( \sqrt{\log(HSA/\delta)} \right)

\leq O \left( \frac{S^2 \log(HSA/\delta)}{n \cdot d_m} \right)

\text{then}

\hat{v}_t^\pi - v_t^\pi = \hat{P}^\pi_{t+1} \hat{v}_t^{\pi_{t+1}} - P^\pi_{t+1} v_t^{\pi_{t+1}}

\leq \left( ||\hat{P}^\pi_{t+1} - P^\pi_{t+1}||_1 ||\hat{v}_t^{\pi_{t+1}}||_\infty + ||P^\pi_{t+1}||_1 ||\hat{v}_t^{\pi_{t+1}} - v_t^{\pi_{t+1}}||_\infty \right) \cdot 1

\leq (H \epsilon_P + ||\hat{v}_t^{\pi_{t+1}} - v_t^{\pi_{t+1}}||_\infty) \cdot 1,

\text{solving recursively, we have}

$$
||\hat{v}_1^\pi - v_1^\pi||_\infty \leq H^2 \epsilon_P \leq O \left( \sqrt{\frac{H^4 S^2 \log(HSA/\delta)}{n \cdot d_m}} \right).

$$

This verifies our claim in discussion section 6. We do point out above standard analysis can be improved by Jiang (2018) Section 2.2 to $\tilde{O}\sqrt{H^4 S/(n \cdot d_m)}$, then in this case our analysis (Lemma 3.4) has an improvement of $H^2 S$ with respect to the modified result.

**J Algorithms**

**Remark J.1.** We can see Algorithm 2 requires the splitting data size $M$ which is undecided by Yin \& Wang (2020) and that makes the hyper-parameter requiring additional concrete specifications to make the data splitting estimator sample efficient. In contrast, OPEMA in Algorithm 1 is defined without ambiguity and can be implemented without extra work.
Algorithm 1 OPEMA

**Input:** Logging data \( \mathcal{D} = \{\{s_t^{(i)}, a_t^{(i)}, r_t^{(i)}\}_{t=1}^H\}_{i=1}^n \) from the behavior policy \( \mu \). A target policy \( \pi \) which we want to evaluate its cumulative reward.

1. Calculate the on-policy estimation of initial distribution \( d_1(\cdot) \) by \( \hat{d}_1(s) := \frac{1}{n} \sum_{i=1}^n 1(s^{(i)} = s) \), and set \( \hat{d}_1^n(\cdot) := \hat{d}_1(\cdot), \hat{d}_1^n(s) := \hat{d}_1(s) \).
2. **for** \( t = 2, 3, \ldots, H \) **do**
3. Choose all transition data at time step \( t \), \( \{s_t^{(i)}, a_t^{(i)}, r_t^{(i)}\}_{i=1}^n \).
4. Calculate the on-policy estimation of \( d_t^n(\cdot) \) by \( \hat{d}_t^n(s) := \frac{1}{n} \sum_{i=1}^n 1(s^{(i)} = s) \).
5. Set the off-policy estimation of \( \hat{P}_t(s_t|s_{t-1}, a_{t-1}) \):
\[
\hat{P}_t(s_t|s_{t-1}, a_{t-1}) := \frac{\sum_{i=1}^n 1([s_t^{(i)}, a_{t-1}^{(i)}, s_{t-1}^{(i)}] = (s_t, s_{t-1}, a_{t-1}))}{n_{s_{t-1}, a_{t-1}}}
\]

when \( n_{s_{t-1}, a_{t-1}} > 0 \). Otherwise set it to be zero.
6. Estimate the reward function
\[
\hat{r}_t(s_t, a_t) := \frac{\sum_{i=1}^n r_t^{(i)} 1(s_t^{(i)} = s_t, a_t^{(i)} = a_t)}{\sum_{i=1}^n 1(s_t^{(i)} = s_t, a_t^{(i)} = a_t)}.
\]

when \( n_{s_t, a_t} > 0 \). Otherwise set it to be zero.
7. **Set** \( \hat{d}_t^n(\cdot, \cdot) \) **according to** \( \hat{P}_t^n \hat{d}_t^n_{t-1} \), where \( \hat{d}_t^n(\cdot, \cdot) \) is the estimated state-action distribution.
8. **end for**
9. Substitute the all estimated values above into \( \hat{v}^\pi = \sum_{t=1}^H \langle \hat{d}_t^n, \hat{r}_t \rangle \) to obtain \( \hat{v}^\pi \), the estimated value of \( \pi \).

Algorithm 2 Data Splitting TMIS in Yin & Wang (2020)

**Input:** Logging data \( \mathcal{D} = \{\{s_t^{(i)}, a_t^{(i)}, r_t^{(i)}\}_{t=1}^H\}_{i=1}^n \) from the behavior policy \( \mu \). A target policy \( \pi \) which we want to evaluate its cumulative reward. Requiring splitting data size \( M \).

1: Randomly splitting the data \( \mathcal{D} \) evenly into \( N \) folds, with each fold \( |\mathcal{D}^{(i)}| = M \), i.e. \( n = M \cdot N \).
2. **for** \( i = 1, 2, \ldots, N \) **do**
3. Use Algorithm 1 to estimate \( \hat{v}^\pi_{(i)} \) with data \( \mathcal{D}^{(i)} \).
4. **end for**
5. Use the mean of \( \hat{v}^\pi_{(1)}, \hat{v}^\pi_{(2)}, \ldots, \hat{v}^\pi_{(N)} \) as the final estimation of \( v^\pi \).